

1 Topological spaces

Definition 1.1. (Topology; Topological space.) Let X be a set. A *topology* \mathcal{T} on X is a collection of subsets of X that satisfies the following properties:

- (T1) The sets \emptyset and X are elements of \mathcal{T} .
- (T2) If $\{U_i\}_{i \in I}$ is any collection of elements of \mathcal{T} , then $\bigcup_{i \in I} U_i$ is in \mathcal{T} .
- (T3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

A set X endowed with a topology \mathcal{T} is called a *topological space* and denoted by (X, \mathcal{T}) , or simply denoted by X when \mathcal{T} is clear from context. The elements of \mathcal{T} are called the *open subsets* of the topological space X .

We have already proved the following result:

Theorem 1.2. (Metrics induce a topology.) Let (X, d) be a metric space. Then the collection \mathcal{T}_d of all open sets in X forms a topology on X .

The topology \mathcal{T}_d is called the *topology induced by the metric d* . We will see that not every topology on a set X necessarily arises from a metric structure on X .

Definition 1.3. (Metrizable topologies.) A topology \mathcal{T} on a set X is said to be *metrizable* if there exists a metric d on X such that \mathcal{T} is the set \mathcal{T}_d of open sets for the metric space (X, d) .

In-class Exercises

- Let X be a set.
 - Let $\mathcal{T} = \{\emptyset, X\}$. Prove that \mathcal{T} is a topology on X . It is called the *indiscrete topology*.
 - Let \mathcal{T} be the collection of all subsets of X . Prove that \mathcal{T} is a topology on X . It is called the *discrete topology*.
- Let X be a set.
 - Show that the discrete topology is metrizable.
 - Suppose that X contains at least 2 elements. Show that the indiscrete topology is not metrizable.
- (A useful criterion for openness).** Let (X, \mathcal{T}) be a topological space. Show that $V \in \mathcal{T}$ (that is, V is open) if and only if for every $x \in V$ there is some set $U_x \subseteq X$ containing x such that $U_x \in \mathcal{T}$ and $U_x \subseteq V$.
- Consider the following definition.

Definition 1.4. (Closed subsets of a topological space.) Let (X, \mathcal{T}) be a topological space. A subset $C \subseteq X$ is called *closed* if its complement is open, that is, if $X \setminus C$ is an element of \mathcal{T} .

- Verify that X and \emptyset are closed.
- Suppose that B and C are closed subsets of X . Verify that $B \cup C$ is closed.
- Suppose that $\{C_i\}_{i \in I}$ is a collection of closed sets in X . Verify that $\bigcap_{i \in I} C_i$ is closed.

5. **(Optional)**. Verify that the following sets are topologies on \mathbb{R} .

- The topology induced by the Euclidean metric
- $\mathcal{T} = \{\mathbb{R}, \emptyset\}$
- $\mathcal{T} = \{\mathbb{R}, (0, 1), \emptyset\}$
- $\mathcal{T} = \{\mathbb{R}, \{0, 1\}, \{0\}, \{1\}, \emptyset\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, \mathbb{R} \setminus A \text{ is finite}\} \cup \{\emptyset\}$
- $\mathcal{T} = \{A \mid A \text{ is a union of intervals of the form } [a, b) \text{ for } a, b \in \mathbb{R}\} \cup \{\emptyset\}$
- $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \in A\} \cup \{\emptyset\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \notin A\} \cup \{\emptyset\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 1 \in A\} \cup \{\emptyset\}$

6. **(Optional)**. Consider the following definitions.

Definition (Coarser topology; finer topology). Let X be a set. Let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on X . If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then the topology \mathcal{T}_1 is said to be *coarser* than \mathcal{T}_2 , and the topology \mathcal{T}_2 is said to be *finer* than the topology \mathcal{T}_1 .

- (a) Let X be a set. Show that the indiscrete topology on X is coarser than any other topology on X .
- (b) Let X be a set. Show that the discrete topology on X is finer than any other topology on X .
- (c) Consider all the topologies on \mathbb{R} in Problem 5. For each pair of topologies, determine either that one is coarser than the other, or show that they are not comparable.

7. **(Optional)**.

- (a) Let $X = \{a\}$. Explain why the only topology on X is $\{\emptyset, X\}$.
- (b) Let $X = \{a, b\}$. Find all possible topologies on X .
- (c) Let $X = \{a, b, c\}$. Find all possible topologies on X .
Hint: Look at the picture on our Canvas homepage.