

1 Subspaces of topological spaces

Definition 1.1. (Subspace topology.) Let (X, \mathcal{T}_X) be a topological space, and let $S \subseteq X$ be any subset. Then S inherits the structure of a topological space, defined by the topology

$$\mathcal{T}_S = \{U \cap S \mid U \in \mathcal{T}_X\}.$$

The topology \mathcal{T}_S on S is called the *subspace topology*.

Example 1.2. Describe the subspace topology on the following subsets of \mathbb{R} , with the topology induced by the Euclidean metric (we call this the “standard topology”).

(a) $S = \{0, 1, 2\}$

(b) $S = (0, 1)$

In-class Exercises

1. Verify that the subspace topology is, in fact, a topology.
2. Let (X, \mathcal{T}_X) be a topological space, and let $S \subseteq X$ be any subset. Let ι_S be the *inclusion map*

$$\begin{aligned}\iota_S : S &\rightarrow X \\ \iota_S(s) &= s\end{aligned}$$

Verify that the subspace topology on S is precisely the set $\{i_S^{-1}(U) \mid U \subseteq X \text{ is open}\}$.

Remark: We haven’t defined these terms, but we can summarize this result by the slogan “the subspace topology on S is the coarsest topology that makes the inclusion maps ι_S continuous”.

3. Let (X, \mathcal{T}_X) be a topological space, and let $S \subseteq X$ be a subset. Let \mathcal{T}_S denote the subspace topology on S .
 - (a) Show by example that an open subset of S (in the subspace topology \mathcal{T}_S) may not be open as a subset of X . In other words, show there could be a subset $U \subseteq S$ with $U \in \mathcal{T}_S$, $U \notin \mathcal{T}_X$.
 - (b) Conversely, suppose that $U \subseteq S$ and U is open in X . Show that U is open in the subspace topology on S . In other words, for $U \subseteq S$, if $U \in \mathcal{T}_X$ then $U \in \mathcal{T}_S$.
 - (c) Suppose that S is an open subset of X . Show that a subset $U \subseteq S$ is open in S (with the subspace topology) if and only if it is open in X . In other words, whenever S is open and $U \subseteq S$, $U \in \mathcal{T}_S$ if and only if $U \in \mathcal{T}_X$.
4. Let (X, \mathcal{T}_X) be a topological space and let $S \subseteq X$ be a subset endowed with the subspace topology \mathcal{T}_S . Show that a set $C \subseteq S$ is closed in S if and only if there is some set $D \subseteq X$ that is closed in X with $C = D \cap S$.

5. **(Optional)**. Let (X, \mathcal{T}_X) be a topological space, and let $Z \subseteq Y \subseteq X$ be subsets. Show that the subspace topology on Z as a subspace of X coincides with the subspace topology on X as a subspace of Y (with the subspace topology as a subset of X). Conclude that there is no ambiguity in how to topologize the subset Z – to refer to its “subspace topology” we do not need to specify whether Y or X is the ambient space.
6. **(Optional)**. Let (X, d) be a metric space, and let \mathcal{T}_d^X be the topology induced by the metric. Let $S \subseteq X$ be a subset. We now have two methods of constructing a topology on S : we can restrict the metric from X to S , and take the topology \mathcal{T}_d^S induced by the metric. We can also take the subspace topology \mathcal{T}_S defined by \mathcal{T}_d^X . Show that these two topologies on S are equal, so there is no ambiguity in how to topologize a subset of a metric space.
7. **(Optional)**. Let (X, d) be a metric space with the metric topology \mathcal{T}_d . Show that the subspace topology on any **finite** subset of X is the discrete topology.
8. **(Optional)**. Let (X, \mathcal{T}) be a topological space, and $S \subseteq X$ a subset endowed with the subspace topology.
 - (a) Suppose X has the discrete topology. Must S have the discrete topology?
 - (b) Suppose X has the indiscrete topology. Must S have the indiscrete topology?
 - (c) Suppose X is metrizable. Is S metrizable?
 - (d) Recall that a topological space is *Hausdorff* if every pair of points have disjoint open neighbourhoods. If X is Hausdorff, then must S be Hausdorff?
 - (e) A space has the T_1 *property* if every singleton subset $\{x\}$ is closed. If X is T_1 , then must S be T_1 ?
 - (f) For which of the above does the converse hold?

Remark: A property is called *hereditary* if, whenever a topological space has the property, all of its subspaces necessarily have the property.

9. **(Optional)**. Consider \mathbb{R} with the standard topology (that is, the topology induced by the Euclidean metric). For each of the following statements, construct a nonempty subset S of \mathbb{R} with that satisfies the description, or prove that none exists.
 - (a) S is an infinite, closed subset of \mathbb{R} , and the subspace topology on S is discrete.
 - (b) S is not a closed subset of \mathbb{R} , and the subspace topology on S is discrete.
 - (c) S has the indiscrete topology.
 - (d) The subspace topology on S consists of exactly 2 open subsets.
 - (e) The subspace topology on S consists of exactly 3 open subsets.