1 Bases for topological spaces

Definition 1.1. (Basis of a topology.) Let $(X, \mathcal{T})$ be a topological space. We say that a collection $\mathcal{B}$ of subsets of X is a *basis* for the topology $\mathcal{T}$ if

- $\mathcal{B} \subseteq \mathcal{T}$, that is, every basis element is open, and
- every element of $\mathcal{T}$ can be expressed as a union of elements of $\mathcal{B}$.

We say that the basis $\mathcal{B}$ *generates* the topology $\mathcal{T}$.

Remark 1.2. By convention, we say that the empty set $\varnothing$ is the union of an empty collection of open sets. So a basis $\mathcal{B}$ does not need to include $\varnothing$.

Example 1.3. Let X be a set.
(a) Find a basis for the discrete topology on X.

(b) Find a basis for the indiscrete topology on X.

In-class Exercises

1. **(The basis criteria).** Let $(X, \mathcal{T})$ be a topological space. Show that a collection $\mathcal{B}$ of subsets of X is a basis for $\mathcal{T}$ if and only if it satisfies the following two conditions:

   (i) Every basis element is an **open** set, that is, $\mathcal{B} \subseteq \mathcal{T}$.

   (ii) For every open set $U \in \mathcal{T}$, and every $x \in U$, there exists some element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

2. Let (X, d) be a metric space. Show that the set of open balls

$$\mathcal{B} = \{ \ B_r(x_0) \mid x_0 \in X, r \in \mathbb{R}, r > 0 \ \}$$

   is a basis for the topology induced by d.

3. Let (X, d_X) and (Y, d_Y) be metric spaces. Recall that $X \times Y$ is then a metric space with metric

$$d_{X\times Y} : \ (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

   (a) Show that the set

$$\mathcal{B} = \{ \ U \times V \mid U \subseteq X \text{ is open, } V \subseteq Y \text{ is open } \}$$

   forms a basis for the topology induced by $d_{X\times Y}$.

   (b) Does every open set in $X \times Y$ have the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open?

4. An advantage of identifying a basis for a topology is that many topological statements can be reduced to statements about the basis. As an example, prove the following theorem.

Theorem 1.4. (Equivalent definition of continuity). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$. Prove that a map $f : X \to Y$ is continuous if and only if for every open set $U \in \mathcal{B}_Y$, the preimage $f^{-1}(U) \subseteq X$ is open.*

5. In our definition of a basis, we began with a space with a given topology, and defined a basis to be a collection of open subsets satisfying certain properties. In many cases, however, we will wish to topologize our set X by first specifying a basis, and using the basis to define a topology on X. Prove the following theorem, which gives conditions on a collection of subsets $\mathcal{B}$ of X that ensure it will generate a valid topology on X.

   Theorem 1.5. (An extrinsic definition of a basis). *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that*

   - $\displaystyle\bigcup_{B \in \mathcal{B}} B = X$
   - *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq (B_1 \cap B_2)$.*

   Let
   $$\mathcal{T} = \{ \ U \mid U \text{ is a union of elements of } \mathcal{B} \ \}.$$
   Then $\mathcal{T}$ is a topology on X, and $\mathcal{B}$ is a basis for $\mathcal{T}$. We say that $\mathcal{T}$ is the topology generated by *the basis $\mathcal{B}$.*

6. **(Optional).**
   (a) Consider the topology on $\mathbb{R}^n$ induced by the Euclidean metric. Prove that the following set is a basis for $\mathbb{R}^n$.
   $$\mathcal{B} = \{ \ B_\epsilon(x) \mid \ \epsilon > 0, \epsilon \text{ is } \textbf{rational}; x \in \mathbb{R}^n, \text{ all coordinates } x_i \text{ of } x \text{ are } \textbf{rational.}\}$$
   (b) Show that $\mathbb{R}^n$ has uncountably many open sets, but that the basis $\mathcal{B}$ is countable.

7. **(Optional).** Let (X, d) be a metric space, and $\mathcal{B}$ a basis for the topology $\mathcal{T}_d$ induced by d.
   (a) Let $S \subseteq X$ be a subset, and $s \in S$. Show that s is an interior point of S if and only if there is some element $B \in \mathcal{B}$ such that $s \in B$ and $B \subseteq S$.
   (b) Deduce that $\displaystyle\text{Int}(S) = \bigcup_{B \in \mathcal{B}, B \subseteq S} B.$

8. **(Optional). Definition (Subbases).** Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X whose union is equal to X. Then the *topology generated by the subbasis* $\mathcal{S}$ is the collection of all arbitrary unions of all finite intersections of elements in $\mathcal{S}$. *Remark*: In contrast to a basis, we are permitted to take finite intersections of sets in a subbasis.
   (a) Show that the set $\mathcal{T}$ generated by a subbasis $\mathcal{S}$ really is a topology, and is moreover the coarsest topology containing $\mathcal{S}$.
   (b) Verify that $\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}$ is a subbasis for the standard topology on $\mathbb{R}$.
   (c) Prove the following proposition.

   Proposition. Let $f : X \to Y$ be a function of topological spaces, and let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if $f^{-1}(U)$ is open for every subbasis element $U \subseteq \mathcal{S}$.