1 Convergent sequences in metric spaces

Definition 1.1. (Convergent sequences in $\mathbb{R}$.) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then we say that the sequence *converges* to $a_\infty \in \mathbb{R}$, and write $\lim\limits_{n\to\infty} a_n = a_\infty$, if ...

Definition 1.2. (Convergent sequences in metric spaces.) Let (X, d_X) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. Then we say that the sequence *converges* to $a_\infty \in X$, and write $\lim\limits_{n\to\infty} a_n = a_\infty$, if ...

Rephrased:

In-class Exercises

1. Prove the following result:

   Theorem (An equivalent definition of convergence.) A sequence $(a_n)_{n\in\mathbb{N}}$ of points in a metric space (X, d) converges to a_∞ if and only if for any open set $U \subseteq X$ which contains a_∞, there exists some $N > 0$ so that $a_n \in U$ for all $n \geq N$.

2. **(Uniqueness of limits in a metric space)**.
   (a) Let (X, d) be a metric space, and let x, y be distinct points in X. Show that there exist **disjoint** open subsets U_x and U_y of X such that $x \in U_x$ and $y \in U_y$.
   Remark: This is called the *Hausdorff* property of metric spaces, and this result shows that metric spaces are T_2*-spaces*.
   (b) Let (X, d) be a metric space. Show that the limit of a sequence, if it exists, is **unique**, in the following sense. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to a point $a_\infty \in X$, and converges to a point $\widetilde{a}_\infty \in X$. Show that $a_\infty = \widetilde{a}_\infty$.

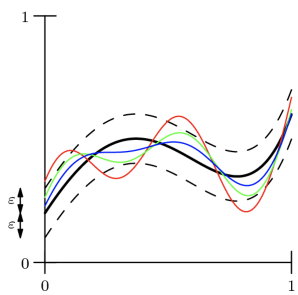
3. **(Optional)** Let (X, d) be a metric space. Let (a_n) be a sequence in X that converges to a point $a_\infty \in X$. Show that the set $\{a_n \mid n \in \mathbb{N}\} \cup \{a_\infty\}$ is a closed subset of X.

4. **(Optional) Definition (Pointwise and Uniform Convergence).** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : X \to Y$.

   - The sequence $(f_n)_{n\in\mathbb{N}}$ *converges at a point* $x \in X$ if the sequence $(f_n(x))_{n\in\mathbb{N}}$ of points in Y converges.
   - The sequence $(f_n)_{n\in\mathbb{N}}$ *converges pointwise* to a function $f_\infty : X \to Y$ if for every point $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ of points in Y converges to the point $f_\infty(x) \in Y$.
   - The sequence $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to a function $f_\infty : X \to Y$ if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ so that $d_Y\big(f_n(x), f_\infty(x)\big) < \epsilon$ for every $n \geq N$ and $x \in X$.

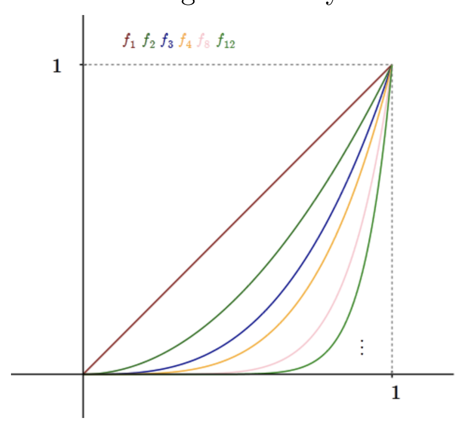
   In other words, if the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f_∞, then for each $\epsilon > 0$ the choice of N may depend on the point $x \in X$. To converge uniformly to f_∞, there must exist a choice of N that is independent of the point x.

   (a) Use the following picture of functions f_1, f_2, f_3, and f_∞ to explain the concept of uniform convergence of functions $\mathbb{R} \to \mathbb{R}$, and how it differs from pointwise convergence.

   (b) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n : X \to Y$ that converges uniformly to a function $f_\infty : X \to Y$. Show that f_∞ is continuous.

   (c) Consider the space $[0, 1]$ with the usual metric, and the sequence of functions $f_n : [0, 1] \to [0, 1]$ defined by $f_n(x) = x^n$. Show that this sequence converges pointwise, but conclude from part (b) that it does not converge uniformly.

   $f_n(x) = x^n$

   (d) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : X \to Y$. Show that uniform convergence implies pointwise convergence.

   (e) Recall the metric on the space $\mathcal{C}(a, b)$ of continuous functions $[a, b] \to \mathbb{R}$,

   $$d_\infty : \mathcal{C}(a, b) \times \mathcal{C}(a, b) \longrightarrow \mathbb{R}$$
   $$d_\infty(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

   Consider a sequence of continuous functions $f_n \in \mathcal{C}(a, b)$. Show that $(f_n)_{n\in\mathbb{N}}$ converges with respect to the metric d_∞ if and only if it converges uniformly.

   (f) **(Challenge)** Is there a metric on $\mathcal{C}(a, b)$ where convergence of a sequence is equivalent to pointwise convergence?