1 Connected topological spaces

Definition 1.1. (Disconnected spaces; connected spaces). A topological space $(X, \mathcal{T})$ is *disconnected* if there exist disjoint nonempty open subsets A and B in X such that $X = A \cup B$. We call the sets A and B a *separation* of X. If no separation of X exists, then X is called *connected*.

A subset S of X is said to be *connected* if it is connected when viewed with the subspace topology $(S, \mathcal{T}_S)$. This means ...

Example 1.2. Determine which of the following topological spaces are connected.
(a) $X = \mathbb{R}$, $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$

(b) $X = \{a, b, c, d\}$, $\mathcal{T} = \Big\{\varnothing, \{a, b\}, \{c\}, \{a, b, c\}, \{d\}, \{a, b, d\}, \{c, d\}, \{a, b, c, d\}\Big\}$

In-class Exercises

1. Show that the following topological spaces (with the Euclidean metric) are disconnected.

   (a) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ (b) $(0, 1) \cup \{5\}$ (c) $\mathbb{Q}$

2. (a) Give an example of a connected topological space X, and a subset $S \subseteq X$ that is disconnected.
   (b) Give an example of a disconnected topological space X, and a subset $S \subseteq X$ that is connected.

3. Prove that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is subset A, with $\varnothing \subsetneq A \subsetneq X$, that is both open and closed.

4. Consider $\{0, 1\}$ as a topological space with the discrete topology. Show that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is a continuous **surjective** function $X \to \{0, 1\}$.

5. Prove the following (often useful) lemma:

   Lemma. Let X be a topological space, and let A, B be a separation of X. Let $S \subseteq X$. If S is connected, then $S \subseteq A$ or $S \subseteq B$.

6. Let X be a topological space, and let A_i, $i \in I$, be a collection of subsets of X. Suppose that A_i is connected for each i.
   (a) Show by example that the union $\bigcup_{i \in I} A_i$ may be disconnected.
   (b) Suppose that $\bigcap_{i \in I} A_i \neq \varnothing$. Show that the union $\bigcup_{i \in I} A_i$ is connected.

7. **(Optional).** Consider $\mathbb{R}$ with the Euclidean metric. Which of the following subsets are connected?

$$\{x \in \mathbb{R} \mid d(x,1) < 1 \text{ or } d(x,-1) < 1\}$$
$$\{x \in \mathbb{R} \mid d(x,1) \leq 1 \text{ or } d(x,-1) < 1\}$$
$$\{x \in \mathbb{R} \mid d(x,1) \leq 1 \text{ or } d(x,-1) \leq 1\}$$

8. **(Optional).** Consider the set $X = \{a,b,c,d\}$. For which of the following topologies $\mathcal{T}$ is the topological space $(X, \mathcal{T})$ connected?

   (a) $\mathcal{T} = \Big\{\varnothing, \{c\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}\Big\}$

   (b) $\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,d\}, \{a,b,d\}, \{a,b,c,d\}\Big\}$

9. **(Optional).** Consider the following topologies on $\mathbb{R}$. Which of these topological spaces are connected?

   (a) indiscrete topology

   (b) discrete topology

   (c) Euclidean topology

   (d) cofinite topology

   (e) $\mathcal{T} = \{\mathbb{R}, (0,1), \varnothing\}$

   (f) $\mathcal{T} = \{\mathbb{R}, \{0,1\}, \{0\}, \{1\}, \varnothing\}$

   (g) $\mathcal{T} = \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$

   (h) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$

   (i) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$

   (j) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$

10. **(Optional).** Let $(X, \mathcal{T})$ be a topological space. Let $\{\ A_n \mid n \in \mathbb{Z}\ \}$ be a family of connected subspaces of X such that $A_n \cap A_{n+1} \neq \varnothing$ for every n. Prove $\bigcup_{n \in \mathbb{Z}} A_n$ is connected.

11. **(Optional).** Let $(X, \mathcal{T})$ be a topological space. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of connected subspaces in X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$. Is $\bigcap_{n \in \mathbb{N}} A_n$ is necessarily connected?

12. **(Optional).** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Suppose $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are both closed or both open, then A and B are connected.