1 Sequences in topological spaces

Definition 1.1. (Convergence topological spaces.) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in a topological space $(X, \mathcal{T})$. Then we say that that $(a_n)_{n\in\mathbb{N}}$ *converges* to a point $a_\infty \in X$ if ...

Example 1.2. Let $X = \{0, 1\}$ with the topology $\mathcal{T} = \Big\{\varnothing, \{0\}, \{0,1\}\Big\}$. Find all limits of the following sequences.

(a) Constant sequence $0, 0, 0, 0, 0, \dots$

(b) Constant sequence $1, 1, 1, 1, 1, \dots$

(c) Alternating sequence $0, 1, 0, 1, 0, \dots$

Example 1.3. Let $X = \mathbb{R}$ with the cofinite topology. Find all limits of the sequence $(n)_{n\in\mathbb{N}}$.

In-class Exercises

1. Let X be a topological space, and let $x \in X$. Show that the constant sequence $a_n = x$ converges to x. Could it also converge to other points of X?

2. (a) Give an example of a topological space $(X, \mathcal{T})$, and a sequence $(a_n)_{n\in\mathbb{N}}$ in X that converges to (at least) two distinct points $a_\infty \in X$ and $\widetilde{a}_\infty \in X$.

   (b) Now suppose that $(X, \mathcal{T})$ is a **Hausdorff** topological space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that, if $(a_n)_{n\in\mathbb{N}}$ converges, then it converges to only one point a_∞.

3. **Definition (Sequential continuity).** Let $f : X \to Y$ be a function of topological spaces. Then f is called *sequentially continuous* if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X and any limit x_∞ of the sequence, the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y converges to the point $f(x_\infty)$.

   You proved that on Homework #3 Problem 4 that if X and Y are metric spaces (or, more generally, metrizable topological spaces), then sequential continuity is equivalent to continuity for a function $f : X \to Y$.

   Prove that any continuous function $f : X \to Y$ of topological spaces is sequentially continuous. It turns out that the converse does not hold in general for abstract topological spaces!

4. Let A be a subset of a topological space $(X, \mathcal{T})$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in A that converge to a point $a_\infty \in X$. Prove that $a_\infty \in \overline{A}$.

5. **(Optional).** For each of the following sequences: find the set of all limits, or determine that the sequence does not converge.
   - Let $X = \{a, b, c, d\}$ have the topology $\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\}$.
     (i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$
   - Let $\mathbb{R}$ have the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.
     (ii) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ (iii) $(n)_{n\in\mathbb{N}}$ (iv) $(-n)_{n\in\mathbb{N}}$
   - Let $\mathbb{R}$ have the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.
     (v) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ (vi) $1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \cdots$

6. **(Optional).** Let $X = \{0, 1\}$ with the topology $\mathcal{T} = \Big\{\varnothing, \{0\}, \{0, 1\}\Big\}$. Prove that every sequence of points in X converges to 1.

7. **(Optional).** In this problem, we will construct an example of a function of topological spaces that is sequentially continuous but not continuous.
   (a) Recall that a set S is *countable* if there exists an injective function $S \to \mathbb{N}$, equivalently, if there is a surjective functions $\mathbb{N} \to S$. Such sets are either finite or countably infinite. Define a topology on $\mathbb{R}$ by
   $$\mathcal{T}_{cc} = \{\varnothing\} \cup \{\ U \subseteq \mathbb{R} \mid\ \mathbb{R} \setminus U \text{ is countable }\}.$$
   Show that $\mathcal{T}_{cc}$ is indeed a topology on $\mathbb{R}$. It is called the *co-countable topology.*
   (b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathbb{R}, \mathcal{T}_{cc})$. Show that $(a_n)_{n\in\mathbb{N}}$ converges if and only if it is *eventually constant.* This means there is some $N \in \mathbb{N}$ and $x \in \mathbb{R}$ so that $a_n = x$ for all $n \geq N$.
   (c) Let $\mathcal{T}_{dsc}$ denote the discrete topology on $\mathbb{R}$. Let $I : \mathbb{R} \to \mathbb{R}$ be the identity map. Show that the following map of topological spaces is **not** continuous:
   $$I : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_{dsc})$$
   (d) Show that $I : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_{dsc})$ is sequentially continuous.

8. **(Optional).**
   (a) Suppose (X, d) is a metric space, and $A \subseteq X$. Prove that $x \in \overline{A}$ if and only if there is some sequence of points $(a_n)_{n\in\mathbb{N}}$ in A that converge to x.
   (b) Consider $\mathbb{R}$ with the cocountable topology, and let $A \subseteq \mathbb{R}$. What is $\overline{A}$ if A is (i) countable, or (ii) uncountable?
   (c) Let $A = (0, 1)$, so $\overline{A} = \mathbb{R}$. Show that, for any $x \in \overline{A} \setminus A$, there is **no** sequence of points in A that converge to x.
   (d) **Definition (First countable spaces).** A topological space $(X, \mathcal{T})$ is called *first countable* if each point $x \in X$ has a *countable neighbourhood basis.* This means, for each $x \in X$, there is a countable collection $\{N_i\}_{i\in\mathbb{N}}$ of neighbourhoods of x with the property that, if N is any neighbourhood of x, then there is some i such that $N_i \subseteq N$.
   Let X be a first countable space, and let $A \subseteq X$. Show that, given any $x \in \overline{A}$, there is some sequence of points in A that converges to x.