Toric Varieties and Lattice Polytopes

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1 Introduction

We will show how to construct spaces called toric varieties from lattice polytopes. Toric fibrations correspond to slices of the polytope, and the Dynkin diagrams of singularities will appear in the lattice polytope. This process has been used to study K3 surfaces and higher-dimensional Calabi-Yau manifolds.

2 Building Toric Varieties

We will construct an algebro-geometric space based on combinatorial data. This process generalizes the construction of complex projective space $\mathbf{P}^n$.

2.1 Cones, Fans, and Polytopes

We begin with a lattice N isomorphic to $\mathbf{Z}^n$. The dual lattice M of N is given by $\mathrm{Hom}(N, \mathbf{Z})$; it is also isomorphic to $\mathbf{Z}^n$. (The alphabet may appear to be going backwards; but this notation is standard in the literature.) We write the pairing of $v \in N$ and $w \in M$ as $\langle v, w\rangle$.

A cone in N is a subset of the real vector space $N_{\mathbf{R}} = N \otimes \mathbf{R}$ generated by nonnegative $\mathbf{R}$-linear combinations of a set of vectors $\{v_1, \dots, v_n\} \subseteq N$. We assume that cones are strongly convex, that is, they contain no line through the origin. Note that each face of a cone is a cone. (Strictly speaking, our "cones in N" are "strongly convex rational polyhedral cones".)

Any cone σ in N has a dual cone $\sigma^\dagger$ in M given by $\{w \in M_{\mathbf{R}} \,|\, \langle v, w\rangle \geq 0 \,\forall\, v \in \sigma\}$.

A fan consists of a finite collection of cones such that each face of a cone in the fan is also in the fan, and any pair of cones in the fan intersects in a common face. Note the analogy to simplicial complexes.

A (convex) polytope in a finite-dimensional vector space is the convex hull of a finite set of points. We are interested in lattice polytopes, for which this finite set of points– the polytope's vertices– are contained in our integer lattice. Given a lattice polytope in N containing 0, we may construct a fan by taking cones over each face of the polytope.

Given a lattice polytope K in N, we define its *polar polytope* K^0 to be $K^0 = \{w \in \mathrm{M} \,|\, \langle v, w\rangle \geq -1 \,\forall\, v \in K\}$. If K^0 is also a lattice polytope, we say that K is a reflexive polytope and that K and K^0 are a mirror pair.

Example 1. The generalized octahedron in N with vertices at $(\pm 1, 0, \dots, 0)$, $(0, \pm 1, \dots, 0), \dots, (0, 0, \dots, \pm 1)$ has the hypercube with vertices at $(\pm 1, \pm 1, \dots, \pm 1)$ as its polar.

A reflexive polytope must contain 0; furthermore, 0 is the only interior lattice point of the polytope. Thus, if we have a reflexive polytope, we may construct a fan by taking every lattice point of the polytope as a generator. This gives us a refinement of the fan constructed by cones over the polytope's faces.

2.2 The Toric Variety

2.2.1 Varieties from Cones

From a cone σ in N, we may obtain a complex affine variety (that is, the common vanishing set of a finite set of complex polynomials $\{f_1, \dots, f_j\}$ in m variables in $\mathbf{C}^m$) according to the following procedure. Let S_σ be the commutative semigroup $\sigma^\dagger \cap M$ (S_σ is closed under the semigroup operation of addition and contains an identity element 0, but does not contain additive inverses). Then $\mathbf{C}[S_\sigma]$, finite formal linear combinations of elements of S_σ with coefficients in $\mathbf{C}$, is a finitely generated commutative $\mathbf{C}$-algebra. If you have studied enough algebraic geometry, you know that this sort of algebra corresponds to an affine variety $U_\sigma = \mathrm{Spec}(\mathbf{C}[S_\sigma])$, and that $\mathbf{C}[S_\sigma]$ can be viewed as the regular functions on this variety. In our case, the correspondence is quite simple: if we can find an isomorphism between $\mathbf{C}[S_\sigma]$ and $\mathbf{C}[X_1, \ldots, X_m]/\{f_1, \ldots, f_j\}$, then we know U_σ is isomorphic to the common vanishing set of the $\{f_1, \ldots, f_j\}$.

Example 2. Let $e_1, \ldots, e_n$ be basis vectors for N, and suppose $e_1, \ldots, e_n$ generate σ. Then the dual basis $e_1^*, \ldots, e_n^*$ generates S_σ, so $\mathbf{C}[S_\sigma] = \mathbf{C}[X_1, \ldots, X_n] = \mathbf{C}[X_1, \ldots, X_n]/\{0\}$. Thus $U_\sigma = Z(\{0\})$, that is, $\mathbf{C}^n$.

Example 3. Let $n = 2$ and let σ be generated by e_1 and $e_1 + 2e_2$. Then $\sigma^\dagger$ is generated by e_2^* and $2e_1^* - e_2^*$. S_σ has e_1^*, e_2^*, and $2e_1^* - e_2^*$ as generators, so $\mathbf{C}[S_\sigma] = \mathbf{C}[X, Y, X^2 Y^{-1}] \cong \mathbf{C}[U, V, W]/(U^2 - VW)$.

Example 4. Let σ be $\{0\}$. Then S_σ is all of M. As a semigroup, S_σ has generators $e_1, \ldots, e_n, -e_1, \ldots, -e_n$. Thus $\mathbf{C}[S_\sigma] = \mathbf{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$. If we think of $\mathbf{C}[S_\sigma]$ as algebraic functions on U_σ, then we see that each element defines a function on the points of $\mathbf{C}^n$ whose coordinates are all nonzero; thus, $U_\sigma = (\mathbf{C}^*)^n$, the complex torus.

2.2.2 Varieties from Fans

Each cone in a fan gives us an affine variety. We may obtain a cone from a fan by gluing these affine varieties together along the open sets corresponding to shared faces.

Example 5. Let $n = 1$ and let Σ be the fan containing the cones $\{0\}$, the non-negative real numbers, and the non-positive real numbers. These cones correspond to rings $\mathbf{C}[X_1, X_1^{-1}]$, $\mathbf{C}[X_1]$, and $\mathbf{C}[X_1^{-1}]$, respectively; the corresponding varieties are $\mathbf{C}^*$, $\mathbf{C}$, and $\mathbf{C}$. Gluing the two copies of $\mathbf{C}$ along $\mathbf{C}^*$ via the map $x \mapsto 1/x$, we obtain the complex projective space $\mathbf{P}^1$.

This gluing construction gives us a covering of our variety by affine charts.

Alternatively, we may obtain global homogeneous coordinates for our space, in a process analogous to the construction of $\mathbf{P}^n$ as a quotient space of $(\mathbf{C}^*)^n$. Let Σ be a fan in N, and let $v_1, \ldots, v_q$ be generators of the one-dimensional cones of N. We assume that Σ is simplicial; the general case is slightly more complicated. Let $Z_\Sigma \subseteq \mathbf{C}^q$ be the set $\cup_I \{(z_1, \ldots, z_q) \,|\, z_i = 0 \,\forall i \in I\}$, where the index I ranges over all sets $I \subseteq \{1, \ldots, q\}$ such that $\{v_i \,|\, i \in I\}$ is *not* a cone in Σ. Our variety is given by $(\mathbf{C} \setminus Z_\Sigma)/\sim$, where the equivalence relation $\sim$ is as follows:

$$(z_1, \ldots, z_q) \sim (\lambda^{a_j^1} z_1, \ldots, \lambda^{a_j^q} z_q) \text{ if } \sum_k a_j^k v_k = 0$$

.

Here $\lambda \in \mathbf{C}^*$ and $a_j^k \in \mathbf{Z}^+$; there are $q - n$ independent sets of relations $\{a_j^1, \ldots, a_j^q\}$.

Example 6. Let $n = 2$, let e_1 and e_2 be generators of N, and let $\diamond$ be the polytope with vertices e_1, e_2, and $-e_1 - e_2$. Let Σ be the fan obtained by taking cones on the faces of $\diamond$. Then the generators of the one-dimensional cones of Σ are just e_1, e_2, and $-e_1 - e_2$. Any two of these generators belong to a cone in Σ, so Z_Σ is $\{(0, 0, 0)\}$. We have exactly one relation, $e_1 + e_2 + (-e1 - e2) = 0$, where the weights are all 1. Thus, $\mathcal{V}_\Sigma$ is simply $\mathbf{P}^2$ with the usual homogeneous coordinates.

Example 7. Let $n = 2$ and let $\diamond$ be the polytope with vertices e_1, $-e_1$, e_2, and $-e_2$. Again, let Σ be the fan obtained by taking cones on the faces of $\diamond$, so the generators of the one-dimensional cones of Σ are e_1, $-e_1$, e_2, and $-e_2$. The generators e_1 and $-e_1$ do not belong to a common cone Σ; neither do e_2 and $-e_2$. Furthermore, no set of three or four one-dimensional generators can span a cone in Σ. Thus, $Z_\Sigma = \{(0, 0, 0, 0)\} \cup \{(z_1, z_2, 0, 0)\} \cup \{(0, 0, z_3, z_4)\} \cup \{(z_1, 0, 0, 0)\} \cup \{(0, z_2, 0, 0)\} \cup \{(0, 0, z_3, 0)\} \cup \{(0, 0, 0, z_4)\}$. We have $4 - 2 = 2$ independent relations, $e_1 + -e_1 = 0$ and $e_2 + -e_2 = 0$. This yields the following equivalence relations:

$$(z_1, z_2, z_3, z_4) \sim (\lambda z_1, \lambda z_2, z_3, z_4)$$

and

$$(z_1, z_2, \lambda z_3, \lambda z_4) \sim (z_1, z_2, z_3, z_4),$$

where $\lambda \in \mathbf{C}^*$. Thus, $\mathcal{V}_\Sigma$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

2.3 Refining Fans

If Σ and T are fans such that the cones in Σ are a subset of the cones in T, we say that T is a refinement of Σ. Refinements of fans correspond to birational maps between their corresponding varieties.

Example 8. Let $n = 2$, and Σ be the fan containing the cone σ generated by e_1 and e_2 and its faces. Let T be the fan containing σ, the cone τ generated by e_1 and $e_1 + e_2$, the cone υ generated by e_2 and $e_1 + e_2$, and their faces. As we have already seen, $\mathcal{V}_\Sigma$ is $\mathbf{C}^2$. $\mathbf{C}[S_\tau] = \mathbf{C}[Y, XY^{-1}]$ and $\mathbf{C}[S_\upsilon] = \mathbf{C}[Y, XY^{-1}]$. These affine sets glue to give us the blow-up of $\mathbf{C}^2$ at 0.

If we construct a fan by taking cones over the faces of a reflexive polytope, the non-vertex points of the polytope correspond to singularities which could be removed by refining the fan– that is, blowing up.

3 Subspaces and Fibrations

We have already seen that faces of cones correspond to open subspaces of our variety. We would like to use our fan to construct closed subspaces.

3.1 Orbit Closures

Let Σ be a fan and let τ be a k-dimensional cone in Σ. Let N_τ be the sublattice generated *as a group* by $\tau \cap N$, and let $N(\tau)$ be the quotient lattice N/N_τ. The star of τ is the set of cones in Σ which have τ as a face. The image of these cones in $N(\tau)$ is a fan in $N(\tau)$; we call this fan $\mathrm{Star}(\tau)$. $\mathrm{Star}(\tau)$ defines an $n-k$-dimensional variety $\mathcal{V}(\tau)$. $\mathcal{V}(\tau)$ is covered by open affine charts corresponding to the cones in the star of τ. We embed $\mathcal{V}(\tau)$ in the toric variety $\mathcal{V}_\Sigma$ by constructing compatible closed embeddings for each chart. We illustrate this process by example.

Example 9. Let $\diamond$ be the polytope in $N \cong \mathbf{Z}^2$ with vertices e_1, e_2, and $-e_1 - e_2$, and let Σ be the fan obtained by taking the cones on the faces of $\diamond$. We have already shown that the variety $\mathcal{V}_\Sigma$ is $\mathbf{P}^2$. Let τ be the cone in Σ generated by e_1. Then $N_\tau = \tau \cap N$ and $N(\tau) = N/N_\tau$ are both isomorphic to $\mathbf{Z}$. $\mathrm{Star}(\tau)$ is a fan in $N(\tau)$ with one cone for each of the two cones in Σ containing τ. We have seen this fan before; it is the fan corresponding to $\mathbf{P}^1$.

We must determine the correct way to embed this $\mathbf{P}^1$ in $\mathbf{P}^2$. We will do so explicitly for just one chart; the others follow similarly. Let σ be the cone in Σ generated by e_1 and e_2. Then $\sigma^\dagger$ is generated by the dual generators e_1^* and e_2^*. The image $\bar{\sigma}$ of σ in $\mathrm{Star}(\tau)$ is generated by $\bar{e}_2$, the image of e_2 in $N(\tau)$. We may write $\mathbf{C}[S_\sigma]$ as $\mathbf{C}[X, Y]$, and $\mathbf{C}[S_{\bar{\sigma}}]$ as $\mathbf{C}[Y]$. The map of $\mathbf{C}$-algebras from $\mathbf{C}[X, Y]$ to $\mathbf{C}[Y]$ induced by $X \mapsto 0$ induces a map of varieties from $\mathbf{C}$ to $\mathbf{C}^2$ given by $z \mapsto (0, z)$.

Thus, the embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ described by τ is just the map in homogeneous coordinates $(z_1, z_2) \mapsto (0, z_1, z_2)$.

3.2 Polynomials in Global Coordinates

We may describe closed hypersurfaces in $\mathbf{P}^n$ by taking the vanishing sets of homogeneous polynomials. Similarly, if we view points in a toric variety in their global homogeneous coordinates $(z_1, \ldots, z_q)$, we may obtain closed hypersurfaces by taking the vanishing sets of appropriate polynomials in the z_i. In particular, if $\diamond$ is a reflexive polytope and Σ is a fan obtained from a lattice triangulation of the faces of $\diamond$, then the zero set of the following polynomial describes a Calabi-Yau hypersurface in the variety $\mathcal{V}_\Sigma$:

$$p = \sum_{x \in \diamond^0 \cap M} c_x \prod_{k=1}^{n} z_k^{\langle v_k, x \rangle + 1}$$

.

(As before, the v_k are generators of the one-dimensional cones of Σ; we may also view them as lattice points in the triangulation of $\diamond$.)

Example 10. Let $n = 2$ and let $\diamond$ be the polytope with vertices e_1, $-e_1$, e_2, and $-e_2$; let Σ be the fan obtained by taking cones on the faces of $\diamond$. The polar polytope $\diamond^0$ is a square with vertices $e_1^* + e_2^*$, $e_1^* - e_2^*$,

$-e_1^* + e_2^*$, and $-e_1^* - e_2^*$; it has nine lattice points. Thus, the polynomial p will have nine terms; for instance, the term corresponding to the lattice point e_2^* is $a_{e_2^*} z_1^{0+1} z_2^{0+1} z_3^{1+1} z_4^{-1+1} = a_{e_2^*} z_1 z_2 z_3^2$.

3.3 Fibrations

Let $\diamond$ be a reflexive polytope in N, and let Σ be a fan obtained from a triangulation of $\diamond$. Suppose there exists an n'-dimensional linear subspace N_{fiber} of N such that $N_{\text{fiber}} \cap \diamond$ is an n'-dimensional reflexive polytope with 0 as its unique interior point. Then we may construct an exact sequence

$$0 \to N_{\text{fiber}} \to N \to N_{\text{base}}.$$

If the image Σ_{base} of Σ in the lattice N_{base} is a fan, this sequence yields a fibration of $\mathcal{V}_\Sigma$: the generic fiber corresponds to the polytope $N_{\text{fiber}} \cap \diamond$, and the base is given by Σ_{base}.

In particular, if $n' = n - 1$, then N_{fiber} is a hyperplane determined by a vector $m_{\text{fiber}} \in M$, and the base space of the fibration is $\mathbf{P}^1$. Homogeneous coordinates for this base space are given by $(z_{\text{up}}, z_{\text{down}})$, where

$$z_{\text{up}} = \prod_{i\colon \langle v_i, m_{\text{fiber}} \rangle > 0} z_i^{\langle v_i, m_{\text{fiber}} \rangle}$$

and

$$z_{\text{down}} = \prod_{i\colon \langle v_i, m_{\text{fiber}} \rangle < 0} z_i^{-\langle v_i, m_{\text{fiber}} \rangle}.$$

Example 11. Let $n = 2$ and let $\diamond$ be the polytope with vertices e_1, $-e_1$, e_2, and $-e_2$. As we saw in a previous example, this polytope yields the toric variety $\mathbf{P}^1 \times \mathbf{P}^1$ with coordinates (z_1, z_2, z_3, z_4) corresponding to the one-dimensional generators e_1, $-e_1$, e_2, and $-e_2$, respectively. If we intersect $\diamond$ with the hyperplane generated by e_1, we obtain a one-dimensional reflexive polytope with vertices e_1 and $-e_1$; this polytope corresponds to $\mathbf{P}^1$, so our fiber is $\mathbf{P}^1$. The hyperplane generated by e_1 is determined by $m_{\text{fiber}} = e_2^* \in \diamond^0$. Then z_{up} and z_{down} are just z_3 and z_4 respectively, so our base $\mathbf{P}^1$ has coordinates (z_3, z_4) and the projection map from $\mathbf{P}^1 \times \mathbf{P}^1$ to the base $\mathbf{P}^1$ is given by $(z_1, z_2, z_3, z_4) \mapsto (z_3, z_4)$.

4 K3 Surfaces and Intersecting Divisors

A suitably generic K3 surface in a toric variety described by a three-dimensional reflexive polytope $\diamond$ with a two-dimensional reflexive polygon slice will inherit an elliptic fibration structure from the fibration on the toric variety.

In the orbit closure construction, we saw that a one-dimensional cone τ in a fan Σ corresponds to an $n-1$-dimensional variety $\mathcal{V}(\tau)$ embedded in $\mathcal{V}_\Sigma$. If Σ is obtained from our maximally triangulated reflexive polytope $\diamond$, then each nonzero lattice point v_i in $\diamond$ corresponds to a one-dimensional cone τ_i and thus to a two-complex-dimensional embedded variety D_i. These varieties represent homology classes in $H_4(\mathcal{V}_\Sigma)$. If our K3 surface is sufficiently generic, its intersection with each of these varieties will be a hypersurface in the K3, generating a homology class in $H_2(K3)$. We are interested in the intersection pairing on these hypersurfaces. We may try to determine the values of this pairing by taking triple intersections $D_i \cdot D_j \cdot K3$ in $\mathcal{V}_\Sigma$. It turns out that $[K3] = [\sum_i D_i]$ in $H_4(\mathcal{V}_\Sigma)$, so we may reduce the problem to considering triple intersections of the D_i. If $D_i \cdot D_j \cdot K3$ is to have a nontrivial intersection, $\text{Star}(\tau_i)$ and $\text{Star}(\tau_j)$ must intersect somewhere other than the origin, so v_i and v_j must be adjacent in the triangulation of $\diamond$. Further calculations show that $D_i \cdot D_j \cdot K3$ is nonzero if and only if v_i and v_j are neighbors along a face of $\diamond$.

Intersection pairings of divisors correspond to the ADE classification of singularities obtained by blowing down these divisors. In our case, the correspondence is particularly simple: if we blow down all of the divisors aside from the divisors D_{fiber} and D_{section} corresponding to our fibration and a section of the fibration, we may read the Dynkin diagrams of the singularities from the edge diagrams in our polytope “above” and “below” the two-dimensional slice!

References

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