1 Continuous functions on metric spaces

Definition 1.1. (Continuous functions $f : \mathbb{R} \to \mathbb{R}$.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is *continuous at a point* $x \in \mathbb{R}$ if ...

The function f is called *continuous* if it is continuous at every point $x \in \mathbb{R}$.

Rephrased:

How can we generalize this definition to general metric spaces?

Definition 1.2. (Continuous functions on metric spaces.) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function. Then f is *continuous at a point* $x \in X$ if ...

The function f is called *continuous* if it is continuous at every point $x \in X$.

Rephrased:

Notation: Let $f : X \to Y$ be a function.
For a subset $A \subseteq X$, the *image* of A is the set $f(A) = \{f(a) \mid a \in A\}$, a subset of Y.
For a subset $B \subseteq Y$, the *preimage* of B is the set $f^{-1}(B) = \{x \mid f(x) \in B\}$, a subset of X.
This definition makes sense (and we use the notation $f^{-1}(B)$) even when f is not invertible.

In-class Exercises

1. In this question, we will prove the following result:

   Theorem (Continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if, given any open set $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open.

   (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Suppose that $U \subseteq Y$ is an open set. Prove that $f^{-1}(U)$ is open.

   (b) Suppose that f is a function with the property that, for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is an open set in X. Show that f is continuous.

2. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between metric spaces. Show that the composite
$$g \circ f : X \to Z$$
is continuous. *Hint:* With our new criterion for continuity, this argument can be quite quick!

3. **(Optional)** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called an *isometric embedding* if it is "distance-preserving" in the sense that
$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

   (a) Give an intuitive description, with pictures, of what it means for a map to be an isometric embedding.

   (b) Show that an isometric embedding is continuous.

   (c) Show that an isometric embedding is always injective.

   (d) Show that map
$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (x, 0) \end{aligned}$$
   is an isometric embedding of $\mathbb{R}$ into $\mathbb{R}^2$ (each with the Euclidean metric).

   (e) Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ given by the map $f(x) = (x, mx + b)$ for $m, b \in \mathbb{R}$, so f maps the real line to the graph of the function $mx + b$. For which values of m and b is this an isometric embedding of Euclidean spaces?

   (f) For functions $f(x) = (x, mx+b)$ that are not isometric embeddings, can you find a different parameterization of this line that is an isometric embedding? In other words, can you find an isometric embedding $g : \mathbb{R} \to \mathbb{R}^2$ whose image is the set $\{(x, mx + b) \mid x \in \mathbb{R}\}$?

   (g) Show that the image of any isometric embedding from $\mathbb{R}$ into $\mathbb{R}^2$ must be a straight line.

   (h) Let $X = \{a, b, c\}$ be a 3-point set. Find examples of metrics on X so that the resulting metric space can and cannot be isometrically embedded in Euclidean space $\mathbb{R}^2$. Can you find necessary and sufficient conditions on the metric on X to guarantee the existence of an isometric embedding of X into $\mathbb{R}^2$?