Fibration Example

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Let's look at a hypersurface in the toric variety $\mathcal{V}_\diamond$ described by the triangle $\diamond$ in the N lattice with vertices at $(0,1)$,$(2,-1)$, and $(-2,-1)$. (This triangle and its dual are illustrated in Avram et al., Figure 1.) The triangle $\diamond$ has eight lattice points aside from the origin. We label them as follows: $v_1 = (1,0)$, $v_2 = (-1,0)$, $v_3 = (0,1)$, $v_4 = 2,-1)$, $v_5 = (-2,-1)$, $v_6 = (1,-1)$, $v_7 = (0,-1)$, $v_8 = (-1,-1)$. Each vertex corresponds to a homogeneous coordinate in the homogeneous-coordinate description of $\mathcal{V}_\diamond$.

Vertices v_1, v_2, and the origin define a one-dimensional reflexive polytope contained in $\diamond$. This data yields a fibration of $\mathcal{V}_\diamond$ with a $\mathbf{P}^1$ fiber; the fiber has homogeneous coordinates (z_1, z_2). According to Perevalov and Skarke Equation (27), the base space of this fibration is another copy of $\mathbf{P}^1$ with homogeneous coordinates $(z_{\text{upper}}, z_{\text{lower}})$, where $z_{\text{upper}} = z_3$ and $z_{\text{lower}} = z_4 z_5 z_6 z_7 z_8$.

We have the following equation for a hypersurface in $\mathcal{V}_\diamond$:

$$p = \sum_{x \in \diamond^0 \cap M} c_x \prod_{k=1}^{n} z_k^{\langle v_k, x\rangle + 1}.$$

We would like to write an equation for the restriction of p to our fiber. To do this, we use Equations (18) and (19) from Kreuzer and Skarke. The first step is to divide the vertices of $\diamond^0$ into equivalence classes according to the rule:

$$x \sim y \text{ if } x - y \in M_{\text{base}}.$$

In our example, M_{base} is generated by $(0,1)$. Thus, the vertices of $\diamond^0$ fall into the following equivalence classes: $[(0,1)] = \{(0,1),(0,0),(0,-1)\}$, $[(1,-1)] = \{(1,-1)\}$, and $[(-1,-1)] = \{(-1,-1)\}$. Equation (19) of Kreuzer and Skarke tells us that we can rewrite p as

$$p = a'_{[(0,1)]} z_1 z_2 + a'_{[(1,-1)]} z_1^2 + a'_{[(-1,-1)]} z_2^2.$$

The coefficients $a'_{[(0,1)]}$, $a'_{[(1,-1)]}$, and $a'_{[(-1,-1)]}$ are as follows:

$$a'_{[(0,1)]} = a_{(0,1)} z_3^2 + a_{(0,0)} z_3 z_4 z_5 z_6 z_7 z_8 + a_{(0,-1)} z_4^2 z_5^2 z_6^2 z_7^2 z_8^2$$

$$a'_{[(1,-1)]} = a_{(1,-1)} z_4^4 z_6^3 z_7^2 z_8$$

$$a'_{[(-1,-1)]} = a_{(-1,-1)} z_5^4 z_6 z_7^2 z_8^3$$

The coefficients depend only on the coordinates $z_3, \ldots, z_8$, which are used to define the base space. However, the coefficients do *not* depend only on the coordinates $(z_{\text{upper}}, z_{\text{lower}})$ of our base space. Consider the point $(\lambda, 1)$ in the base space. This point corresponds to many different choices of the $z_3, \ldots, z_8$. For example, we may take $z_3 = \lambda$, $z_4 = 2$, $z_5 = 1/2$, and $z_6 = z_7 = z_8 = 1$. In this case, p becomes

$$p_1 = (a_{(0,1)}\lambda^2 + a_{(0,0)}\lambda + a_{(0,-1)}) z_1 z_2 + a_{(1,-1)} 16 z_1^2 + a_{(-1,-1)} \frac{1}{16} z_2^2.$$

But if $z_3 = \lambda$, $z_4 = 1/2$, $z_5 = 2$, and $z_6 = z_7 = z_8 = 1$, then p becomes

$$p_2 = (a_{(0,1)}\lambda^2 + a_{(0,0)}\lambda + a_{(0,-1)}) z_1 z_2 + a_{(1,-1)} \frac{1}{16} z_1^2 + a_{(-1,-1)} 16 z_2^2.$$

The polynomials p_1 and p_2 are different (except for certain very special choices of the coefficients $a_{(1,-1)}$ and $a_{(-1,-1)}$) and vanish on different subsets of the fiber. Thus, we have defined two different hypersurfaces in the fiber which correspond to the *same* base point $(\lambda, 1)$.

References

[1] A. C. Avram, M. Kreuzer, M. Mandelberg and H. Skarke, *Searching for K3 Fibrations*, hep-th/9610154 v1, 20 October 1996.

[2] Maximilian Kreuzer and Harold Skarke, *Calabi-Yau Fourfolds and Toric Fibrations*, hep-th 9701175 v1, 29 January 1997.

[3] Eugene Perevalov and Harald Skarke, *Enhanced Gauge Symmetry in Type II and F-Theory Compactifications: Dynkin Diagrams from Polyhedra*, hep-th/9704129 v2, 12 May 1997.