1 Complete metric spaces

Definition 1.1. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

Example 1.2. Give an example of a metric space that is not complete.

In-class Exercises

1. Suppose that a metric space (X, d) is sequentially compact. Show that (X, d) is complete.

2. (a) Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Show that it is contained in some closed and bounded subset of $\mathbb{R}^n$.

   (b) Prove the following theorem.

   > **Theorem ($\mathbb{R}^n$ is complete).** The space $\mathbb{R}^n$ is complete with respect to the Euclidean metric.

   Hint: You can quote the following results from Homework 5:

   - Cauchy sequences are bounded.
   - A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

3. **(Optional)** Let X be a nonempty set with the discrete metric. Under what conditions is X complete?

4. **(Optional)** Let X and Y be metric spaces. Suppose that X is complete.

   (a) Let $f : X \to Y$ be a continuous function. Must its image $f(X)$ be complete?

   (b) Suppose $f : X \to Y$ is a homeomorphism. Must Y be complete?

   (c) Suppose that $f : X \to Y$ is an isometric embedding. Must $f(X)$ be complete?

5. **(Optional)** A subset A of a metric space X is *dense* if $\overline{A} = X$. An *isometry* is a bijective isometric embedding.

   > **Definition (Completion).** Let X be a metric space. The *completion* of X is a complete metric space Y along with an isometric embedding $h : X \to Y$ such that $h(X)$ is dense in Y.

   In this question, we will construct the completion of X, and verify that it is unique up to isometry.

   (a) Let A be a dense subset of metric space Z. Show that, if every Cauchy subsequence in A converges in Z, then Z is complete.

(b) Let (X, d) be a metric space. Let $\tilde{X}$ denote the set of Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in X. Let Y denote the equivalences classes defined by the equivalence relation on $\tilde{X}$,

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \qquad \Longleftrightarrow \qquad d(x_n, y_n) \xrightarrow{n\to\infty} 0.$$

Verify that this is indeed an equivalence relation.

(c) For $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, let $[\mathbf{x}]$ and $[\mathbf{y}]$ denote the corresponding equivalence classes. Define

$$D : Y \times Y \to \mathbb{R}$$
$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Show that D is well-defined, that is, its value does not depend on the choice of representative of the equivalence class.

(d) Show that D defines a metric on Y.

(e) Define

$$h : X \to Y$$
$$x \mapsto [(x)_{n\in\mathbb{N}}]$$

sending a point x to the constant sequence $(x)_{n\in\mathbb{N}}$. Show that h is an isometric embedding.

(f) Show that, for any Cauchy sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the sequence $(h(x_n))_{n\in\mathbb{N}}$ converges in Y to $[\mathbf{x}]$. Conclude that $h(X)$ is dense ins Y.

(g) Further conclude that every Cauchy sequence in $h(X)$ must converge in Y, and thus by part (a) the space (Y, D) is complete. This shows that Y is the completion of X.

(h) Show that this completion is unique, in the sense of the following theorem.

> **Theorem (The completion of X is unique up to isometry).** Let $h : X \to Y$ and $h' : X \to Y'$ be isometric embeddings of the metric space (X, d) into complete metric spaces (Y, D) and (Y', D'), respectively, with dense image. Then there is an isometry of $(\overline{h(X)}, D)$ and $(\overline{h'(X)}, D')$ that equals $h' \circ h^{-1}$ when restricted to the subspace $h(X)$.