1 Open and closed sets

Going forward, we will implicitly assume $\mathbb{R}^n$ has the Euclidean metric unless otherwise stated.

Definition 1.1. (Open ball of radius r about x_0.) Let (X, d) be a metric space, and $x_0 \in X$. Let $r \in \mathbb{R}$, $r > 0$. We define the *open ball of radius r about x_0* as the subset of X

$$B_r(x_0) = \{x \in X \mid d(x_0, x) < r\} \subseteq X.$$

Example 1.2. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$. What is $B_1(0)$? What is $B_2(-6)$?

Example 1.3. Let $X = \mathbb{R}^2$ with the usual Euclidean metric. Draw $B_2(1, 0)$.

Definition 1.4. (Interior points; open sets in a metric space.) Let (X, d) be a metric space, and let $U \subseteq X$ be a subset of X. A point $x \in U$ is called an *interior point of U* if there is some radius $r_x \in \mathbb{R}$, $r_x > 0$, so that $B_{r_x}(x) \subseteq U$.

The set $U \subseteq X$ is called *open* if every point $x \in U$ is an interior point of U.

Example 1.5. Show that the interval $[0, 1) \subseteq \mathbb{R}$ is not open.

Example 1.6. Show that the interval $(0, 1) \subseteq \mathbb{R}$ is open.

Proposition 1.7. *Let* (X, d) *be a metric space,* $x_0 \in X$ *and* $0 < r \in \mathbb{R}$*. Then the ball* $B_r(x_0)$ *is an open subset of* X*.*

Proof.

Definition 1.8. (Closed sets in a metric space.) A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is open.

Warning: Despite the English connotations of the words ‘closed’ and ‘open’, mathematically, ‘closed’ does **not** mean “not open”, and vice versa.

In-class Exercises

1. Let (X, d) be a metric space. Solve (with justification) the following:

   (a) Is X open? (b) Is $\varnothing$ open? (c) Is X closed? (d) Is $\varnothing$ closed?

2. Consider $\mathbb{R}$ with the Euclidean metric. Find an example of a subset of $\mathbb{R}$ that is ...

   (a) open and not closed,
   (b) closed and not open,
   (c) both open and closed,
   (d) neither open nor closed.

3. Let $X = \mathbb{R}^2$. Sketch the balls $B_1(0,0)$ and $B_2(0,0)$ for each of the following metrics on $\mathbb{R}^2$. Denote $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$.

   (a) $d(\overline{x}, \overline{y}) = ||\overline{x} - \overline{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
   (b) $d(\overline{x}, \overline{y}) = |x_1 - y_1| + |x_2 - y_2|$
   (c) $d(\overline{x}, \overline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
   (d) $d(\overline{x}, \overline{y}) = \begin{cases} 0, & \overline{x} = \overline{y} \\ 1, & \overline{x} \neq \overline{y} \end{cases}$

4. Let $\{U_i\}_{i \in I}$ denote a collection of open sets in a metric space (X, d).

   (a) Prove that the union $\bigcup_{i \in I} U_i$ is an open set. Do not assume that I is necessarily finite, or countable!
   (b) Show by example that the intersection $\bigcap_{i \in I} U_i$ may not be open. (This means, give an example of a metric space (X, d) and a collection of open sets $U_i \subseteq X$, and prove that $\bigcap_{i \in I} U_i$ is not open).
   (c) Now assume we have a **finite** collection $\{U_i\}_{i=1}^n$ of open sets in a metric space. Prove that the intersection $\bigcap_{i=1}^n U_i$ is open.

5. **(Optional).**

   (a) Rigorously verify that the sets $\{1\}$, $[1, \infty)$, and $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\} \cup \{0\}$ are all closed subsets of $\mathbb{R}$ (with the Euclidean metric).
   (b) Consider $\mathbb{R}$ with the Euclidean metric. Is the subset $\mathbb{Q} \subseteq \mathbb{R}$ open? Is it closed?
   (c) Recall that $\mathcal{C}(0,2)$ is the set of continuous functions from the closed interval $[0,2]$ to $\mathbb{R}$, and that

   $$\begin{aligned} d_\infty : \mathcal{C}(0,2) \times \mathcal{C}(0,2) &\longrightarrow \mathbb{R} \\ d(f,g) &= \sup_{x \in [0,2]} |f(x) - g(x)| \end{aligned}$$

   defines a metric on $\mathcal{C}(0,2)$. Determine whether the subset $\{f(x) \in \mathcal{C}(0,2) \mid f(1) = 0\}$ is closed, open, neither, or both.

6. **(Optional).** Let X be a **finite** set, and let d be any metric on X. What can you say about which subsets of X are open? Which subsets of X are closed?

7. **(Optional).** Let (X, d) be a metric space, and let $x \in X$. Prove that the singleton set $\{x\}$ is closed.