1 Compact spaces: more properties and consequences

In-class Exercises

1. Prove the following result. This theorem is a major reason we care about compactness!

   Theorem (Generalized Extreme Value Theorem). Let X be a nonempty compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). Then $\sup(f(X)) < \infty$, and there exists some $z \in X$ such that $f(z) = \sup(f(X))$. That is, f achieves its supremum on X.

2. (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.

   (b) Prove that any compact metric space is sequentially compact.

   Combined with Homework #5 Problem 5, this exercise proves:

   Theorem (Compactness vs sequential compactness in metric spaces). Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

   (Neither direction of this theorem holds, however, for arbitrary topological spaces!)

   Combined with Worksheet #8, Problem 2, this exercise proves:

   Theorem (Compactness in $\mathbb{R}^n$). Endow $\mathbb{R}^n$ with the Euclidean metric. A subspace $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two nonempty topological spaces. Suppose that their Cartesian product $X \times Y$ is compact with respect to the product topology $\mathcal{T}_{X\times Y}$. Prove that X and Y are compact.

4. **(Optional).** The following problem (combined with Problem 3) will prove the theorem,

   Theorem 1.1. (Products of compact spaces). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be nonempty topological spaces. Then $X \times Y$ is compact with respect to the product topology $\mathcal{T}_{X\times Y}$ if and only if both X and Y are compact.*

   Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be nonempty compact topological spaces. Let $\mathcal{U}$ be any open cover of $X \times Y$ (with the product topology).
   For this exercise, we will call a subset $A \subseteq X$ *good* if $A \times Y$ is covered by a finite subcollection of open sets in $\mathcal{U}$. Our goal is to show that X is good.

   (a) Suppose that $A_1, \ldots, A_r$ is a finite collection of good subsets of X. Show that their union is good.

   (b) Fix $x \in X$. For each $y \in Y$, explain why it is possible to find open sets $U_y \in X$ and $V_y \in Y$ so that $(x, y) \in U_y \times V_y$ and $U_y \times V_y$ is contained in some open set in $\mathcal{U}$.

   (c) Explain why there is a finite list of points $y_1, \ldots, y_n \in Y$ so that the sets $\{V_{y_1}, \ldots, V_{y_n}\}$ cover Y.

(d) Define

$$U_x = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}.$$

Show that U_x is a good set, and is an open subset of X containing x. This shows that every element $x \in X$ is contained in a good open set U_x.

(e) Consider the collection of open subsets $\{U_x \mid x \in X\}$ of X. Use the fact that X is compact to conclude that X is good.

5. **(Optional).**

   Definition (Lindelöf). A topological space X is called *Lindelöf* if every open cover of X has a countable subcover.

   Suppose that X is a Lindelöf space and Y is a compact space. Prove that the product $X \times Y$, with the product topology, is Lindelöf.

6. **(Optional).** Recall that a map of topological spaces is called *closed* if the image of every closed set in the domain is a closed subset of the codomain.

   Let X and Y be topological spaces, and endow their product $X \times Y$ with the product topology. We saw on Worksheet #7 Problem 4 that the projection map $\pi_X : X \times Y \to X$ need not be closed in general. Prove that, if Y is compact, then π_X is a closed map.