Mirror, Mirror

String Theory and Pairs of Polyhedra

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Math Reviews & University of Wisconsin–Eau Claire

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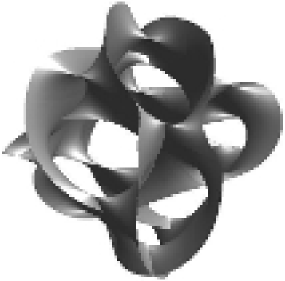


What do these people have in common?

Figure: Maxim Kontsevich

Figure: Andrew Strominger

Figure: Kumrun Vafa

Figure: Yuri Milner

Figure: Mark Zuckerberg

A Prize

The \$3 million Fundamental Prizes in Mathematics and Physics!

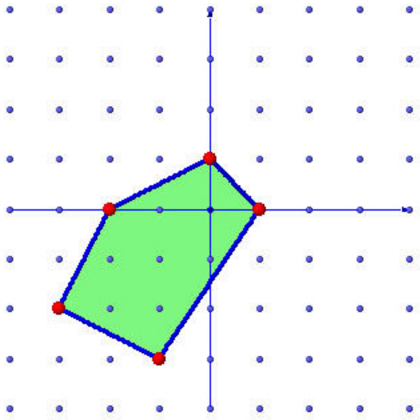
What is Mirror Symmetry?

Figure: Maxim Kontsevich

"Numerous contributions which have taken the fruitful interaction between modern theoretical physics and mathematics to new heights, including the development of homological mirror symmetry."

Lattice Polygons

The points in the plane with integer coordinates form a lattice N.
A lattice polygon is a convex polygon in the plane which has vertices in the lattice.



Fano Polygons

We say a lattice polygon is Fano if it has only one lattice point, the origin, in its interior.

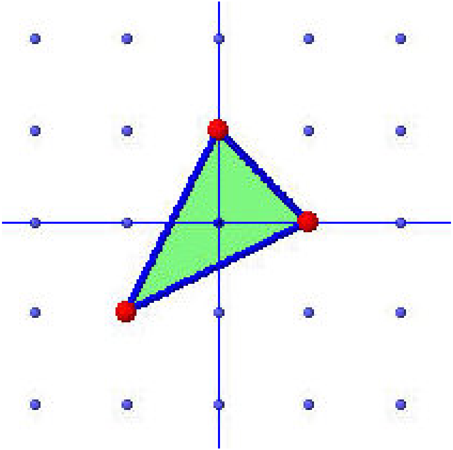


Figure: A Fano triangle

How Many Polygons?

Question

How many Fano polygons are there?

Infinite Families

There are infinite families of Fano polygons.

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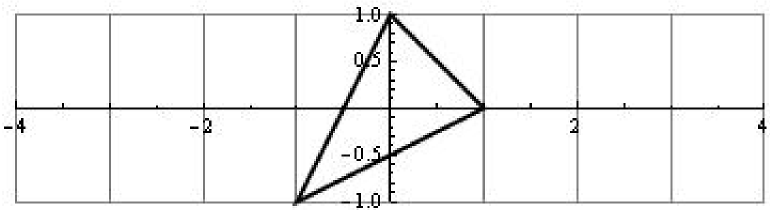
For instance, the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

yields an infinite family of polygons.



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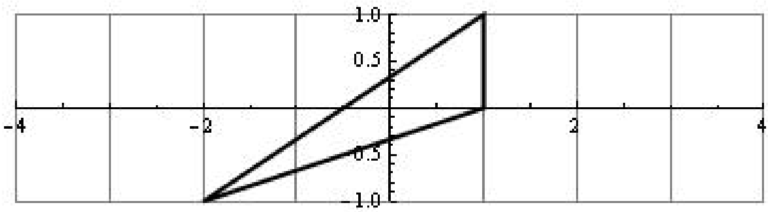

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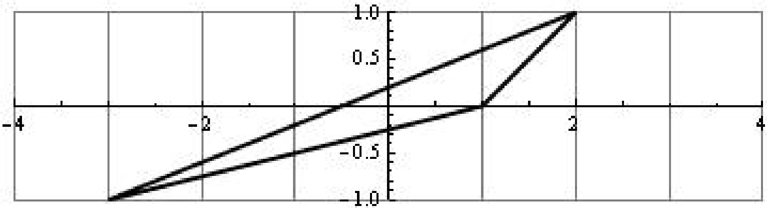

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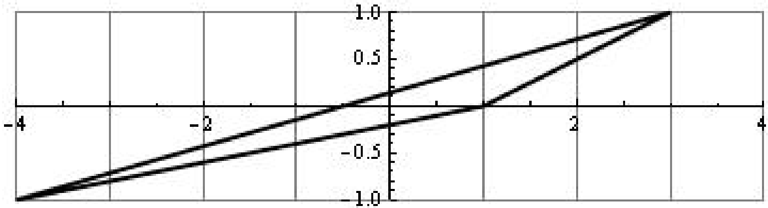

Polygon Symmetries

When should we consider two Fano polygons equivalent?

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- When they are related by a shear that preserves the lattice.

Polygon Symmetries

When should we consider two Fano polygons equivalent?

- When they are related by a rotation that preserves the lattice.
- When they are related by a reflection that preserves the lattice.
- When they are related by a shear that preserves the lattice.
- When they are related by a finite composition of these maps.

Symmetries and Matrices

We can describe rotations, reflections, shears, and their compositions using matrices with integer coordinates:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$ad - bc = \pm 1$$

Classifying Fano Polygons

- We can classify Fano polygons up to equivalence
- There are 16 equivalence classes of Fano polygons

16 Fano Polygons

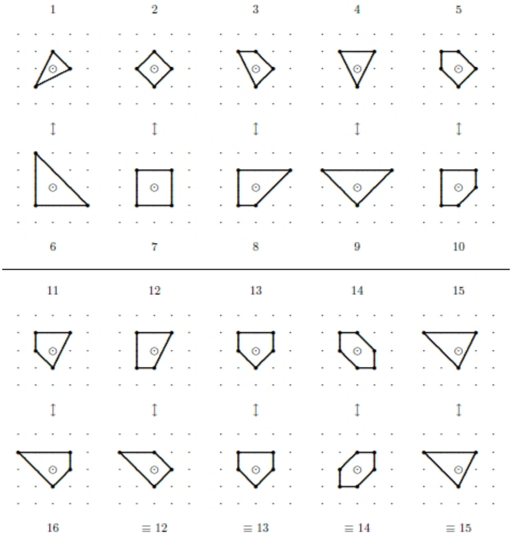
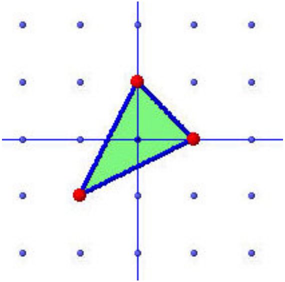


Figure: F. Rohsiepe, "Elliptic Toric K3 Surfaces and Gauge Algebras"

Describing a Fano Polygon



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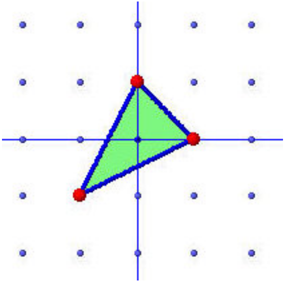

- List the vertices

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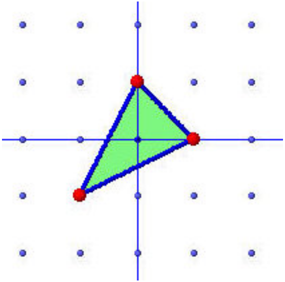

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$$\{(0,1),(1,0),(-1,-1)\}$$

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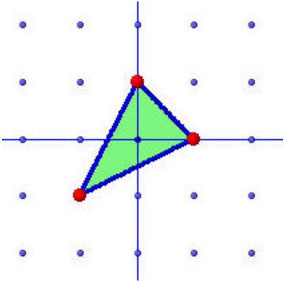

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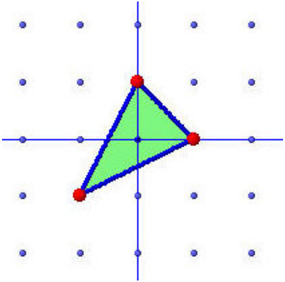

- List the vertices

$$\{(0,1),(1,0),(-1,-1)\}$$

- List the equations of the edges

$$\begin{aligned} -x-y &= -1 \\ 2x-y &= -1 \\ -x+2y &= -1 \end{aligned}$$

A Dual Lattice

- Let M be another copy of the points in the plane with integer coordinates.
- We refer to the plane containing N as $N_{\mathbb{R}}$, and the plane containing M as $M_{\mathbb{R}}$.
- The dot product lets us pair points in $N_{\mathbb{R}}$ with points in $M_{\mathbb{R}}$:

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2$$

Polar Polygons

Edge equations define new polygons

Let M be another copy of the points in the plane with integer coordinates. If we start with a lattice polygon Δ in N which contains $(0,0)$, we can construct a polar polygon Δ° in the vector space $M_\mathbb{R}$ using the coefficients of our edge equations.

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$$\begin{aligned} -1x - 1y &= -1 \\ 2x - 1y &= -1 \\ -1x + 2y &= -1 \end{aligned}$$

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$$2x - 1y = -1 \qquad (2,-1)$$

$$-1x + 2y = -1 \qquad (-1,2)$$

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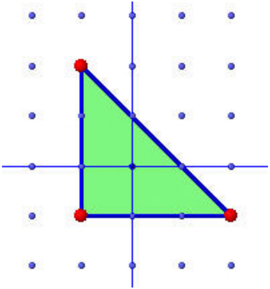


Figure: Our triangle's polar polygon

Mirror Pairs

If Δ is a Fano polygon, then:

- Δ° is a lattice polygon
- In fact, Δ° is another Fano polygon
- $(\Delta^\circ)^\circ = \Delta$.

We say that . . .

- Δ is a reflexive polygon.
- Δ and Δ° are a mirror pair.

A Polygon Duality

Mirror pair of triangles

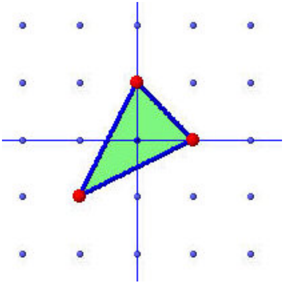


Figure: 3 boundary lattice points

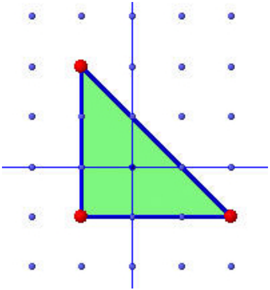


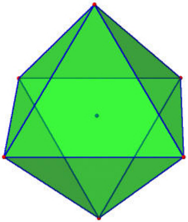
Figure: 9 boundary lattice points

$$3 + 9 = 12$$

Other Dimensions

- A polytope is the k-dimensional generalization of a polygon or polyhedron.
- We construct a polytope by taking the convex hull of a finite set of vertices.
- The facets of a polytope are equations of the form

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = c.$$



Polar Polytopes

Let N be the lattice of points with integer coordinates in the k-dimensional space $\mathbb{R}^k$. A lattice polytope has vertices in N. As before, we have a dual lattice M in another copy of $\mathbb{R}^k$.

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Definition

Let Δ be a lattice polytope in N which contains $(0, \ldots, 0)$. Then we can write the facet equations for Δ in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = -1.$$

The polar polytope Δ° is the polytope with vertices given by the facet equations of Δ:

$$(a_1, a_2, \ldots, a_k).$$

Reflexive Polytopes

Definition

A lattice polytope Δ is reflexive if Δ° is also a lattice polytope.

- If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- Δ and Δ° are a mirror pair.

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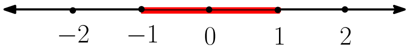

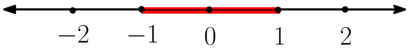

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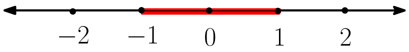
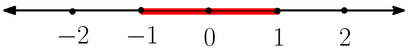
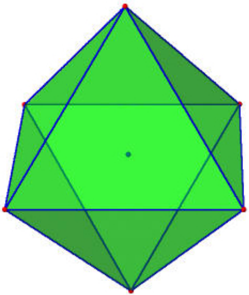
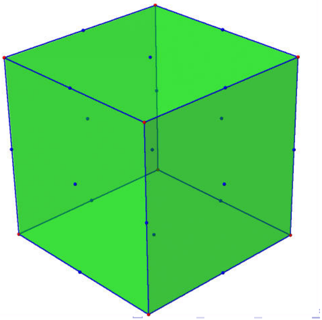


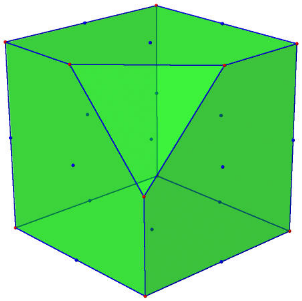






Fano vs. Reflexive

- Every reflexive polytope is Fano
- In dimensions $n \geq 3$, not every Fano polytope is reflexive



Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	
2	
3	
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	
3	
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	4,319
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5	

Classifying Reflexive Polytopes

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Dimension	Reflexive Polytopes
1	1
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Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
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4	473,800,776
5	??

Where's the Physics?

The physicists Maximilian Kreuzer and Harald Skarke classified reflexive polytopes. What were they looking for?

Figure: Vienna String Theory Group

A Quick Tour of Twentieth-Century Physics

- General relativity

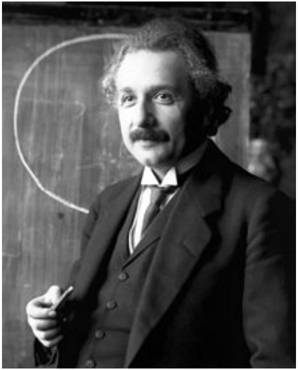


Figure: Albert Einstein

- Quantum mechanics

Figure: Fermilab

General Relativity

Features

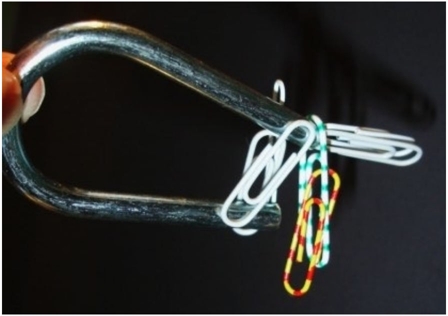
Figure: S. Bush et al.

- Measurements of time and distance depend on your relative speed.
- We specify events using coordinates in space-time.
- Space-time is curved.
- The curvature of space-time produces the effects of gravity.
- Useful for understanding large, massive objects such as stars and galaxies.

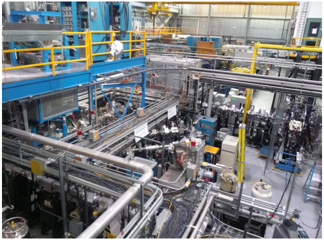
General Relativity

Question

- Why is the force of gravity so weak compared to other forces?



Quantum Physics



- The smallest components of the universe behave randomly.
- Sometimes they act like waves and sometimes they act like particles.
- There are 61 elementary particles: electrons, neutrinos, quarks, photons, gluons, etc.
- Useful for understanding small objects at high energies.

Quantum Physics

Questions

- Why are there so many elementary particles?
- Why does the Standard Model depend on so many parameters?

Where's the Theory of Everything?

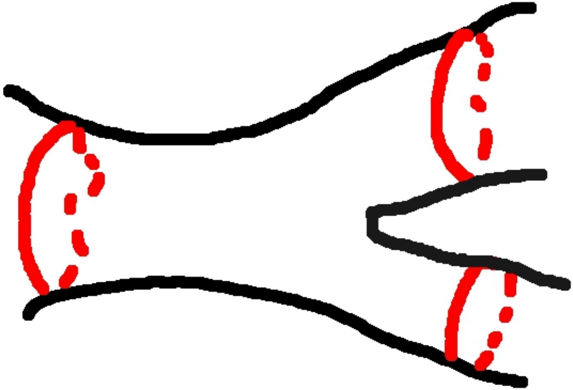
Can we build a theory of quantum gravity?

Challenge

Quantum fluctuations in "empty" space create infinite energy!

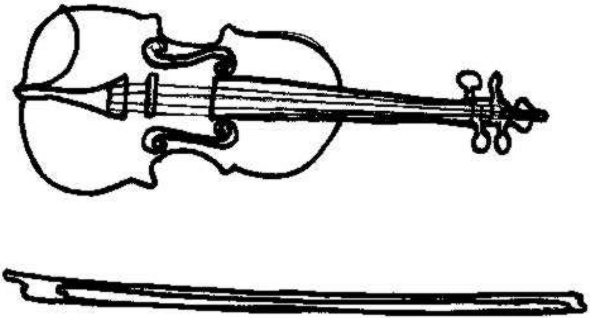
Are Strings the Answer?

String Theory proposes that “fundamental” particles are strings.



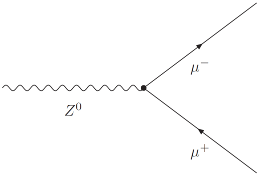
Vibration Distinguishes Particles

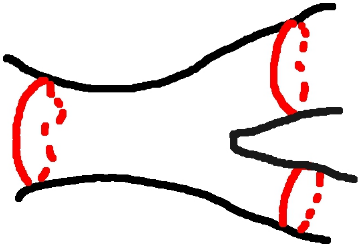
- Particles such as electrons and photons are strings vibrating at different frequencies.



Finite Energy

String theory "smears" the energy created by creation and destruction of particles, producing finite space-time energy.





Extra Dimensions

For string theory to work as a consistent theory of quantum mechanics, it must allow the strings to vibrate in extra, compact dimensions.

Is Gravity Leaking?

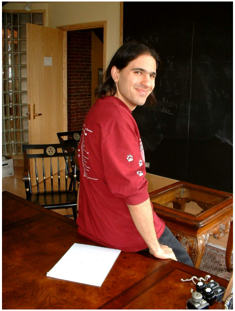


Figure: Nima Arkani-Hamed

If the electromagnetic force is confined to 4 dimensions but gravity can probe the extra dimensions, would this describe the apparent weakness of gravity?

T-Duality

Pairs of Universes

An extra dimension shaped like a circle of radius R and an extra dimension shaped like a circle of radius α'/R yield indistinguishable physics! (The slope parameter α' has units of length squared.)

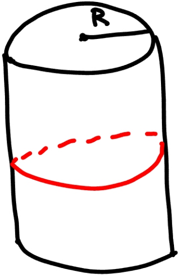


Figure: Large radius, few windings

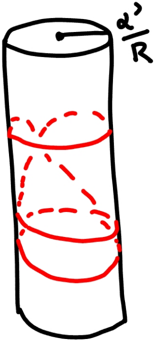


Figure: Small radius, many windings

Building a Model

At every point in 4-dimensional space-time, we should have 6 extra dimensions in the shape of a Calabi-Yau manifold.

A-Model or B-Model?

Choosing Complex Variables

- $z = a + ib$, $w = c + id$
- $z = a + ib$, $\overline{w} = c - id$

Mirror Symmetry

Physicists say . . .

- Calabi-Yau manifolds appear in pairs (V, V°).
- The universes described by V and V° have the same observable physics.

Mirror Symmetry for Mathematicians

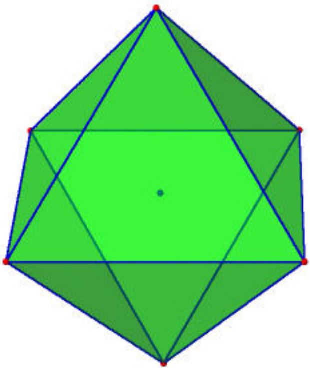
The physicists' prediction led to mathematical discoveries!

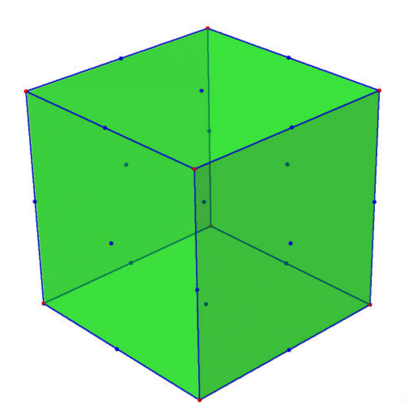
Mathematicians say . . .

- Calabi-Yau manifolds appear in paired families $(V_\alpha, V_\alpha^\circ)$.
- The families V_α and V_α° have dual geometric properties.

Batyrev's Insight

We can write equations for mirror families of Calabi-Yau manifolds using reflexive polytopes.





Mirror Polytopes Yield Mirror Spaces

$$
\begin{array}{ccc}
\text{polytope} & \longleftrightarrow & \text{polar polytope} \\
\downarrow & & \downarrow \\
\text{Laurent polynomial} & \longleftrightarrow & \text{mirror Laurent polynomial} \\
\downarrow & & \downarrow \\
\text{space} & \longleftrightarrow & \text{mirror space}
\end{array}
$$

A Recipe for a Space

- Each dimension of our polytope gives us a variable
- Each lattice point in our polynomial gives us exponents for our variables
- We add all the variables to obtain a polynomial
- Solutions to this equation are the space we want!
- Using the polar dual polytope gives us the dual space.

Example

The One-Dimensional Reflexive Polytope

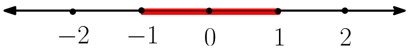


Figure: Δ

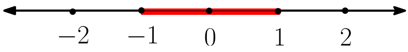


Figure: Δ°

- Standard basis vectors in $N \leftrightarrow$ variables z_i

$$(1) \leftrightarrow z_1$$

Example

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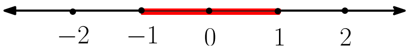

Figure: Δ

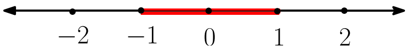

Figure: Δ°

- Standard basis vectors in $N \leftrightarrow$ variables z_i

$$(1) \leftrightarrow z_1$$

- Lattice points in $\Delta^\circ \leftrightarrow$ monomials defined on $(\mathbb{C}^*)^n$

$$\begin{aligned}(-1) &\leftrightarrow z_1^{-1}\\ (0) &\leftrightarrow z_1^0 = 1\\ (1) &\leftrightarrow z_1^1 = z_1\end{aligned}$$

Example

Continued

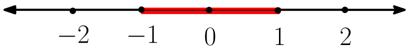

Figure: Δ

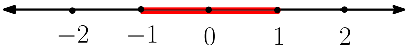

Figure: Δ°

- $\Delta^\circ \leftrightarrow$ Laurent polynomials p_α

$$\Delta^\circ \leftrightarrow p_\alpha = \alpha_{(-1)} z_1^{-1} + \alpha_{(0)} + \alpha_{(1)} z_1^{1}$$

Each choice of parameters $(\alpha_{(-1)}, \alpha_{(0)}, \alpha_{(1)})$ defines a Laurent polynomial.

From Polynomials to Spaces

The solutions to the Laurent polynomials p_α describe geometric spaces.

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Example: One Dimensional Polytope

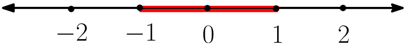

Figure: Δ

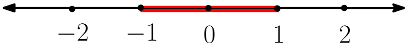

Figure: Δ°

Solutions to $\alpha_{(-1)} z_1^{-1} + \alpha_{(0)} + \alpha_{(1)} z_1^{1} = 0$ define pairs of nonzero points.

- $-z_1^{-1} + z_1 = 0$

From Polynomials to Spaces

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Example: One Dimensional Polytope

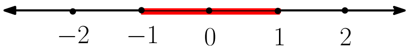
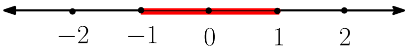


Figure: Δ



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  $z_1 = \pm 1$

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Example: One Dimensional Polytope

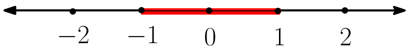


Figure: Δ

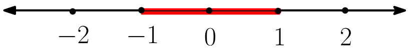


Figure: Δ°

Solutions to $\alpha_{(-1)}z_1^{-1} + \alpha_{(0)} + \alpha_{(1)}z_1^1 = 0$ define pairs of nonzero points.

- $-z_1^{-1} + z_1 = 0$
  $z_1 = \pm 1$
- $z_1^{-1} + z_1 = 0$

From Polynomials to Spaces

The solutions to the Laurent polynomials p_α describe geometric spaces.

Example: One Dimensional Polytope

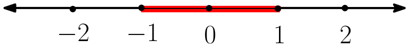


Figure: Δ

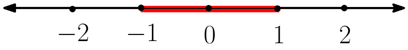


Figure: Δ°

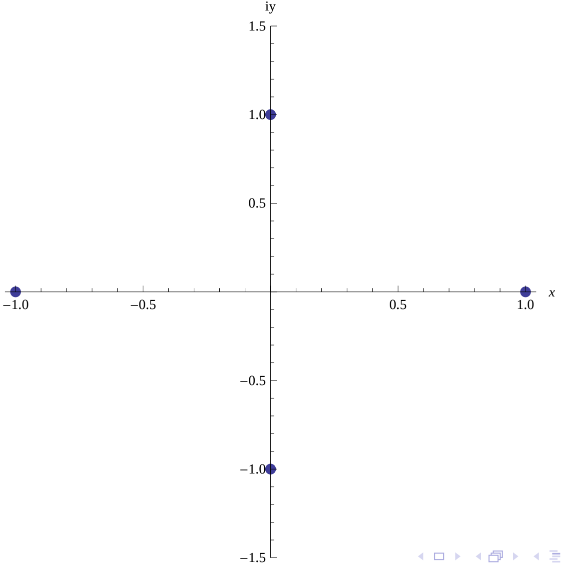
Solutions to $\alpha_{(-1)} z_1^{-1} + \alpha_{(0)} + \alpha_{(1)} z_1^{1} = 0$ define pairs of nonzero points.

- $-z_1^{-1} + z_1 = 0$
  $z_1 = \pm 1$
- $z_1^{-1} + z_1 = 0$
  $z_1 = \pm i$

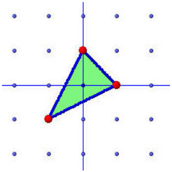
Example: One-Dimensional Polytope

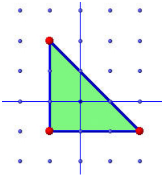
Continued

We can graph our points in the complex plane.



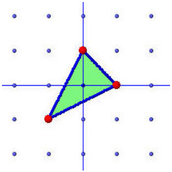
Example: Two-Dimensional Polytopes

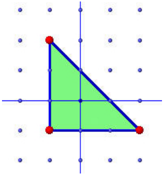




$$\alpha_{(-1,2)} z_1^{-1} z_2^{2} + \cdots + \alpha_{(2,-1)} z_1^{2} z_2^{-1} = 0$$

Example: Two-Dimensional Polytopes





$$\alpha_{(-1,2)} z_1^{-1} z_2^2 + \cdots + \alpha_{(2,-1)} z_1^2 z_2^{-1} = 0$$

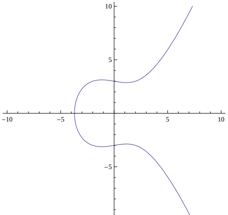


Figure: Real part of a curve

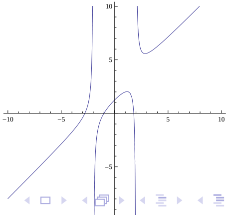
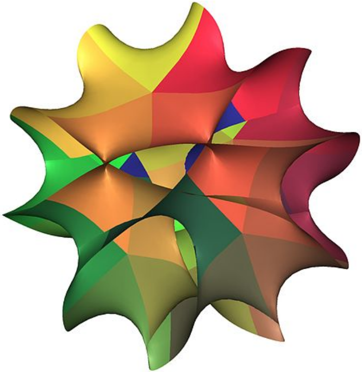


Figure: Another real curve

Example: Four-Dimensional Polytopes

Let Δ be the four-dimensional polytope with vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-1,-1,-1)$. Then Δ defines a three complex-dimensional or six real-dimensional Calabi-Yau manifold!



Compactifying

Our Laurent polynomials p_α define spaces which are not compact: $||z_i||$ can be infinitely large. We can solve this problem by adding in some "points at infinity" using a standard procedure from algebraic geometry together with the data of our polytope.

Calabi-Yau Varieties

The resulting compact spaces V_α are Calabi-Yau varieties of dimension $d = k - 1$.

- When $k = 2$, for generic choice of α, the V_α are elliptic curves.
- When $k = 3$, for generic choice of α, the V_α are K3 surfaces.
- When $k = 4$, for generic choice of α, the V_α are 3-dimensional Calabi-Yau varieties.

Mirror Symmetry

If we start with the polar polytope, we obtain a second family of geometric spaces which is the mirror family of the first.

$$\begin{array}{ccc} \text{polytope} & \longleftrightarrow & \text{polar polytope} \\ \downarrow & & \downarrow \\ \text{Laurent polynomials } p_\alpha & \longleftrightarrow & \text{mirror Laurent polynomials } p_\alpha^\circ \\ \downarrow & & \downarrow \\ \text{spaces } V_\alpha & \longleftrightarrow & \text{mirror spaces } V_\alpha^\circ \end{array}$$

For Further Reading

- Charles Doran and U.W.
  "From Polygons to String Theory."
  Mathematics Magazine, December 2012.
- Brian Greene.
  The Elegant Universe.
  New York: Vintage Books, 1999, 2003.
- Bjorn Poonen and Fernando Rodriguez-Villegas.
  "Lattice Polygons and the Number 12."
  The American Mathematical Monthly, March 2000.