Polarized Families of K3 Hypersurfaces

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Abstract

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We study families of K3 surfaces whose Picard groups contain specific primitive sublattices. We begin by reviewing proposals for mirror symmetry of K3 surfaces. In Chapter 2, we describe the connection between a finite group of symplectic symmetries of a K3 surface and a sublattice of its Picard group. We show how to compute the invariants of this sublattice, demonstrate that many examples of families of K3 surfaces with high Picard rank already studied in the literature may be united by this framework, and construct moduli spaces of K3 surfaces with symplectic symmetries. In Chapter 3, we compute the Picard-Fuchs equations of a particular family of K3 surfaces polarized by the lattice $H \oplus E_8 \oplus E_8$, and relate the result to the isogeny of elliptic curves.

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DEDICATION

For Susan, who owes me a book.

Chapter 1

K3 SURFACES AND MIRROR SYMMETRY

1.1 Introduction

A K3 surface is a simply connected compact complex surface with trivial canonical bundle. All K3 surfaces are diffeomorphic.

K3 surfaces are two-dimensional Calabi-Yau manifolds. Mirror symmetry constructions for Calabi-Yau manifolds relate the complex structure of a family of Calabi-Yau manifolds to the symplectic structure of a different family of Calabi-Yau manifolds. Mirror symmetry has been studied extensively in both the math and physics literature, particularly in the three-dimensional case. (See [CK99] for a mathematically oriented overview.)

The earliest mirror symmetry construction is due to Greene and Plesser. [GP90] Let us recall this construction, following the exposition in [CK99].

Example 1.1.1. Let $\mathcal{F}$ be the family of quintic hypersurfaces in $\mathbf{P}^4$. Then a generic member V of $\mathcal{F}$ is a smooth Calabi-Yau threefold. Let us consider the pencil $\mathcal{F}_t$ given by the equation $x^5 + y^5 + z^5 + v^5 + w^5 - 5t(xyzvw) = 0$ in homogeneous coordinates $[x, y, z, v, w]$. A subgroup $G \cong (\mathbf{Z}/5\mathbf{Z})^3$ of the big torus in $\mathbf{P}^4$ acts on $\mathcal{F}_t$ as follows. Let μ be a primitive fifth root of unity. Then for any integers $(a_1, a_2, a_3, a_4, a_5)$ such that $\sum_i a_1 \equiv 0 \pmod 5$, the assignment $[x, y, z, v, w] \mapsto [\mu^{a_1}x, \mu^{a_2}y, \mu^{a_3}z, \mu^{a_4}v, \mu^{a_5}w]$ fixes each member V_t of $\mathcal{F}_t$, yielding an action of $(\mathbf{Z}/5\mathbf{Z})^4$. Taking the quotient by the diagonal action $[x, y, z, v, w] \mapsto [\mu^a x, \mu^a y, \mu^a z, \mu^a v, \mu^a w]$, we obtain an action of $G \cong (\mathbf{Z}/5\mathbf{Z})^3$ on $\mathcal{F}_t$ which restricts to an automorphism of each Calabi-Yau threefold V_t.

Let $\widetilde{\mathbf{P}^4/G}$ be a minimal resolution of the quotient $\mathbf{P}^4/G$, and let $\check{\mathcal{F}}_t$ be the image of $\mathcal{F}_t$ in $\mathbf{P}^4/G$. Then $\mathcal{F}$ and $\check{\mathcal{F}}_t$ are mirror families. One consequence of the mirror relationship involves the Hodge numbers of members of each family: if X and $\check{X}$ are smooth members of $\mathcal{F}$ and $\check{\mathcal{F}}_t$ respectively, then $h^{1,1}(X) = h^{2,1}(\check{X}) = 1$ and $h^{2,1}(X) = h^{1,1}(\check{X}) = 101$.

The method of Example 1.1.1 may be extended to families of Calabi-Yau threefolds in appropriately chosen weighted projective spaces.

Batyrev constructed mirrors for families of hypersurfaces in $n+1$-dimensional toric varieties ($n \geq 3$) corresponding to reflexive polytopes, generalizing the Greene and Plesser construction. Again, we have a relationship between Hodge numbers of an n-dimensional Calabi-Yau variety X in one family and another Calabi-Yau $\check{X}$ in the mirror family: $h^{1,1}(X) = h^{n-1,1}(\check{X})$ and $h^{n-1,1}(X) = h^{1,1}(\check{X})$. [Bat94]

Since any two K3 surfaces have the same Hodge numbers, we need a more refined version of mirror symmetry in two dimensions. In this case, the Picard group $\text{Pic}(X)$ of a K3 surface X takes the place of $H^{2-1,1}(X) = H^{1,1}(X)$. We may identify the Picard group of a K3 surface with the Néron-Severi group generated by complex hypersurfaces up to homological equivalence. This free abelian group has a lattice structure inherited from the cup product on $H^*(X)$. We review the properties of even lattices in Section 1.2.

If $\mathcal{F}$ and $\check{\mathcal{F}}$ are mirror families of K3 surfaces and X and $\check{X}$ are generic elements of $\mathcal{F}$ and $\check{\mathcal{F}}$ respectively, we expect that $\text{rank Pic}(X) + \text{rank Pic}(\check{X}) = 20$. Dolgachev gave a detailed formulation of mirror symmetry for K3 surfaces in terms of their moduli spaces, relating not just the ranks of the Picard groups, but their lattice structure. [Dol96] We discuss his construction in Section 1.3.

One might try to obtain more concrete examples of mirror families of K3 surfaces by extending the constructions of Greene and Plesser or Batyrev to the K3 case. Rohsiepe linked Batyrev's construction of mirror families for hypersurfaces in toric varieties with the Dolgachev approach. [Roh04] We will consider his argument in Section 1.4.

Thus, studying mirror symmetry leads us to investigate *polarized* families of K3 surfaces, that is, families of K3 surfaces which admit a particular lattice as a primitive sublattice of the Picard group. (We say a sublattice M of a lattice L is *primitive* if L/M is a free abelian group.) In Chapter 2, we will describe the connection between a finite group of symmetries of a K3 surface and a sublattice of its Picard group, and show that many examples of families of K3 surfaces with high Picard rank already studied in the literature may be united by this framework. In Chapter 3, we compute the Picard-Fuchs equations of a particular family of K3 surfaces polarized by the lattice $H \oplus E_8 \oplus E_8$, and relate the result to the isogeny of elliptic curves.

1.2 Even Lattices

1.2.1 Properties of Lattices

This section collects a number of elementary properties of lattices. A more detailed discussion of basic lattice properties may be found in [BHPVdV04]. [Nik80b] gives a more advanced treatment.

A *lattice* L is a finitely generated free $\mathbf{Z}$-module, together with an integral bilinear form $\langle\, ,\, \rangle$. (We will occasionally write $\langle x, y\rangle$ as $x \cdot y$.) We shall restrict our attention to lattices with symmetric bilinear form; these are called *euclidean* lattices. If $\langle x, x\rangle$ is even for every $x \in L$, we say that the lattice L is even; otherwise, we say that L is odd.

Let $e_1, \ldots, e_n$ be a basis for L. We call the matrix $(\langle e_i, e_j\rangle)$ an *intersection matrix* of L. The determinant of the intersection matrix is called the *discriminant* of L; we write the discriminant as $d(L)$. Lattices with nonzero discriminant are called *nondegenerate* lattices; lattices with discriminant ± 1 are called *unimodular* lattices. The *signature* (l^+, l^-) of L is the signature of $(\langle e_i, e_j\rangle)$. We say L is *positive definite* if $l^- = 0$, *negative definite* if $l^+ = 0$, and *indefinite* if both l^+ and l^- are nonzero.

We call $L^* := \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ the *dual* of L. There is a natural map ϕ from L to L^* given by $\phi(x) = \langle\, , x\rangle$. We may extend the bilinear form $\langle\, ,\, \rangle$ to a bilinear form on L^* which takes values in $\mathbf{Q}$. This extension induces a bilinear form b_L on L^*/L which takes values in $\mathbf{Q}/\mathbf{Z}$. When L is even, $\langle x, x\rangle$ induces a quadratic form q_L with values in $\mathbf{Q}/2\mathbf{Z}$. We refer to these forms as *discriminant forms*. The possible discriminant forms have been classified; a summary of the classification appears in [Nik80b], and a more leisurely discussion may be found in [Bel97].

Let M be a sublattice of L. If M and L have the same rank, we have the following useful lemma:

Lemma 1.2.1. [BHPVdV04] Let L be a nondegenerate lattice, and let M be a sublattice of L satisfying $\operatorname{rank} M = \operatorname{rank} L$. Then,

$$(L : M)^2 = d(M)/d(L)$$

, where $(L : M)$ is the order of the finite abelian group L/M.

Recall that a sublattice M of L is a *primitive* sublattice if the quotient L/M is a free module. Let $M^\perp = \{x \in L \mid \langle m, x\rangle = 0 \,\forall\, m \in M\}$; we call $M^\perp$ the *perpendicular complement* of M in L. Note that $M^\perp$ is always a primitive sublattice of L. If L is nondegenerate, then $\operatorname{rank} M + \operatorname{rank} M^\perp = \operatorname{rank} L$. However, $M \oplus M^\perp$ need not equal L, even when M is primitive:

Lemma 1.2.2. Let M be a primitive sublattice of a unimodular lattice L. Then $|d(M)| = |d(M^\perp)|$, and $M \oplus M^\perp = L$ if and only if M is unimodular.

(See [BHPVdV04] for a brief proof of Lemma 1.2.2.)

If M is a primitive sublattice of L, then every basis of M extends to a basis of L, so the natural map $L^* \to M^*$ is surjective. When L is unimodular, we may obtain a homomorphism $\psi : M^*/M \to (M^\perp)^*/M^\perp$ by extending an element of M^* to L^* and then restricting to $(M^\perp)^*$. If the restriction $\langle\, ,\, \rangle|M$ is nondegenerate, then ψ is an isomorphism and $b_M \cong -b_{M^\perp}$. (If L and M are even, we also have $q_M \cong -b_{q^\perp}$.)

1.2.2 Existence and Embeddings of Even Lattices

In this section, we give simple criteria for the existence and uniqueness of even lattices with prescribed properties. We also discuss the existence and uniqueness of primitive embeddings of even lattices in even unimodular lattices. In many cases, we will give only sufficient conditions for existence or uniqueness; [Nik80b] gives stronger conditions.

J. Milnor gave conditions for existence and uniqueness of unimodular lattices: [Nik80b]

Theorem 1.2.1. An even unimodular lattice of signature (l^+, l^-) exists if and only if $l^+ - l^- \equiv 0 \pmod 8$. An odd unimodular lattice exists as long as l^+ or l^- is greater than zero.

Theorem 1.2.2. An indefinite unimodular lattice L is determined up to isomorphism by its parity (odd or even) and its signature.

W. Durfee characterized lattices with isomorphic discriminant quadratic forms: [Nik80b]

Theorem 1.2.3. Two even lattices M_1 and M_2 have isomorphic discriminant quadratic forms q_{M_1} and q_{M_2} if and only if there exist even unimodular lattices L_1 and L_2 such that

$$M_1 \oplus L_1 \cong M_2 \oplus L_2.$$

We may use Theorem 1.2.3 together with Theorem 1.2.1 to define the signature of a quadratic form:

Definition 1.2.1. Let q be a quadratic form defined on $\mathbf{Q}/2\mathbf{Z}$, and let L be any even lattice such that $q = q_L$. Then the *signature* $\operatorname{sign} q$ is given by $l^+ - l^- \pmod 8$, where (l^+, l^-) is the signature of L.

For any finitely generated abelian group A, let us write $l(A)$ for the minimal number of generators of A.

Theorem 1.2.4. [Nik80b] An even lattice M with signature (m^+, m^-) and discriminant quadratic form q defined on a finite abelian group A_q exists if $\operatorname{sign} q = m^+ - m^-$ and $m^+ + m^- > l(A_q)$.

Remark 1.2.1. Clearly, the condition that $\operatorname{sign} q = m^+ - m^-$ is necessary. The requirement that $m^+ + m^- > l(A_q)$ may be weakened to $m^+ + m^- \geq l(A_q)$ with the addition of certain more complicated conditions discussed in [Nik80b].

The following theorems give conditions for the existence of a primitive embedding of an even lattice M in an even unimodular lattice L.

Theorem 1.2.5. [Nik80b] Let M be an even lattice with signature (m^+, m^-) and discriminant quadratic form q_M, and let L be an even unimodular lattice with signature (l^+, l^-). The following properties are equivalent:

1. A primitive embedding of M into L exists.

2. An even lattice with signature $(l^+ - m^+, l^- - m^-)$ and discriminant quadratic form $-q_M$ exists.

3. An even lattice with signature $(l^- - m^-, l^+ - m^+)$ and discriminant quadratic form q_M exists.

Theorem 1.2.6. [Nik80b] Let M be an even lattice with signature (m^+, m^-) and discriminant quadratic form q_M, and let L be an even unimodular lattice with signature (l^+, l^-). Then a primitive embedding of M into L exists if $l^+ - m^+ \geq 0$, $l^- - m^- \geq 0$, and $l^+ + l^- - m^+ - m^- > l(M^*/M)$.

Theorem 1.2.7. [Nik80b] Let (m^+, m^-) and (l^+, l^-) be two pairs of nonnegative integers. The following are equivalent:

1. Every even lattice of signature (m^+, m^-) has a primitive embedding into an even unimodular lattice of signature (l^+, l^-).

2. $l^+ - l^- \equiv 0 \pmod 8$, $m^+ \leq l^+$, $m^- \leq l^-$, and $m^+ + m^- \leq \frac{1}{2}(l^+ + l^-)$.

The following theorem gives a criterion for the uniqueness of an indefinite lattice:

Theorem 1.2.8. [Nik80b] An indefinite even lattice with signature (m^+, m^-) and discriminant quadratic form q_M is unique up to isomorphism if $m^+ + m^- \geq 2 + l(M^*/M)$.

One may weaken the requirement that $m^+ + m^- \geq 2 + l(M^*/M)$; we give an easily checked condition. For any prime p, let $(M^*/M)_p$ be the p-component of the finite abelian group M^*/M.

Theorem 1.2.9. [Nik80b] An indefinite even lattice with signature (m^+, m^-) is unique if the following conditions hold:

1. $m^+ + m^- \geq 3$
2. For each $p \neq 2$, either $m^+ + m^- \geq 2 + l((M^*/M)_p)$, or $M^*/M \cong (\mathbf{Z}/p^k\mathbf{Z})^2 \oplus A^{(p)}$ for some abelian group $A^{(p)}$
3. For $p = 2$, either $m^+ + m^- \geq 2 + l((M^*/M)_2)$ or $M^*/M \cong (\mathbf{Z}/2\mathbf{Z})^3 \oplus A^{(2)}$ for some abelian group $A^{(2)}$

An analogue of Theorem 1.2.6 gives a simple condition for the existence of a *unique* primitive embedding in a unimodular lattice:

Theorem 1.2.10. [Nik80b] Let M be an even lattice with signature (m^+, m^-) and let L be an even unimodular lattice of signature (l^+, l^-). If $l^+ - m^+ > 0$, $l^- - m^- > 0$, and $l^+ + l^- - m^+ - m^- \geq 2 + l(M^*/M)$, then there exists a unique primitive embedding of M in L.

We may use Conditions (1.5) and (1.6) of [Nik80a] to weaken Theorem 1.2.10 still further:

Theorem 1.2.11. [Nik80a, Nik80b]

Let M be an even lattice with signature (m^+, m^-) and let L be an even unimodular lattice of signature (l^+, l^-). Suppose the following conditions are satisfied:

1. $l^+ - m^+ > 0$, $l^- - m^- > 0$, and $l^+ + l^- - m^+ - m^- \geq 3$

2. For all primes $p \neq 2$, either $l^+ + l^- - m^+ - m^- \geq 2 + l((M^*/M)_p)$ or $M^*/M \cong (\mathbf{Z}/p\mathbf{Z})^2 \oplus A^{(p)}$ for some abelian group $A^{(p)}$

3. For $p = 2$, either $m^+ + m^- \geq 2 + l((M^*/M)_2)$ or $M^*/M \cong (\mathbf{Z}/2\mathbf{Z})^3 \oplus A$ for some abelian group A

Then there exists a unique primitive embedding of M in L.

If K and M have the same rank, and there exists a lattice embedding $K \hookrightarrow M$, we say that M is an *overlattice* of K. Then $K \hookrightarrow M \hookrightarrow M^* \hookrightarrow K^*$, so $M/K \subset M^*/K \subset K^*/K$.

Theorem 1.2.12. [Nik80b] Let M and K be even lattices, and suppose M is an overlattice of K. Then $M^*/M = (M^*/K)/(M/K)$, and $(M/K)^\perp = M^*/K$. The discriminant quadratic form q_M is given by $q_M = (q_K|(M/K)^\perp)/(M/K)$.

To analyze all (perhaps non-primitive) embeddings of a lattice K in another lattice L, we must first enumerate the overlattices of K, then compute the primitive embeddings of each overlattice in L.

1.3 Polarized K3 Surfaces and Mirror Moduli Spaces

In this section, we review the moduli space constructions involved in Dolgachev's mirror symmetry prescription for K3 surfaces. [Dol96] An alternative description with an emphasis on ample K3 surfaces may be found in [BHPVdV04].

For any K3 surface X, the intersection product induces a lattice structure on $H^2(X, \mathbf{Z})$; as a lattice, $H^2(X, \mathbf{Z})$ is isomorphic to $L = H \oplus H \oplus H \oplus E_8 \oplus E_8$. We call a choice of isomorphism $\phi : H^2(X, \mathbf{Z}) \to L$ a *marking* of X, and refer to the pair $(X; \phi)$ as a *marked K3 surface.*

Let us write $\langle\, , \,\rangle$ for the bilinear form on L; we set $L_{\mathbf{R}} = L \otimes \mathbf{R}$ and $L_{\mathbf{C}} = L \otimes \mathbf{C}$. For any nonzero element ω of $L_{\mathbf{C}}$, let $[\omega]$ be the corresponding element of the projective space $\mathbf{P}(L_{\mathbf{C}})$. Let $\Omega = \{[\omega] \in \mathbf{P}(L_{\mathbf{C}}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$.

Let $(X; \phi)$ be a marked K3 surface, and let ω_X be a nowhere-vanishing holomorphic two-form on X. (The form ω_X is unique up to a scalar multiple.) The image of ω_X under $\phi_{\mathbf{C}}$ determines a point $[\phi_{\mathbf{C}}(\omega_X)]$ in $\mathbf{P}(L_{\mathbf{C}})$. Since $\omega_X \wedge \omega_X = 0$ and $\omega_X \wedge \bar{\omega_X} > 0$, $[\phi_{\mathbf{C}}(\omega_X)]$ is an element of Ω, which we refer to as the *period point.*

Theorem 1.3.1. *Weak Torelli Theorem.* Two K3 surfaces X and X' are isomorphic if and only if there exist markings for each surface such that the corresponding period points are the same.

Theorem 1.3.2. All points of Ω occur as period points of marked K3 surfaces.

Theorem 1.3.3. There exists a universal marked family of K3 surfaces. The base space $\mathcal{N}$ is a non-Hausdorff "smooth analytic space" of dimension 20.

The period points of marked K3 surfaces yield a *period map* $\tau_{\mathcal{N}} : \mathcal{N} \to \Omega$.

Remark 1.3.1. A detailed exposition of the above standard theorems may be found in [BHPVdV04].

Let M be an even, nondegenerate lattice of signature $(1, t)$. We assume that $t \leq 19$.

Definition 1.3.1. An *M-polarized* K3 surface (X, j) is a K3 surface X together with a primitive lattice embedding $j : M \hookrightarrow \mathrm{Pic}(X)$.

Let $B(M)$ be the cone in $M_{\mathbf{R}} = M \otimes \mathbf{R}$ given by $\{x \in M_{\mathbf{R}} \mid \langle x, x \rangle > 0\}$. The cone $B(M)$ has two components; let us fix one of these components, and call it $B(M)^+$. Let $\Delta(M) = \{\delta \in M \mid \langle \delta, \delta \rangle = -2\}$. We may partition $\Delta(M)$ as $\Delta(M)^+ \amalg \Delta(M)^-$, where $\Delta(M)^+$ is closed under positive integer linear combinations of its elements, and $\Delta(M)^- = \{-\delta \mid \delta \in \Delta(M)^+\}$. Let $C(M)^+$ be the subset of M given by $\{h \in B(M)^+ \cap M \mid \langle h, \delta \rangle > 0, \forall \delta \in \Delta(M)^+\}$. We may use this data to define ample and pseudo-ample M-polarized K3 surfaces:

Definition 1.3.2. A *pseudo-ample* M-polarized K3 surface is an M-polarized K3 surface (X, j) such that all divisors in $j(C(M)^+)$ are pseudo-ample. (A divisor is *pseudo-ample* if it is numerically effective and has positive self-intersection.) An M-polarized K3 surface (X, j) is *ample* if all divisors in $j(C(M)^+)$ are ample.

Let us fix a primitive embedding $i_M : M \hookrightarrow L$.

Definition 1.3.3. A *marked M-polarized* K3 surface is a marked K3 surface (X, ϕ) such that $\phi^{-1}(M) \subset \mathrm{Pic}(X)$. The restriction $j_\phi := \phi^{-1}|M : M \to \mathrm{Pic}(X)$ yields an M-polarized K3 surface (X, j_ϕ).

The *M-polarized* K3 surfaces have period points in a subset Ω_M of Ω given by

$$\Omega_M = \{[\omega] \in \Omega \mid \langle [\omega], m \rangle, \forall m \in M\}.$$

Let $\mathcal{K}_M$ be the subspace of the moduli space $\mathcal{N}$ of marked K3 surfaces mapped to Ω_M by the period map $\tau_{\mathcal{N}}$.

Theorem 1.3.4. [Dol96] The restriction of the period map $\tau_{\mathcal{N}} : \mathcal{K}_M \to \Omega_M$ to the moduli space $\mathcal{K}_M^{pa}$ of marked pseudo-ample M-polarized K3 surfaces is surjective.

We wish to define a subset of Ω_M corresponding to ample M-polarized K3 surfaces. Set $N = M^{\perp} \subset L$, and let $\Delta(N) = \{\delta \in N \mid \langle \delta, \delta \rangle = -2\}$. For any $\delta \in \Delta(N)$, let $H_\delta = \{z \in N_{\mathbf{C}} \mid \langle z, \delta \rangle = 0\}$. We define Ω_M^a as follows:

$$\Omega_M^a = \Omega_M - (\bigcup_{\delta \in \Delta(N)} H_\delta \cap \Omega_M).$$

Let $\mathcal{K}_M^a$ be the subspace of $\mathcal{N}$ mapped to Ω_M^a by $\tau_{\mathcal{N}}$. Then $\mathcal{K}_M^a$ is a moduli space of marked ample M-polarized K3 surfaces. [Dol96]

Let $\Gamma(M) = O(L, M)$ be the group of isometries of L preserving M. This group acts properly and discontinuously on Ω_M, inducing an action on $\mathcal{K}_M^{pa}$; similarly, the restriction of the $\Gamma(M)$ action to Ω_M^a induces an action on $\mathcal{K}_M^a$. Let $\mathbf{K}_M$ be given by $\mathcal{K}_M^{pa}/\Gamma(M)$, and let $\mathbf{K}_M^a$ be given by $\mathcal{K}_M^a/\Gamma(M)$. Then $\mathbf{K}_M$ may be given the structure of a quasi-projective algebraic variety; $\mathbf{K}_M^a$ also has the structure of a variety. [Dol96]

Suppose that our primitive embedding $i_M : M \hookrightarrow L$ is unique up to a lattice isomorphism of L. (We may guarantee a unique primitive embedding by choosing an M which satisfies the conditions in Theorem 1.2.10). Then $\mathbf{K}_M$ is a moduli space of pseudo-ample M-polarized K3 surfaces, and $\mathbf{K}_M^a$ is a moduli space of ample M-polarized K3 surfaces.

Let $H(m)$ be the lattice with intersection matrix $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$, where m is a positive integer.

Definition 1.3.4. We say a primitive sublattice M of L is *m-admissible* if $M^{\perp} = J \oplus \check{M}$, where J is isomorphic to $H(M)$. In this situation, we call $\check{M}$ the *mirror* of M.

Definition 1.3.5. The moduli space $\mathbf{K}_{\check{M}}$ is called the *mirror moduli space* of $\mathbf{K}_M$.

When $m = 1$, we may choose $J \cong H$ such that $M = J \oplus \check{M}$ and $M^{\perp} = J \oplus M$. In this case, the mirror of $\check{M}$ is M, so we obtain a duality. [Dol96]

1.4 Mirror Polytopes

In this section, we review the proposal of [Roh04] for mirror symmetry of K3 surfaces realized as hypersurfaces in toric varieties.

1.4.1 Toric Varieties

We begin by recalling some standard constructions involving toric varieties.

Let $N \cong \mathbf{Z}^n$ be a lattice with dual lattice M. Given a lattice polytope $\diamond$ in N, we define its *polar polytope* $\diamond^0$ to be $\diamond^0 = \{w \in \mathrm{M} \,|\, \langle v, w \rangle \geq -1 \,\forall\, v \in K\}$. If $\diamond^0$ is also a lattice polytope, we say that $\diamond$ is a reflexive polytope and that $\diamond$ and $\diamond^0$ are a mirror pair.

Example 1.4.1. The generalized octahedron in N with vertices at $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, \ldots, 0), \ldots, (0, 0, \ldots, \pm 1)$ has the hypercube with vertices at $(\pm 1, \pm 1, \ldots, \pm 1)$ as its polar.

A reflexive polytope must contain 0; furthermore, 0 is the only interior lattice point of the polytope. We may obtain a fan R by taking cones over the faces of $\diamond$. Let Σ be a simplicial refinement of R such that the one-dimensional cones of Σ are generated by the nonzero lattice points v_k, $k = 1 \ldots q$ of $\diamond$. Then the resulting variety $V = \mathcal{V}(\Sigma)$ is an orbifold; if $n = 3$, V is smooth. Generic representatives X of the anticanonical class of V are Calabi-Yau varieties; if $n = 3$, then the representatives are K3 surfaces. If we perform the same operations on the polar polytope $\diamond^0$, we obtain another family of Calabi-Yau varieties $\check{X}$. We shall refer to these two families as *mirror families.* [CK99]

We may obtain global homogeneous coordinates for V by a process analogous to the construction of $\mathbf{P}^n$ as a quotient space of $(\mathbf{C}^*)^n$. Let $Z_\Sigma \subseteq \mathbf{C}^q$ be the set $\cup_I \{(z_1, \ldots, z_q) \,|\, z_i = 0 \,\forall i \in I\}$, where the index I ranges over all sets $I \subseteq \{1, \ldots, q\}$

such that $\{v_i \,|\, i \in I\}$ is *not* a cone in Σ. Our variety is given by $(\mathbf{C}^q \setminus Z_\Sigma)/\sim$, where the equivalence relation $\sim$ is as follows:

$$(z_1, \ldots, z_q) \sim (\lambda^{a_j^1} z_1, \ldots, \lambda^{a_j^q} z_q) \text{ if } \sum_k a_j^k v_k = 0.$$

Here $\lambda \in \mathbf{C}^*$ and $a_j^k \in \mathbf{Z}^+$; there are $q - n$ independent sets of relations $\{a_j^1, \ldots, a_j^q\}$. In global homogeneous coordinates, a Calabi-Yau variety X in V is described by the polynomial

$$p = \sum_{x \in \diamond^0 \cap M} c_x \prod_{k=1}^{n} z_k^{\langle v_k, x\rangle + 1}.$$

If X is described by a polynomial p and the products $z_i\,\partial p/\partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on X, we say p is *nondegenerate* and X is *regular*. [Mav00]

1.4.2 Mirror Constructions

The nonzero lattice points v_k of $\diamond$ correspond to irreducible torus-invariant divisors W_k in V. (In global homogeneous coordinates, these are just the hypersurfaces $z_k = 0$.) Since V is obtained from a simplicial fan, the divisors W_k generate $\mathrm{Pic}(V) \otimes \mathbf{Q}$ subject to certain relations; in particular, rank $\mathrm{Pic}(V) = q-3$. When V is smooth, the divisors generate $\mathrm{Pic}(V)$.

Generically, the intersection of a divisor W_k with a Calabi-Yau hypersurface X of V is empty when the corresponding lattice point v_k is in the interior of a codimension-one face of $\diamond$. If v_k is on the boundary of a codimension-one face, then the intersection of W_k and X may form a single divisor of X; alternatively, $W_k \cap X$ may split into several irreducible components. In fact, W_k splits when the corresponding lattice point v_k is interior to a codimension-two face θ of $\diamond$ and the dual face $\hat{\theta}$ also has interior points. In this case, $W_k \cap X$ has $l(\hat{\theta}) - 1$ components W_{kj}, where $l(\hat{\theta})$ is the number of lattice points in the dual edge $\hat{\theta}$.

Batyrev used these counts to show that, when $n \geq 4$, the Hodge numbers $h^{1,1}(X)$ and $h^{n-2,1}(X)$ are given by the following formulas: [Bat94]

$$h^{1,1}(X) = l(\diamond) - n - 1 - \sum_{\mathrm{codim}\Gamma=1} l^*(\Gamma) + \sum_{\mathrm{codim}\Gamma=2} l^*(\Gamma) l^*(\hat{\Gamma}) \tag{1.1}$$

$$h^{n-2,1}(X) = l(\diamond^0) - n - 1 - \sum_{\mathrm{codim}\Gamma^0=1} l^*(\Gamma^0) + \sum_{\mathrm{codim}\Gamma^0=2} l^*(\Gamma^0) l^*(\hat{\Gamma^0}) \tag{1.2}$$

Here $l(\diamond)$ denotes the number of lattice points in $\diamond$, Γ is a face of $\diamond$ of the given codimension and Γ^0 is a face of $\diamond^0$, $\hat{\Gamma}$ indicates the dual face, and $l^*(\Gamma)$ is the number of points in the relative interior of the face.

Let $\check{X}$ be a generic Calabi-Yau hypersurface in the family obtained from $\diamond^0$. Interchanging the roles of $\diamond$ and $\diamond^0$ in the above formulas, we obtain the following theorem:

Theorem 1.4.1. [Bat94] The Hodge numbers of X and $\check{X}$ are related by $h^{1,1}(X) = h^{n-2,1}(\check{X})$ and $h^{n-2,1}(X) = h^{1,1}(\check{X})$.

Now, let us restrict to $n = 3$. In this case, the generic hypersurfaces X are smooth K3 surfaces, and the Picard group Pic(X) is a sublattice of $H^{1,1}(X, \mathbf{Z})$. Let $\iota : X \to V$ be the inclusion map; we define the so-called *toric divisors* as $\mathrm{Pic}_{\mathrm{tor}}(X) = \iota^*(\mathrm{Pic}(V))$. We shall refer to the sum $\delta = \sum_{\mathrm{codim}\Gamma=2} l^*(\Gamma) l^*(\hat{\Gamma})$ as the *toric correction term*. The toric divisors together with the divisors W_{kj} generate a group of rank $\delta + \mathrm{rank}\,\mathrm{Pic}_{\mathrm{tor}}(X)$ which we shall call $\mathrm{Pic}_{\mathrm{cor}}(X)$.

Oguiso showed that any analytic neighborhood in the base of a one-parameter, non-isotrivial family of K3 surfaces has a dense subset where the Picard ranks of the corresponding surfaces are greater than the minimum Picard rank of the family. [Ogu00] Thus, in the case of K3 surfaces, one might expect the equality of Equation 1.1 to be replaced by an inequality:

Proposition 1.4.1. [Roh04] Let X be a regular K3 hypersurface in V. Then,

$$\mathrm{rank}\ \mathrm{Pic}_{\mathrm{tor}}(X) = l(\diamond) - 4 - \sum_{\mathrm{codim}\Gamma=1} l^*(\Gamma)$$

and

$$\mathrm{rank}\ \mathrm{Pic}(X) \geq \mathrm{rank}\ \mathrm{Pic}_{\mathrm{tor}}(X) + \delta$$

The analogue of Theorem 1.4.1 for a K3 hypersurface is:

Theorem 1.4.2. [Roh04] Let X be a regular K3 hypersurface in V, and let $\check{X}$ be a regular K3 hypersurface in the mirror family. Then,

$$\mathrm{rank}\ \mathrm{Pic}_{\mathrm{tor}}(X) + \mathrm{rank}\ \mathrm{Pic}_{\mathrm{tor}}(\check{X}) + \delta = 20.$$

(Recall that any K3 surface X has $h^{1,1}(X) = 20$.)

Rohsiepe claimed that $(\mathrm{Pic}_{\mathrm{tor}}(X))^{\perp} \cong H \oplus \mathrm{Pic}_{\mathrm{cor}}(\check{X})$ and $(\mathrm{Pic}_{\mathrm{cor}}(X))^{\perp} \cong H \oplus \mathrm{Pic}_{\mathrm{tor}}(\check{X})$.

Chapter 2

SYMPLECTIC GROUP ACTIONS

2.1 Symplectic Actions

Let X be a K3 surface, and let G be a finite group acting on X by automorphisms. The action of G on X induces an action on the cohomology of X. We assume G acts symplectically: that is, G acts as the identity on $H^{2,0}(X)$. In this case, the minimum resolution Y of the quotient X/G is itself a K3 surface.

Nikulin classified the finite abelian groups which act symplectically on K3 surfaces by analyzing the relationship between X and Y. In the abelian case, Nikulin also described moduli spaces of K3 surfaces with G actions; these topological spaces are subspaces of the moduli space of marked K3 surfaces. [Nik80a] Mukai showed that any finite group G with a symplectic action on a K3 surface is a subgroup of a member of a list of eleven groups, and gave an example of a symplectic action of each of these maximal groups. [Muk88] Xiao gave an alternate proof of the classification by listing the possible types of singularities, and Kondō showed that the action of G on the K3 lattice extends to an action on a Niemeier lattice. [Xia96, Kon98]

The Picard group of X has a primitive sublattice S_G determined by the action of G. The rank of S_G varies from 8 to 19, depending on G. Thus, K3 surfaces which admit symplectic group actions provide a rich source of examples of families of K3 surfaces with high-rank Picard groups. The monodromy and mirror symmetry properties of algebraic K3 surfaces which admit a sublattice S_G of rank 18, and therefore have a Picard group of rank 19, have been extensively studied. (cf. [NS01, Smi07, KD08]) Conversely, if the structure of $\mathrm{Pic}(X)$ is known, one may examine its sublattices to detect symplectic group actions on X. Morrison used the structure of S_G for $G = \mathbf{Z}/2\mathbf{Z}$ to study K3 surfaces which admit Shioda-Inose structures. [Mor84] Recently, Garbagnati and Sarti have computed S_G for all possible abelian groups with symplectic action, correcting an earlier computation of Nikulin's; Garbagnati has also studied S_G for dihedral groups, and Hashimoto calculated the invariants of S_G for the permutation group $G = \mathcal{S}_5$. [GS07, Gar08b, Gar08a, Gar09, HT09]

In Section 2.2, we discuss the relationship between the lattice S_G and the singularities of X/G for any symplectic G-action, and show how to compute the rank and discriminant of S_G. We apply these techniques to K3 surfaces realized as hypersurfaces in toric varieties in Section 2.3. In Section 2.4, we show that the maps between X, Y, and X/G can be generalized to the realm of moduli spaces, and describe moduli spaces of K3 surfaces with symplectic G-action. Our proof extends the discussion in [Nik80a] to the case that G is not abelian. The key observation is that we may work backwards from a K3 surface Y endowed with a set of exceptional curves to the K3 surface X.

2.2 A Sublattice of the Picard Group

Let X be a K3 surface, and let G be a finite group acting symplectically on X. The cup product induces a bilinear form $\langle\, , \rangle$ on $H^2(X, \mathbf{Z}) \cong H \oplus H \oplus H \oplus E_8 \oplus E_8$. Using this form, we define $S_G = (H^2(X, \mathbf{Z})^G)^\perp$. The Picard group of X, $\mathrm{Pic}(X)$, consists of $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$; the group $\mathcal{T}(X) \subseteq H^2(X, \mathbf{Z})$ of transcendental cycles is defined as $(\mathrm{Pic}(X))^\perp$. Nikulin showed that the groups $\mathrm{Pic}(X)$ and S_G are related:

Proposition 2.2.1. [Nik80a] $S_G \subseteq \mathrm{Pic}(X)$ and $\mathcal{T}(X) \subseteq H^2(X, \mathbf{Z})^G$.

Nikulin also proved that S_G is a negative definite lattice. [Nik80a] In this section, we show how to compute the rank and discriminant of S_G, and relate S_G to the singularity structure of X/G.

The number of fixed points of an element g of a group G acting symplectically on a K3 surface X depends only on the order of g. [Nik80a]

Proposition 2.2.2. [Muk88, Ogu03] Let $m(n)$ be the number of elements in G of order n, and let $f(n)$ be the number of fixed points of an element of order n. Then,

$$\mathrm{rank}\ H^*(X, \mathbf{Z})^G = \frac{1}{|G|}(24 + \sum_{n=2}^{8} m(n) f(n)).$$

Since G acts as the identity on $H^0(X, \mathbf{Z})$ and $H^4(X, \mathbf{Z})$ as well as $H^{2,0}(X)$ and $H^{0,2}(X)$, we also know that $\mathrm{rank}\ H^*(X, \mathbf{Z})^G \geq 4$.

Because G acts symplectically on X, X/G has a minimal resolution Y which is also a K3 surface. Let $\{p_i\}$ be the singular points of X/G. The inverse image in Y of p_i is a configuration Ψ_i of (-2)-curves of ADE type; let c_i be the number of curves in this configuration. The configurations Ψ_i generate a lattice K in $\mathrm{Pic}(Y) \subset H^2(Y, \mathbf{Z})$ of rank $\sum_i c_i$. Let M be the minimal primitive sublattice of $H^2(Y, \mathbf{Z})$ containing K. Then M also has rank $\sum_i c_i$, and $H^2(Y, \mathbf{Z})/M$ is a free abelian group. Xiao showed that M is uniquely determined by the Ψ_i. [Xia96]

Remark 2.2.1. If G is isomorphic to Q_8, the group of unit quaternions, or T_{24}, the binary tetrahedral group of order 24, then K may be one of two different lattices, depending on the action of G. In all other cases, K (and thus M) is uniquely determined by G. [Xia96]

Let $\{q_{ij}\}$ be the inverse images in X of p_i, and let G_i be the stabilizer group of any q_{ij}; set $N_i = |G_i|$.

Proposition 2.2.3. [Xia96]

$$\sum_i c_i = \frac{24(|G|-1)}{|G|} - \sum_{i=1}^{k} \frac{N_i - 1}{N_i}.$$

Proposition 2.2.4. rank $S_G = \sum_i c_i$.

[Nik80a] discusses this proposition in the case that G is abelian. We use Propositions 2.2.2 and 2.2.3 to give a brief proof for any G.

Proof. We calculate:

$$\begin{aligned}
\text{rank } H^*(X, \mathbf{Z})^G + \sum_i c_i &= 24 - \text{rank } S_G + \sum_i c_i \\
&= \frac{1}{|G|}(24 + \sum_{n=2}^{8} m(n)f(n)) + \frac{24(|G|-1)}{|G|} - \sum_{i=1}^{k} \frac{N_i - 1}{N_i} \\
&= 24 + \frac{1}{|G|} \sum_{n=2}^{8} m(n)f(n) - \sum_{i=1}^{k} \frac{N_i - 1}{N_i}.
\end{aligned}$$

Thus, it suffices to show that

$$\sum_{n=2}^{8} m(n)f(n) = \sum_{i=1}^{k} \frac{|G|}{N_i}(N_i - 1).$$

$\sum_{n=2}^{8} m(n)f(n)$ counts each non-identity element g of G once for each point of X which g fixes. $N_i - 1$ counts the non-identity elements of the stabilizer group G_i.

The point p_i has precisely $\frac{|G|}{N_i}$ preimages q_{ij} in X; by definition, the elements of G_i fix the q_{ij}. Summing over all singular points p_i, we see that $\sum_{i=1}^{k} \frac{|G|}{N_i}(N_i - 1)$ also counts every element of G other than the identity once for each point of X which that element fixes. □

Though the lattices S_G and M are primitive sublattices of the K3 lattice $H \oplus H \oplus H \oplus E_8 \oplus E_8$ and have the same rank, they are not isomorphic: Nikulin showed that S_G contains no elements with square -2. [Nik80a] Instead, the relationship between S_G and M is given by the fact that $S_G = (H^2(X, \mathbf{Z})^G)^{\perp}$ and the following exact sequence.

Theorem 2.2.1. There exists an exact sequence

$$0 \longrightarrow M/K \longrightarrow H^2(Y, \mathbf{Z})/K \xrightarrow{\theta} (H^2(X, \mathbf{Z}))^G \longrightarrow H^3(G, \mathbf{Z}) \longrightarrow 0$$

where $\langle \theta(m), \theta(n) \rangle = |G| \langle m, n \rangle$.

Proof. Let $X' = X - (\cup_{i,j} q_{ij})$ and let $Y' = Y - (\cup_i \Psi_i)$. Since X is a simply connected complex surface, X' is also simply connected; since $Y' = X'/G$, X' is the universal covering space of Y'. By [CE99], Application XVI.1, there exists an exact sequence

$$0 \longrightarrow H^2(G, \mathbf{Z}) \longrightarrow H^2(Y', \mathbf{Z}) \xrightarrow{\theta} (H^2(X', \mathbf{Z}))^G \longrightarrow H^3(G, \mathbf{Z}) \xrightarrow{\zeta} H^3(Y', \mathbf{Z}).$$

Since θ is induced by the quotient map $X' \to Y'$, $\langle \theta(m), \theta(n) \rangle = |G| \langle m, n \rangle$. Xiao showed that $H^2(G, \mathbf{Z}) = M/K$ and $H^2(Y', \mathbf{Z}) = H^2(Y, \mathbf{Z})/K$; Nikulin observed that $H^2(X, \mathbf{Z}) = H^2(X', \mathbf{Z})$. [Xia96, Nik80a] Since G is a finite group, $H^3(G, \mathbf{Z})$ is a finite abelian group. We shall show that $H^3(Y', \mathbf{Z})$ is a free abelian group, so ζ must be the zero map.

Let N_i be a tubular neighborhood of the configuration of exceptional curves Ψ_i in Y, and let L_i be the boundary of N_i. Consider the Mayer-Vietoris sequence

$$\ldots \longrightarrow H^3(Y,\mathbf{Z}) \longrightarrow H^3(Y',\mathbf{Z}) \oplus_i \bigoplus H^3(N_i,\mathbf{Z}) \longrightarrow \bigoplus_i H^3(L_i,\mathbf{Z}) \longrightarrow H^4(Y,\mathbf{Z}) \longrightarrow \ldots$$

Since Y is a K3 surface, $H^3(Y) = 0$ and $H^4(Y) = \mathbf{Z}$. Because N_i is a tubular neighborhood of an ADE configuration of curves, N_i is homotopy equivalent to a bouquet of c_i 2-spheres, so $H^3(N_i) = 0$. Since L_i is a smooth real 3-manifold, $H^3(L_i) = \mathbf{Z}$. Furthermore, the map $\bigoplus_i H^3(L_i) \to H^4(Y)$ is given by $f : \mathbf{Z} \to \mathbf{Z}^n$, where $f((x_1,\ldots,x_{c_i})) = x_1 + .. + x_{c_i}$. Thus, $H^3(Y')$ is isomorphic to the kernel of f, a free abelian group of rank $c_i - 1$.

□

Remark 2.2.2. Garbagnati proved a variant of Theorem 2.2.1 in the case that G is an abelian group, correcting Nikulin's claim that θ is surjective. [Gar08b]

Lemma 2.2.1. [Nik80a] Let $J = \mathrm{Im}(\theta)$. Then the lattice discriminants $d(J)$ and $d(M)$ are related by

$$d(J) = -\frac{|G|^{22-\mathrm{rank}(M)}}{d(M)}.$$

Furthermore, suppose M^*/M is isomorphic to $\mathbf{Z}/a_1\mathbf{Z} \times \mathbf{Z}/a_2\mathbf{Z} \times \cdots \times \mathbf{Z}/a_k\mathbf{Z}$, where $a_i \geq 2$ and $a_i | a_{i+1}$ for each i, let $b_i = |G|/a_i$, and let $m = \mathrm{rank}\, M$. Then J^*/J is isomorphic to $\mathbf{Z}/b_1\mathbf{Z} \times \mathbf{Z}/b_2\mathbf{Z} \times \cdots \times \mathbf{Z}/b_k\mathbf{Z} \times (\mathbf{Z}/|G|\mathbf{Z})^{22-m-k}$.

Example 2.2.1. Let X be a K3 surface which admits a symplectic action by the permutation group $G = \mathcal{S}_4$. Then $\mathrm{Pic}(X)$ admits a primitive sublattice S_G which has rank 17 and discriminant $d(S_G) = -2^6 \cdot 3^2$.

Proof. Xiao computed that when $G = \mathcal{S}_4$, K is the rank 17 lattice given by $(A_3)^2 \oplus (A_2)^3 \oplus (A_1)^5$, and $M/K \cong \mathbf{Z}/(2\mathbf{Z})$. [Xia96] Next we use the fact that if lattices L and L' have the same rank, and $L \subset L'$, then the discriminants $d(L)$ and $d(L')$ are related by $d(L)/d(L') = [L' : L]^2$, where $[L' : L]$ is the index of L in L' as an abelian group. Since $d(K) = -2^9 \cdot 3^3$, we see that $d(M) = -2^7 \cdot 3^3$. By Lemma 2.2.1, the discriminant $d(J) = 2^8 \cdot 3^2$. The cohomology group $H^3(\mathcal{S}_4, \mathbf{Z})$ is isomorphic to $\mathbf{Z}/(2\mathbf{Z})$, so $[(H^2(X, \mathbf{Z}))^G : J] = 2$ and $d((H^2(X, \mathbf{Z}))^G) = 2^6 \cdot 3^2$. Since S_G is the perpendicular complement of $(H^2(X, \mathbf{Z}))^G$ in the unimodular K3 lattice $H \oplus H \oplus H \oplus E_8 \oplus E_8$, we conclude that $d(S_G) = -d((H^2(X, \mathbf{Z}))^G) = -2^6 \cdot 3^2$. □

Example 2.2.2. Let X be a K3 surface which admits a symplectic action by the Chevalley group $G = L_2(7) \cong PSL(2, \mathbf{F}_7)$. Then $(H^2(X, \mathbf{Z}))^G$ has rank 3 and discriminant 196.

Proof. The table in [Xia96] shows that K is the rank 19 lattice given by $A_6 \oplus (A_3)^2 \oplus (A_2)^3 \oplus A_1$, and $M \cong K$. Thus, $d(M) = -7 \cdot 4^2 \cdot 3^3 \cdot 2$. The order of $L_2(7)$ is $2^3 \cdot 3 \cdot 7$, so by Lemma 2.2.1, the discriminant $d(J) = 2^4 \cdot 7$. We may use the SAGE computer algebra system to show that $H^3(G, \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$. [SAG] Thus, $[(H^2(X, \mathbf{Z}))^G : J] = 2$, so $d(H^2(X, \mathbf{Z})) = (2^4 \cdot 7^2)/2^2 = 196$. □

Remark 2.2.3. The result of Example 2.2.2 is the "Key Lemma" of [OZ02]; that paper gives a longer proof using Niemeier lattices.

Example 2.2.3. Let X be a K3 surface which admits a symplectic action by the alternating group $G = \mathcal{A}_5$. Then the embedding of the lattice S_G in $H^2(X, \mathbf{Z})$ is unique up to an overall isometry of $H^2(X, \mathbf{Z})$.

Proof. Xiao showed that K is the rank 18 lattice given by $(A_4)^2 + (A_2)^3 + (A_1)^4$, and $K \cong M$. [Xia96] Thus, the group M^*/M is isomorphic to $(\mathbf{Z}/5\mathbf{Z})^2 \times (\mathbf{Z}/3\mathbf{Z})^3 \times (\mathbf{Z}/2\mathbf{Z})^4$, which we may rewrite as $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/30\mathbf{Z} \times \mathbf{Z}/30\mathbf{Z}$. The alternating group $\mathcal{A}_5$ has order 60, so by Lemma 2.2.1, J^*/J is isomorphic to $\mathbf{Z}/30\mathbf{Z} \times \mathbf{Z}/10\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Let $L^G = (H^2(X, \mathbf{Z}))^G$. The cohomology group $H^3(\mathcal{A}_5, \mathbf{Z})$ is isomorphic to $\mathbf{Z}/(2\mathbf{Z})$, so $L^G/J \cong \mathbf{Z}/(2\mathbf{Z})$. By Theorem 1.2.12, $L^{G^*}/L^G \subseteq (J^*/J)/(L^G/J)$, so L^{G*}/L^G is isomorphic to a subset of $\mathbf{Z}/30\mathbf{Z} \times \mathbf{Z}/10\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \cong (\mathbf{Z}/2\mathbf{Z})^3 \times \mathbf{Z}/3\mathbf{Z} \times (\mathbf{Z}/5\mathbf{Z})^2$. Recall that S_G^*/S_G is isomorphic to L^{G*}/L^G. Using Proposition 2.2.4 and [Xia96], we see that S_G has signature $(0, 18)$. Thus, S_G satisfies the conditions of Theorem 1.2.11, so S_G has a unique embedding in $H^2(X, \mathbf{Z})$. □

2.3 Toric Examples

In this section, we apply the results of Section 2.2 to generate families of K3 hypersurfaces in toric varieties with high Picard rank.

Let $\diamond$ be a reflexive polytope in a lattice $N \cong \mathbf{Z}^3$, let Σ be a simplicial refinement of the fan over the faces of $\diamond$, and let V be the smooth variety obtained from Σ. Demazure and Cox showed that the automorphism group A of V is generated by the big torus $T \cong (\mathbf{C}^*)^3$, symmetries of the fan Σ induced by lattice automorphisms, and one-parameter families derived from the "roots" of V. [CK99] We are interested in finite subgroups of A which act symplectically on K3 hypersurfaces X in V. To determine when a subgroup acts symplectically, we need an explicit description of a generator of $H^{2,0}(X)$.

Proposition 2.3.1. [Mav00] Let X be a regular K3 hypersurface in V described in global homogeneous coordinates by a polynomial p. Choose an integer basis $m_1, \ldots, m_n$ for the dual lattice M. For any n-element subset $I = \{i_1, \ldots, i_n\}$ of $\{1, \ldots, q\}$, let $\det v_I = \det(\langle m_j, v_{i_k}\rangle_{1 \le j, i_k \le n})$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_n}$, and $\hat{z}_I = \prod_{i \notin I} z_i$. Let Ω be the 3-form on V given in global homogeneous coordinates by $\sum_{|I|=n} \det v_I \hat{z}_I dz_I$. Then $\omega := \mathrm{Res}(\Omega/p)$ generates $H^{2,0}(X)$.

2.3.1 Finite Torus Actions

We begin by analyzing finite subgroups of the big torus T.

Proposition 2.3.2. Let X be a regular K3 hypersurface in V described in global homogeneous coordinates by a polynomial p, and represent $g \in T$ by a diagonal matrix $\Delta \in GL(q, \mathbf{C})$. Suppose $g^* p = \lambda p$, $\lambda \in \mathbf{C}^*$, and $\det(\Delta) = \lambda$. Then the induced action of g on the cohomology of X fixes the holomorphic 2-form ω of X.

Proof. Let Ω be the 3-form on V defined in Proposition 2.3.1. Then $g^*(\Omega) = \det(\Delta)\,\Omega$, so $g^*(\Omega/p) = (\lambda/\lambda)(\Omega/p) = (\Omega/p)$. Thus, g fixes the generator $\mathrm{Res}(\Omega/p)$ of $H^{2,0}(X)$. □

Remark 2.3.1. If $V = \mathbf{P}^3$, then $g^*\Omega = \det(\Delta)\,\Omega$ for any automorphism g of V induced by a matrix $\Delta \in GL(4, \mathbf{C})$; cf. Lemma 2.1 of [Muk88].

K3 hypersurfaces which admit finite torus actions have enhanced Picard rank.

Proposition 2.3.3. Let X be a representative of the anticanonical class of V, and assume X is a smooth K3 surface. Let G be a finite subgroup of T which acts symplectically on X. Then,

$$\mathrm{rank}\ \mathrm{Pic}(X) \geq \mathrm{rank}\ \mathrm{Pic}_{\mathrm{tor}}(X) + \mathrm{rank}\ S_G.$$

Proof. Since G is a subgroup of T, the divisors W_k of V are stable under the action of G. Thus, G fixes the divisors $\{W_\alpha \cap X \mid \alpha \in A\}$ and $\{\sum_{j=1}^{l(\theta^0)-1} W_{\beta j} | \, \beta \in B\}$ of X. Therefore, $\mathrm{Pic}_{\mathrm{tor}}(X) \subseteq H^2(X, \mathbf{Z})^G$. The proposition then follows from the facts that $S_G = (H^2(X, \mathbf{Z})^G)^\perp$ and that S_G is negative definite. □

Example 2.3.1. [Roh04] Consider the pencil of quartics in $\mathbf{P}^3$ described by $x^4 + y^4 + z^4 + w^4 - 4t(xyzw) = 0$. For generic t, the corresponding hypersurface X is a regular K3 surface. We have $\mathrm{rank}\ \mathrm{Pic}(X) \geq 19$.

Proof. $\mathbf{P}^3$ corresponds to the reflexive polytope $\diamond$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(-1, -1, -1)$. The only other lattice point of $\diamond$ is $(0, 0, 0)$, so $\mathrm{rank}\ \mathrm{Pic}_{\mathrm{tor}}(X) = 4 - 3 = 1$.

The group $(\mathbf{Z}/(4\mathbf{Z}))^2$ acts on X by $x \mapsto \lambda x$, $y \mapsto \mu y$, $z \mapsto \lambda^{-1}\mu^{-1}z$, where λ and μ are fourth roots of unity. By Proposition 2.3.2, this action is symplectic. Nikulin showed that $\mathrm{rank}\ S_G = 18$. [Nik80a]

By Proposition 2.3.3, $\mathrm{rank}\ \mathrm{Pic}(X) \geq 1 + 18 = 19$. □

Example 2.3.2. Let us consider the family of K3 surfaces in $\mathbf{WP}(1,1,1,3)$ given by $x^6 + y^6 + z^6 + w^2 - txyzw = 0$. If X is a regular K3 surface in this family, then rank $\mathrm{Pic}(X) \geq 19$.

Proof. $\mathbf{WP}(1,1,1,3)$ corresponds to the reflexive polytope $\diamond$ with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(-1,-1,-3)$. The only other lattice points of $\diamond$ are $(0,0,-1)$, which is interior to a face, and the origin. Thus, rank $\mathrm{Pic}_{\mathrm{tor}}(X) = 4 - 3 = 1$.

The group $\mathbf{Z}/(6\mathbf{Z}) \times \mathbf{Z}/(2\mathbf{Z})$ acts on X by $x \mapsto x$, $y \mapsto \lambda y$, $z \mapsto \lambda^{-1}\mu^{-1}z$, and $w \mapsto \mu w$, where λ is a sixth and μ a square root of unity. By Proposition 2.3.2, this action is symplectic. Nikulin showed that rank $S_G = 18$. [Nik80a] Thus, by Proposition 2.3.3, rank $\mathrm{Pic}(X) \geq 1 + 18 = 19$. □

2.3.2 Fan Symmetries

Let us now consider the automorphisms of V induced by symmetries of the fan Σ. Since Σ is a refinement of R, the fan consisting of cones over the faces of $\diamond$, the group of symmetries of Σ must be a subgroup H' of the group H of symmetries of $\diamond$ (viewed as a lattice polytope). We will identify a family $\mathcal{F}_\diamond$ of K3 surfaces in V on which H' acts by automorphisms, and then compute the induced action of G on the $(2,0)$ form of each member of the family.

Let $h \in H'$, and let X be a K3 surface in V defined by a polynomial p in global homogeneous coordinates. Then h maps lattice points of $\diamond$ to lattice points of $\diamond$, so we may view h as a permutation of the global homogeneous coordinates z_i: h is an automorphism of X if $p \circ h = p$. Alternatively, since H is the automorphism group of both $\diamond$ and its polar dual polytope $\diamond^0$, we may view h as an automorphism of $\diamond^0$: from this vantage point, we see that h acts by a permutation of the coefficients c_x of p, where each coefficient c_x corresponds to a point $x \in \diamond^0$. Thus, if h is to preserve X, we must have $c_x = c_y$ whenever $h(x) = y$. We may define a family of K3 surfaces fixed by H' by requiring that $c_x = c_y$ for any two lattice points $x, y \in \diamond^0$ which lie in the same orbit of H':

Proposition 2.3.4. Let $\mathcal{F}_\diamond$ be the family of K3 surfaces in V defined by the following family of polynomials in global homogeneous coordinates:

$$p = (\sum_{q \in \mathcal{O}} c_q \sum_{x \in \mathcal{O}} \prod_{k=1}^{n} z_k^{\langle v_k, x\rangle + 1}) + \prod_{k=1}^{n} z_k,$$

where $\mathcal{O}$ is the set of orbits of nonzero lattice points in $\diamond^0$ under the action of H'. Then H' acts by automorphisms on each K3 surface X in $\mathcal{F}_\diamond$.

Proposition 2.3.5. Let X be a regular K3 surface in the family $\mathcal{F}_\diamond$, and let $h \in H' \subset \mathbf{GL}(3, \mathbf{Z})$. Then $h^*(\omega) = (\det\, h)\omega$.

Proof. Once again, we use the fact that we may view h as either an automorphism of the lattice N which maps $\diamond$ to itself, or as an automorphism of the dual lattice M which restricts to an automorphism of $\diamond^0$. (If we fix a basis $\{n_1, n_2, n_3\}$ of N, take the dual basis $\{m_1, m_2, m_3\} = \{n_1^*, n_2^*, n_3^*\}$ on M, and treat h as a matrix, then h acts on M by the inverse matrix.) By Proposition 2.3.1, each choice of basis for M yields a generator of $H^{3,0}(V)$. Thus, if Ω is the generator of $H^{3,0}(V)$ corresponding to a fixed choice of integer basis m_1, m_2, m_3, we see that we may obtain a new generator Ω' of $H^{3,0}(V)$ by applying the change of basis h^{-1} to M. Recall that $\Omega = \sum_{|I|=3} \det v_I \hat{z}_I dz_I$, where $\det v_I = \det\,(\langle m_j, v_{i_k}\rangle_{1 \leq j, i_k \leq 3})$.

We compute:

$$\Omega' = \sum_{|I|=3} \det\,(h^{-1}(v_I))\hat{z}_I dz_I \tag{2.1}$$

$$= \sum_{|I|=3} \det\,(h^{-1}) \det v_I \hat{z}_I dz_I \tag{2.2}$$

$$= \det h \sum_{|I|=3} \det v_I \hat{z}_I dz_I \tag{2.3}$$

since $\det h = \pm 1$.

By Proposition 2.3.4, $h^*(p) = p$, so $h^*(\omega) = \mathrm{Res}(\Omega'/p) = (\det h)\omega$. □

Thus the group G of orientation-preserving automorphisms of $\diamond$ which preserve Σ acts symplectically on regular members of $\mathcal{F}_\diamond$.

The largest group which occurs as the orientation-preserving automorphism group of a three-dimensional lattice polytope is S_4. There are three distinct pairs of isomorphism classes of reflexive polytopes which have this symmetry group. In the following examples, we analyze families derived from these pairs of polytopes.

Example 2.3.3. Let $\diamond$ be the cube with vertices of the form $(\pm 1, \pm 1, \pm 1)$. The dual polytope $\diamond^0$ is an octahedron, with vertices $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. We may choose our fan Σ such that the group of lattice automorphisms of $\diamond$ preserves Σ. The group G of orientation-preserving automorphisms of $\diamond$ is isomorphic to S_4. $\mathcal{F}_\diamond$ is a one-parameter family, and if X is a regular member of $\mathcal{F}_\diamond$, $\mathrm{rank}\,\mathrm{Pic}(X) \geq 19$.

Proof. The action of G on $\diamond^0$ has two orbits: the origin, and the vertices of the octahedron. Thus, $\mathcal{F}_\diamond$ is a one-parameter family. Using Example 2.2.1, we conclude that for any smooth member of $\mathcal{F}_\diamond$, $\mathrm{rank}\, S_G = 17$.

Let X be a regular member of $\mathcal{F}_\diamond$. We wish to determine which of the divisors of X inherited from the ambient toric variety V are in $H^2(X, \mathbf{Z})^G$. The action of G on the lattice points of $\diamond$ has four orbits: the origin, the vertices of the cube, the interior points of edges, and interior points of faces. Let $v_1, \ldots, v_8$ be the vertices of the cube and $v_9, \ldots, v_{20}$ be the interior points of edges; let $W_1, \ldots, W_{20}$ be the corresponding torus-invariant divisors of the toric variety V. Since $v_1, \ldots, v_8$ and $v_9, \ldots, v_{20}$ are orbits of the action of G, $W_1 + \cdots + W_8$ and $W_9 + \cdots + W_{20}$ are elements of $\mathrm{Pic}(V)$ which are fixed by G. These two divisors span a rank-two lattice in $\mathrm{Pic}(V)$. Since there are no lattice points strictly in the interior of the edges of $\diamond^0$

and none of the points $v_1, \ldots, v_{20}$ lies in the relative interior of a facet of $\diamond$, $W_k \cap X$ is connected and nonempty for $1 \leq k \leq 20$ and the divisors $W_1 \cap X + \cdots + W_8 \cap X$ and $W_9 \cap X + \cdots + W_{20} \cap X$ span a rank-two lattice in $\mathrm{Pic}(X)$. This rank-two lattice is contained in $H^2(X, \mathbf{Z})^G$.

Since S_G is the perpendicular complement of $H^2(X, \mathbf{Z})^G$, rank $\mathrm{Pic}(X) \geq 17 + 2 = 19$. □

Remark 2.3.2. This family is analyzed in [HLOY04].

Example 2.3.4. Let $\diamond$ be the octahedron with vertices $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, $(1,-1,1)$, $(1,1,-1)$, and $(-1,-1,-1)$. The polar dual $\diamond^0$ has vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,1,1)$, $(1,-1,-1)$, $(0,0,-1)$, $(0,-1,0)$, and $(-1,0,0)$. We may choose our fan Σ such that the group of lattice automorphisms of $\diamond$ preserves Σ. The group G of orientation-preserving automorphisms of $\diamond$ is isomorphic to S_4. $\mathcal{F}_\diamond$ is a one-parameter family. If X is a regular member of $\mathcal{F}_\diamond$, rank $\mathrm{Pic}(X) \geq 19$.

Proof. The action of G on $\diamond^0$ has two orbits, the origin and the polytope's vertices, so $\mathcal{F}_\diamond$ is a one-parameter family. As in the previous example, Example 2.2.1 shows that for any smooth member of $\mathcal{F}_\diamond$, rank $S_G = 17$.

Let X be a regular member of $\mathcal{F}_\diamond$. As before, we determine which of the divisors of X inherited from the ambient toric variety V are in $H^2(X, \mathbf{Z})^G$. The action of G on the lattice points of $\diamond$ has three orbits: the origin, the octahedron's vertices, and the interior points of edges. Let $v_1, \ldots, v_6$ be the vertices and $v_7, \ldots, v_{18}$ be the interior points of edges; let $W_1, \ldots, W_{18}$ be the corresponding torus-invariant divisors of V. Then $W_1 + \cdots + W_6$ and $W_7 + \cdots + W_{18}$ are elements of $\mathrm{Pic}(V)$ fixed by the action of G. These two divisors span a rank-two lattice in $\mathrm{Pic}(V)$. Since there are no lattice points strictly in the interior of the edges of $\diamond^0$ and the facets of $\diamond$ have no points in their relative interiors, $W_k \cap X$ is connected and nonempty for $1 \leq k \leq 18$ and the divisors

$W_1 \cap X + \cdots + W_6 \cap X$ and $W_7 \cap X + \cdots + W_{18} \cap X$ span a rank-two lattice in $\mathrm{Pic}(X)$. This rank-two lattice is contained in $H^2(X, \mathbf{Z})^G$. Thus, rank $\mathrm{Pic}(X) \geq 17 + 2 = 19$.

□

Example 2.3.5. Let $\diamond$ be a three-dimensional reflexive polytope with fourteen vertices and twelve faces. Up to lattice isomorphism, $\diamond$ is unique; moreover, $\diamond$ has the most vertices of any three-dimensional reflexive polytope. We may choose our fan Σ such that the group of lattice automorphisms of $\diamond$ preserves Σ. The group G of orientation-preserving automorphisms of $\diamond$ is isomorphic to S_4, and $\mathcal{F}_\diamond$ is a one-parameter family. If X is a regular member of $\mathcal{F}_\diamond$, rank $\mathrm{Pic}(X) \geq 19$.

Proof. The lattice points of $\diamond^0$ consist of vertices and the origin, and G acts transitively on the vertices of $\diamond^0$, so $\mathcal{F}_\diamond$ is a one-parameter family. As above, Example 2.2.1 shows that for any smooth member of $\mathcal{F}_\diamond$, $\operatorname{rank} S_G = 17$.

Let X be a regular member of $\mathcal{F}_\diamond$. Once again, we determine which of the divisors of X inherited from the ambient toric variety V are in $H^2(X, \mathbf{Z})^G$. The action of G on the lattice points of $\diamond$ has three orbits; one orbit contains the origin, another contains eight vertices, and the last contains the remaining six vertices. Let $\{v_1, \ldots, v_8\}$ and $\{v_9, \ldots, v_{14}\}$ be the vertex orbits; let $W_1, \ldots, W_{14}$ be the corresponding torus-invariant divisors of V. Then $W_1 + \cdots + W_8$ and $W_9 + \cdots + W_{14}$ are elements of $\mathrm{Pic}(V)$ fixed by the action of G; these two divisors span a rank-two lattice in $\mathrm{Pic}(V)$. Since there are no lattice points strictly in the interior of the edges of $\diamond^0$ and the facets of $\diamond$ have no points in their relative interiors, $W_k \cap X$ is connected and nonempty for $1 \leq k \leq 14$ and the divisors $W_1 \cap X + \cdots + W_8 \cap X$ and $W_9 \cap X + \cdots + W_{14} \cap X$ span a rank-two lattice in $\mathrm{Pic}(X)$. This rank-two lattice is contained in $H^2(X, \mathbf{Z})^G$, so rank $\mathrm{Pic}(X) \geq 17 + 2 = 19$.

□

Remark 2.3.3. An explicit analysis of the same family appears in [Ver96].

In the previous examples, we identified reflexive polytopes $\diamond^0$ on which the orientation preserving automorphism group G acted transitively on non-origin lattice points, and then analyzed the resulting one-parameter family. There is one other three-dimensional reflexive polytope with this property, up to isomorphism: the tetrahedron with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$.

Example 2.3.6. Let $\diamond$ be the tetrahedron with vertices $(3,-1,-1)$, $(-1,3,-1)$, $(-1,-1,3)$, and $(-1,-1,-1)$. We may choose our fan Σ such that the group of lattice automorphisms of $\diamond$ preserves Σ. The polar dual $\diamond^0$ has vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$; taking cones over the faces of $\diamond^0$ yields the fan for projective space $\mathbf{P}^3$. The group G of orientation-preserving automorphisms of $\diamond$ is isomorphic to the alternating group $\mathcal{A}_4$, and $\mathcal{F}_\diamond$ is a one-parameter family. If X is a regular member of $\mathcal{F}_\diamond$, rank $\mathrm{Pic}(X) \geq 19$.

Proof. The lattice points of $\diamond^0$ consist of vertices and the origin, so since G acts transitively on the vertices of $\diamond^0$, $\mathcal{F}_\diamond$ is a one-parameter family. Proposition 2.2.4 and [Xia96]'s table show that for any smooth member of $\mathcal{F}_\diamond$, $\operatorname{rank} S_G = 16$.

Let X be a regular member of $\mathcal{F}_\diamond$. We must determine which of the divisors of X inherited from the ambient toric variety V are in $H^2(X,\mathbf{Z})^G$. The action of G on the lattice points of $\diamond$ has five orbits: the origin, the vertices, the central point of each edge, the other interior edge points, and the points on the interior of each face. Let $v_1,\dots,v_4$ be the vertices, $v_5,\dots,v_{10}$ the central points of edges, and $v_{11},\dots,v_{22}$ be the other interior edge points; let $W_1,\dots,W_{22}$ be the corresponding torus-invariant divisors of V. Then $W_1+\dots+W_4$, $W_5+\dots+W_{10}$, and $W_{11}+\dots+W_{22}$ are elements of $\mathrm{Pic}(V)$ fixed by the action of G. These three divisors span a rank-three lattice in $\mathrm{Pic}(V)$. Since there are no lattice points strictly in the interior of the edges of $\diamond^0$ and none of the points $v_1,\dots,v_{22}$ lies in the relative interior of a facet of $\diamond$, $W_k \cap X$ is connected and nonempty for $1 \leq k \leq 22$ and the divisors $W_1 \cap X + \dots + W_4 \cap X$,

$W_5 \cap X + \cdots + W_{10} \cap X$, and $W_{11} \cap X + \cdots + W_{22} \cap X$ span a rank-three lattice in $\mathrm{Pic}(X)$. This rank-three lattice is contained in $H^2(X, \mathbf{Z})^G$, so rank $\mathrm{Pic}(X) \geq 16 + 3 = 19$. $\square$

Remark 2.3.4. We may also use Proposition 1.4.1 to show that $\mathrm{Pic}(X) \geq 19$ without investigating the group of automorphisms. [NS01] contains a detailed analysis of this family.

2.4 Moduli Spaces

Let X be a K3 surface, and let G be a finite group acting symplectically on X. As a lattice, $H^2(X, \mathbf{Z})$ is isomorphic to $L = H \oplus H \oplus H \oplus E_8 \oplus E_8$. Let Y be a minimal resolution of the quotient X/G, and let c_j be the exceptional divisors in Y. Since Y is also a K3 surface, $H^2(Y, \mathbf{Z})$ is isomorphic to L. Let M be the primitive sublattice of L generated by the c_j under this isomorphism (that is, the smallest primitive sublattice which contains these divisors). Recall that M is uniquely determined by the c_j and our choice of isomorphism.

Nikulin showed that, when G is abelian, this picture can be extended to moduli spaces of Kähler K3 surfaces. [Nik80a] We shall extend his arguments to the case of non-abelian G.

Definition 2.4.1. [Nik80a] A *condition* T is a primitive sublattice K in L and a finite subset $\{c_i\}$ of K such that $c_i^2 = -2$ for each i. A *marked K3 surface with condition* T is a K3 surface X together with an isometry $\alpha : H^2(X, \mathbf{Z}) \to L$ such that $\alpha^{-1}(K) \subset H^{1,1}(X)$ and $\alpha^{-1}(c_i)$ is represented by a nonsingular rational curve on X for each i.

Remark 2.4.1. A nonsingular rational curve with self-intersection -2 in a K3 surface is uniquely determined by its homology class.[BHPVdV04, Nik80a] We will often identify the cohomology classes $\alpha^{-1}(c_i)$ with the corresponding curves.

Let us consider the moduli space $\mathcal{M}_T \subset \mathcal{M}$ consisting of all marked Kähler K3 surfaces with condition $T = \{c_j\} \subset M \subset L$.

Remark 2.4.2. Note that by taking M to be isomorphic to a primitive sublattice of $H^2(Y, \mathbf{Z})$, we have fixed the primitive embedding of M in L up to automorphisms of L.

Theorem 2.4.1. $\mathcal{M}_T$ has precisely two path-connected components.

Remark 2.4.3. Nikulin proved Theorem 2.4.1 under the assumption that rank $M \leq 18$ by constructing a path between any two elements in the same component.[Nik80a] We give a more general argument based on the discussion in [Nik80a] and the bijectivity of the refined period map $\tau_{\mathcal{M}}$.

Proof. Let $(K\Omega)^0_M$ be the subset of $(K\Omega)^0$ given by $\{(\kappa, [\omega]) \in (K\Omega)^0 \mid M \subset H^{1,1}_{(\kappa,[\omega])}\}$. Let $m \in M$ correspond to a marked K3 surface $(X; \phi)$, and let $\tau_{\mathcal{M}}(m) = (\kappa, [\omega])$. Let Δ_m be the set given by $\{\delta \in \phi_{\mathbf{C}}(H^{1,1}(X)) \mid \langle \delta, \delta \rangle = -2\}$, and let Δ^+_m be the subset of Δ_m given by $\langle \kappa, \delta \rangle > 0$. Let $(K\Omega)^0_T$ be the subset of $(K\Omega)^0_M$ such that $c_i \in \Delta^+_m$ and c_i is an irreducible element of Δ^+_m for each i.

Next, we use the following propositions.

Proposition 2.4.1. [Nik80a] Let $m \in \mathcal{M}$. Then $m \in \mathcal{M}_T$ if and only if $\tau_{\mathcal{M}}(m) \in K\Omega)^0_T$.

Proposition 2.4.2. [Nik80a] Let M be a negative definite lattice with rank $M \leq 19$. Then $(K\Omega)^0_M$ is a closed smooth complex subspace of $(K\Omega)^0$. The connected components of $(K\Omega)^0_M$ are $((K\Omega)^0_M)^{(\pm)P}$, where P is a continuous choice of partition of Δ_m into Δ^+_m and $-\Delta^+_m$. Furthermore, $((K\Omega)^0_M)^{(\pm)P} - ((K\Omega)^0_T)^{(\pm)P}$ is a closed subset of $((K\Omega)^0_M)^{(\pm)P}$ which is the union of at most countably many closed complex subspaces of $((K\Omega)^0_M)^{(\pm)P}$.

Proposition 2.4.2 implies that $((K\Omega)^0_T)^{(\pm)P}$ is connected and path-connected. As in the abelian case, we have only two possible choices of partition P, corresponding to a designation of effective divisors. (See [Nik80a]; the argument is a direct consequence of the Riemann-Roch theorem.) Thus, $(K\Omega)^0_T$ has two components. Since the refined period map $\tau_{\mathcal{M}}$ is injective and surjective (cf. [BHPVdV04]), Proposition 2.4.1 implies that $\mathcal{M}_T$ also has two path-connected components.

□

Let $C_j = \alpha_m^{-1}(c_j)$ and let $Y'_m = Y_m - (\cup_j C_j)$. Let X'_m be the universal covering space of Y'_m, and let $H \cong \pi_1(Y'_m)$ be the group of covering transformations. The covering spaces of the complements of ADE configurations of rational curves on K3 surfaces have been classified:

Theorem 2.4.2. [SZ01, Cam04] Let $\tilde{\Delta}$ be an ADE configuration of smooth rational curves on a K3 surface K. Let $K' = K - \tilde{\Delta}$, and let J' be the universal covering space of K'. Then J' and $\pi_1(K')$ satisfy one of the following conditions:

1. $J' \cong K'$ and $\pi_1(K')$ is trivial.

2. J' is isomorphic to the complement of a discrete set of points A in $\mathbf{C}^2$, and $\pi_1(K')$ is infinite. Furthermore, there exists a map f from $\mathbf{C}^2 - A$ to a two-dimensional complex torus T and a map g from T to K' such that g is the quotient of T by a finite group of automorphisms Γ and $g \circ f$ is the covering map.

3. J' is isomorphic to a K3 surface with a finite set of points removed, and the group of covering transformations (which is naturally isomorphic to $\pi_1(K')$) acts symplectically on this surface.

Theorem 2.4.3. Suppose there exists $\mu \in \mathcal{M}_T$, corresponding to a marked K3 surface (Y_μ, α_μ), such that Y_μ is the resolution of the quotient of a K3 surface X_μ by a symplectic G-action. Let $q \in \mathcal{M}_T$, and let (Y_q, α_q) be the corresponding marked K3 surface. Then there exists a K3 surface X_q and a symplectic action of G on X_q such that Y_q is a resolution of X_q/G.

Proof. For any $m \in \mathcal{M}_T$, we may choose a neighborhood U_m of m such that for all m' in U_m, there exists a diffeomorphism $\phi : Y_m \to Y_{m'}$ such that $\phi^*(\alpha_{m'}^{-1}(M)) = \alpha_m^{-1}(M)$ and (since rational curves in K3 surfaces are uniquely determined by their homology classes) $\phi^*(\alpha_{m'}^{-1}(\{c_j\})) = \alpha_m^{-1}(\{c_j\})$. Thus $Y_m - \alpha_m^{-1}(\{c_j\})$ and $Y_{m'} - \alpha_{m'}^{-1}(\{c_j\})$ are isomorphic, and $\pi_1(Y_m - \alpha_m^{-1}(\{c_j\})) = \pi_1(Y_{m'} - \alpha_{m'}^{-1}(\{c_j\}))$.

By Theorem 2.4.1, $\mathcal{M}_T$ has two components, $\mathcal{M}_T^+$ and $\mathcal{M}_T^-$; if $m \in \mathcal{M}_T^+$ corresponds to the marked K3 surface (Y_m, α_m), we may obtain a new marked K3 surface $(Y_{m'}, \alpha_{m'})$ with $m' \in \mathcal{M}_T^-$ by setting $Y_m = Y_{m'}$ and $\alpha_{m'} = -\alpha_m$. [Nik80a] By our assumption, the marked K3 surface (Y_μ, α_μ) satisfies $\pi_1(Y_\mu - \alpha_\mu^{-1}(\{c_j\})) = G$. Setting $Y_{\mu'} = Y_\mu$ and $\alpha_{\mu'} = -\alpha_\mu$, we see that $(Y_{\mu'}, \alpha_{\mu'})$ is a marked K3 surface with condition T in the other component of $\mathcal{M}_T$ which also satisfies $\pi_1(Y_{\mu'} - \alpha_{\mu'}^{-1}(\{c_j\})) = G$. Thus, there exists a path in $\mathcal{M}_T$ from q to either μ or μ'. Covering this path by a finite number of the neighborhoods U_m, we see that $\pi_1(Y_q - \alpha_q^{-1}(\{c_j\}))$ is isomorphic to either $\pi_1(Y_\mu - \alpha_\mu^{-1}(\{c_j\}))$ or $\pi_1(Y_{\mu'} - \alpha_{\mu'}^{-1}(\{c_j\}))$, so $\pi_1(Y_q - \alpha_q^{-1}(\{c_j\})) = G$. By Theorem 2.4.2, the covering space of $Y_q - \alpha_q^{-1}(\{c_j\})$ is isomorphic to a K3 surface X_q with a finite number of points removed, and G acts symplectically on X_q. Thus, Y_q is the resolution of X_q/G, as desired. □

Starting with Y_q, we obtained a pair $(X_q, i_q : G \hookrightarrow \operatorname{Aut} G)$.

Definition 2.4.2. We say that two points $m, m' \in \mathcal{M}_T$ *determine the same action of* G on the two-dimensional integral cohomology of K3 surfaces if there exist corresponding pairs $(X_m, i_m : G \hookrightarrow \operatorname{Aut} G)$, $(X_{m'}, i_{m'} : G \hookrightarrow \operatorname{Aut} G)$ and an isomorphism $\phi : H^2(X_m, \mathbf{Z}) \to H^2(X_{m'}, \mathbf{Z})$ which preserves the cup product and satisfies the relation

$$i_{m'}(g)^* = \phi \cdot i_m(g) \cdot \phi^{-1}$$

for any $g \in G$.

The condition that points determine the same action of G defines an equivalence relation on $\mathcal{M}_T$.

Theorem 2.4.4. Let $m_0 \in \mathcal{M}_T$, and suppose m_0 corresponds to the pair $(X_m, i_m : G \hookrightarrow \operatorname{Aut} G)$. The set of points in $\mathcal{M}_T$ which determine the same action of G as m_0 is open.

Proof. We construct an open neighborhood of m_0 in which the action coincides with the action determined by m_0. Since this property depends on the X_m, we would like to construct another moduli space which will parameterize the X_m.

Fix a marking $\beta_{m_0} : H^2(X, \mathbf{Z}) \to L$. Then the map $i_m : G \hookrightarrow \text{Aut } X_m$ induces an action of G on L. We would like to say that G embeds in Aut L, but we have a slight notational difficulty: the multiplication in G is from left to right, but the group operation in Aut L is composition of functions, which goes from right to left. Following [Nik80a], we write $\phi : \dot{G} \hookrightarrow \text{Aut } L$, where $\dot{G}$ is the same group as G but with all multiplication in the reverse order. We have the relation $\beta_{m_0} \circ i_{m_0}(g)^* \circ \beta_{m_0}^{-1} = \phi(g)$, $g \in G$.

The triple $(X_{m_0}, i_{m_0}, \beta_{m_0})$ defines a point μ_0 in the moduli space $\mathcal{M}_{G,\phi}$, where $\mathcal{M}_{G,\phi}$ is the moduli space of marked K3 surfaces with algebraic automorphism group G and action ϕ on the integral cohomology. The usual map $u : \mathcal{X} \to \mathcal{M}$ (where $\mathcal{M}$ is the moduli space of all marked K3 surfaces) restricts to a map $u_{G,\phi} : \mathcal{X}_{G,\phi} \to \mathcal{M}_{G,\phi}$. Following [Nik80a], we obtain a neighborhood V of μ_0 in $\mathcal{M}_{G,\phi}$, a corresponding neighborhood $\mathcal{X}^V_{G,\phi}$ in $\mathcal{X}_{G,\phi}$, and a resolution $\mathcal{Y}^V_{G,\phi}$ of $\mathcal{X}^V_{G,\phi}/G$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{X}^V_{G,\phi} & \xrightarrow{\ \pi\ } & \mathcal{X}^V_{G,\phi}/G & \xleftarrow{\ \sigma\ } & \mathcal{Y}^V_{G,\phi} \\ & {\scriptstyle u_{G,\phi}} \searrow & \downarrow & \swarrow {\scriptstyle v} & \\ & & V & & \end{array}$$

Each curve E_j in Y_{m_0} extends uniquely to an effective divisor $\mathbf{E}_j$ on $\mathcal{Y}^V_{G,\phi}$. For each $\mu \in V$, $\mathbf{E}_j \cdot Y_\mu = E^\mu_j$ is a nonsingular rational curve on Y_m, where $\{E^\mu_j\}$ is the set of components of the curves obtained from the resolution of singularities of X_μ/G. We set $\mathcal{X}'_{G,\phi} = \mathcal{X}_{G,\phi} - \{\text{fixed points of } G\}$ and $\mathcal{Y}'_{G,\phi} = \mathcal{Y}_{G,\phi} - \cup \mathbf{E}_j$, obtaining a new commutative diagram:

$$\begin{array}{ccccccc} \mathcal{X}^V_{G,\phi} & \longrightarrow & (\mathcal{X}^V_{G,\phi})' & \xrightarrow{\pi'} & (\mathcal{Y}^V_{G,\phi})' & \longleftarrow & \mathcal{Y}^V_{G,\phi} \\ & {\scriptstyle u_{G,\phi}}\searrow & \searrow{\scriptstyle u'_{G,\phi}} & & \swarrow{\scriptstyle v'} & \swarrow{\scriptstyle v} & \\ & & & V & & & \end{array}$$

These maps induce corresponding maps on G-sheaves:

$$R^2 u_{G,\phi_*}\mathbf{Z} \xrightarrow{i^*} R^2 u_{G,\phi_*}{}'\mathbf{Z} \xleftarrow{\pi'^*} R^2 v_{G,\phi_*}{}'\mathbf{Z} \xleftarrow{j^*} R^2 v_{G,\phi_*}\mathbf{Z}$$

[Nik80a] showed that there exists a map

$$\theta = (i^*)^{-1} \circ \pi'^* \circ \bar{j}^* : R^2 v_{G,\phi_*}\mathbf{Z}/ \oplus \mathbf{Z}\mathbf{E}_j \to (R^2 v_{G,\phi_*}\mathbf{Z})^G$$

which satisfies $\theta(x) \cdot \theta(y) = |G|(x \cdot y)$ for $x, y \in (\oplus \mathbf{Z}\mathbf{E}_j)^\perp$ and fits into an exact sequence

$$0 \to \ker\theta \longrightarrow R^2 v_*\mathbf{Z}/ \oplus \mathbf{Z}E_j \xrightarrow{\theta} (R^2 u_{G,\phi_*}\mathbf{Z})^G.$$

[Nik80a] also showed that $\ker\theta$ is the torsion subsheaf of $R^2 v_*\mathbf{Z}/ \oplus \mathbf{Z}E_j$.

Over μ_0, we may use the markings α_{m_0} and β_{m_0} to obtain the exact sequence

$$0 \longrightarrow M/ \oplus \mathbf{Z}c_j \longrightarrow L/ \oplus \mathbf{Z}c_j \xrightarrow{\beta_{m_0}\circ\theta\circ\alpha_{m_0}^{-1}} L^{\phi(G)}.$$

Remark 2.4.4. [Nik80a] claimed that θ is a surjective map when G is abelian. As Garbagnati and Sarti observed, this is not the case: the discrepancy is given by Theorem 2.2.1. [Gar08b, GS07]

Note that θ restricts to an injective map $\theta : M^\perp \hookrightarrow L^{\phi(G)}$. Proposition 2.2.4 implies that $M^\perp$ and $L^{\phi(G)}$ have the same rank, so we may extend θ to an isomorphism from $M^\perp \otimes \mathbf{C}$ to $L^{\phi(G)} \otimes \mathbf{C}$. The theorem now follows from [Nik80a]'s argument in the abelian case. □

Corollary 2.4.1. All points of $\mathcal{M}_T$ determine the same action.

Proof. This follows from Theorem 2.4.1 using the argument in [Nik80a]. □

Together, Theorem 2.4.3 and Corollary 2.4.1 show that we may classify symplectic actions on K3 surfaces by classifying the conditions T which are obtained from symplectic actions. Xiao listed the ADE configurations corresponding to finite groups which can act symplectically. [Xia96] In most cases, a group G corresponds to a single ADE configuration; the exceptions are Q_8, the group of unit quaternions, and T_{24}, the binary tetrahedral group of order 24, each of which corresponds to two different configurations. Nikulin showed by direct computation that when G is abelian, the primitive lattice M generated by the singular curves has a unique embedding in the K3 lattice, so T is uniquely determined by G. This need not hold for a non-abelian G.

Does every embedding of a symplectic ADE configuration in the K3 lattice yield a symplectic group action? Theorem 2.4.2 tells us that we may approach this question by analyzing the possible fundamental groups of the complement of a given configuration.

Let T^2 be a two-dimensional complex torus, and let Γ be a finite group of automorphisms of T^2. [Fuj88] classified the possible finite groups Γ, and [Ber88] and [ÖS99] classified the resulting singularities of T^2/Γ:

Group	ADE Configuration
C_2	$16A_1$
C_3	$9A_2$
C_4	$4A_3 + 6A_1$
C_6	$A_5 + 4A_2 + 5A_1$
Q_8	$4D_4 + 3A_1$
Q_{12}	$D_5 + 3A_3 + 2A_2 + A_1$
T_{24}	$A_5 + 2A_3 + 4A_2$
T_{24}	$E_6 + D_4 + 4A_2 + A_1$

(Here C_k is the cyclic group of order k, Q_8 and Q_{12} are binary dihedral groups, and T_{24} is the binary tetrahedral group.)

Xiao's list of K3 singularities obtained from group actions is disjoint from the list above. [Xia96]

We next consider whether there exists an ADE configuration Δ which can be obtained in two ways: from a singular K3 surface whose smooth part has trivial fundamental group, and as the ADE singularity of another K3 surface whose smooth part has non-trivial fundamental group. Most of the cases can be eliminated using the following lemma, as stated by [SZ01]:

Lemma 2.4.1. [Xia96] Let $\tilde{\Delta}$ be an ADE configuration of rational curves on a K3 surface, let $\mathbf{Z}[\Delta]$ be the sublattice of the K3 lattice L generated by the curves in $\tilde{\Delta}$, and let M_Δ be the smallest primitive sublattice of L containing $\mathbf{Z}[\Delta]$. Then the dual of the abelianisation of $\pi_1(X - \tilde{\Delta})$ is canonically isomorphic to $M_\Delta/\mathbf{Z}[\Delta]$. In particular, if $\pi_1(X - \tilde{\Delta})$ is trivial, then $\mathbf{Z}[\Delta]$ embeds in L as a primitive sublattice.

[Xia96] computed $M_\Delta/\mathbf{Z}[\Delta]$ for each ADE configuration which can occur as the exceptional divisor of a resolution of the quotient of a K3 surface by a group of symplectic automorphisms. Using Lemma 2.4.1, we conclude that none of the configurations in [Xia96]'s list can yield a trivial fundamental group, save possibly the following list (corresponding to perfect groups):

Group	ADE Configuration
$\mathcal{A}_5$	$2A_4 + 3A_2 + 4A_1$
$L_2(7)$	$A_6 + 2A_3 + 3A_2 + A_1$
$\mathcal{A}_6$	$2A_4 + 2A_3 + 2A_2 + A_1$
M_{20}	$D_4 + 2A_4 + 3A_2 + A_1$

(Here $\mathcal{A}_5$ and $\mathcal{A}_6$ are alternating groups, $L_2(7)$ is the Chevalley group $PSL(2, \mathbf{F}_7)$, and M_{20} is a subgroup of the Mathieu group M_{24} which is isomorphic to the semidirect product $(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathcal{A}_5$. [Muk88])

Symplectic actions of these groups have been extensively studied using Niemeier lattices. Mukai studied the lattice invariants of S_G when $G = M_{20}$ in an appendix to [Kon98]; Oguiso and Zhang investigated finite non-symplectic extensions of an $L_2(7)$ action in [OZ02]; Keum, Oguiso, and Zhang studied extensions of $\mathcal{A}_6$ actions in [KOZ05] and [KOZ07]; and Hashimoto considered actions induced by $\mathcal{A}_5 \hookrightarrow \mathcal{S}_5$ in [HT09].

When $G = \mathcal{A}_5$, the lattice $M = M_\Delta$ has rank 18 and discriminant group $M^*/M \cong (\mathbf{Z}/5\mathbf{Z})^2 \oplus (\mathbf{Z}/3\mathbf{Z})^3 \oplus (\mathbf{Z}/2\mathbf{Z})^4$. Therefore, the primitive embedding of M in the K3 lattice L is unique up to isometries of L by Theorem 1.2.11, and $\mathcal{A}_5$ corresponds to a single condition T and moduli space $\mathcal{M}_T$.

For each of the groups $\mathcal{A}_6$, $L_2(7)$, and M_{20}, the lattice M has rank 19; thus, its orthogonal complement in L will be a positive definite lattice of rank 3. Since multiple positive definite lattices may have the same lattice invariants, the embedding of M in L need not be unique; thus, we may obtain multiple moduli spaces $\mathcal{M}_T$ for these groups. Enumerating the embeddings of M in L for these groups, and determining whether each embedding corresponds to a symplectic group action, is an interesting question for further research.

Chapter 3

THE PICARD-FUCHS EQUATION OF A POLARIZED FAMILY

3.1 Picard–Fuchs Equations

A *period* is the integral of a differential form with respect to a specified homology class. For instance, evaluating a function on a specified point of a manifold gives us the period corresponding to a zero-form integrated over a class in the zeroth homology. The *Picard-Fuchs differential equation* of a family of varieties is a differential equation that describes the way the value of a period changes as we move through the family. One may use solutions to the Picard-Fuchs differential equation for a family of Calabi-Yau varieties to describe the *mirror map* from the family to the corresponding mirror family.

This chapter discusses joint work with Adrian Clingher, Charles Doran, and Jacob Lewis first presented in [CDLW07]. We begin by reviewing a technique known as the *Griffiths-Dwork Technique* for computing Picard-Fuchs equations. We then apply the technique to a particular family of polarized K3 surfaces possessed of a symplectic involution, illustrating the connections between the lattice structure, Picard-Fuchs equations, and geometric properties of our family.

3.2 The Griffiths–Dwork Technique

The *Griffiths-Dwork technique* provides an algorithm for computing Picard-Fuchs equations for families of hypersurfaces in projective space. The technique has been generalized to hypersurfaces in weighted projective space and in some toric varieties. Unlike other methods for computing Picard-Fuchs equations, the Griffiths-Dwork technique allows the study of arbitrary rational parametrizations.

We review the Griffiths-Dwork technique in Section 3.2.1. In Section 3.2.2, we apply the technique to families of elliptic curves in $\mathbb{P}^2$ in Weierstrass normal form.

3.2.1 Griffiths–Dwork and Residues

Let X be a hypersurface in $\mathbb{P}^n$ given by a homogeneous polynomial Q in coordinates $[x_0, \dots, x_n]$, and let $\iota : X \to \mathbb{P}^n$ be the inclusion map. Let $\mathcal{H}(X)$ be the de Rham cohomology of rational n-forms on $\mathbb{P}^n - X$. We may write any representative of $\mathcal{H}(X)$ as $P\Omega_0/Q^k$, where $\Omega_0 = \sum_{i=0}^n (-1)^i x^i \mathrm{d}x^0 \wedge \dots \widehat{\mathrm{d}x^i} \dots \wedge \mathrm{d}x^n$ is the usual holomorphic form on $\mathbb{P}^n$ and P is a homogeneous polynomial of degree $\deg P = k \deg Q - (n+1)$.

Let the Jacobian ideal $J(Q)$ be the ideal generated by the partial derivatives $\frac{\partial Q}{\partial x_i}$. If we have an element of $\mathcal{H}(X)$ of the form $\frac{K}{Q^{k+1}}\Omega_0$ where $K = \sum_i A_i \frac{\partial Q}{\partial x_i}$ is a member of the Jacobian ideal, then we may reduce the order of the pole:

$$\frac{\Omega_0}{Q^{k+1}} \sum_i A_i \frac{\partial Q}{\partial x_i} = \frac{1}{k} \frac{\Omega_0}{Q^k} \sum_i \frac{\partial A_i}{\partial x_i} + \text{exact terms} \qquad (3.1)$$

Let γ be a cycle in X, and let $T(\gamma)$ be a small tubular neighborhood of γ in $\mathbb{P}^n - X$. Then we may define the *residue map* $\mathrm{Res} : \mathcal{H}(X) \to H^{n-1}(X, \mathbb{C})$ by

$$\frac{1}{2\pi i} \int_{T(\gamma)} \frac{P\Omega_0}{Q^k} = \int_\gamma \mathrm{Res}(\frac{P\Omega_0}{Q^k}) \qquad (3.2)$$

Let H be the hyperplane class in $H^{n-1}(\mathbb{P}^n, \mathbb{C})$. We refer to the perpendicular complement of $\iota^*(H)$ in $H^{n-1}(X, \mathbb{C})$ as the *primitive cohomology* of X, and denote it by

$PH(X)$. The residue map is an isomorphism onto the primitive cohomology. [Gri69]

Now, consider a family of hypersurfaces $X_{t_1 \cdots t_j}$ given by polynomials $Q_{t_1 \cdots t_j}$, where $t_1, \ldots, t_j$ are independent parameters. We may define a corresponding family of cycles $\gamma(t_1, \cdots, t_j)$. For $(t_1, \ldots, t_j)$ in a sufficiently small neighborhood of a fixed parameter value $(t'_1, \ldots, t'_j)$, $T(\gamma(t_1, \ldots, t_j))$ is homologous to $T(\gamma(t'_1, \ldots, t'_j))$ in $H_n(\mathbb{P}^n - X, \mathbb{C})$. Thus, we may differentiate as follows:

$$\begin{aligned} \frac{\partial}{\partial t_i} \int_{T(\gamma(t_1,\ldots,t_j))} \frac{P\Omega_0}{Q(t)^k} &= \frac{\partial}{\partial t_i} \int_{T(\gamma(t'_1,\ldots,t'_j))} \frac{P\Omega_0}{Q(t)^k} \\ &= -k \int_{T(\gamma(t'_1,\ldots,t'_j))} \frac{P\Omega_0}{Q(t)^{k+1}} \frac{\partial Q}{\partial t_i} \end{aligned} \tag{3.3}$$

If $r = \dim_{\mathbb{C}}(H_{n-1}(X)) = \dim_{\mathbb{C}}(H^{n-1}(X, \mathbb{C}))$, only $r-1$ derivatives can be linearly independent. Therefore the periods must satisfy a linear differential equation with coefficients in $\mathbb{Q}(t_1, \ldots, t_j)$ of order at most r — this is a Picard–Fuchs differential equation. One may compute the Picard–Fuchs equation by systematically taking derivatives of $\int_{T(\gamma(t_1,\ldots,t_j))} \frac{P\Omega_0}{Q(t)^k}$ with respect to the various parameters and using 3.1 to rewrite the results in terms of a standard basis for $H^{n-1}(X, \mathbb{C})$. This method is known as the *Griffiths–Dwork technique.* (See [CK99] or [DGJ08] for a more detailed discussion.)

3.2.2 Griffiths–Dwork for the Weierstrass Form

Consider the hypersurface

$$Q = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3,$$

the Weierstrass form for a family of elliptic curves. We illustrate here the Griffiths–Dwork technique, first treating g_2 and g_3 as independent parameters. Equation 3.3 tells us that we may differentiate under the integral sign:

$$\frac{\partial}{\partial g_2}\int\frac{\Omega_0}{Q} = -\int\frac{xz^2\Omega_0}{Q^2} \tag{3.4}$$
$$\frac{\partial}{\partial g_3}\int\frac{\Omega_0}{Q} = -\int\frac{z^3\Omega_0}{Q^2}$$

A Groebner basis computation shows that xz^2 and z^3 are equivalent modulo the Jacobian ideal $J(Q)$. Using Equation 3.1 to reduce the pole order, we find that

$$\frac{\partial}{\partial g_2}\int\frac{\Omega_0}{Q} + \frac{3g_3}{2g_2}\frac{\partial}{\partial g_3}\int\frac{\Omega_0}{Q} = \frac{-1}{4g_2}\int\frac{\Omega_0}{Q} \tag{3.5}$$

Now, suppose g_2 and g_3 are both functions of a single parameter t. We compute:

$$\begin{aligned}\frac{d}{dt}\int\frac{\Omega_0}{Q} &= \left(\frac{\partial}{\partial g_2}\int\frac{\Omega_0}{Q}\right)\frac{\partial g_2}{\partial t} + \left(\frac{\partial}{\partial g_3}\int\frac{\Omega_0}{Q}\right)\frac{\partial g_3}{\partial t}\\ &= -g_2'(t)\int\frac{xz^2\Omega_0}{Q^2} - g_3'(t)\int\frac{z^3\Omega_0}{Q^2}\end{aligned} \tag{3.6}$$

$$\begin{aligned}\frac{d^2}{dt^2}\int\frac{\Omega_0}{Q} &= 2g_2'(t)\int\frac{xz^2(g_2'(t)xz^2+g_3'(t)z^3)}{Q^3}\Omega_0 - g_2''(t)\int\frac{xz^2}{Q^2}\Omega \qquad (3.7)\\ &+ 2g_3'(t)\int\frac{z^3(g_2'(t)xz^2+g_3'(t)z^3)}{Q^3}\Omega_0 - g_3''(t)\int\frac{z^3}{Q^2}\Omega\\ &= 2(g_2'(t))^2\int\frac{(xz^2)^2}{Q^3}\Omega_0 + 4g_2'(t)g_3'(t)\int\frac{(xz^2)(z^3)}{Q^3}\Omega_0 \qquad (3.8)\\ &+ 2(g_3'(t))^2\int\frac{(z^3)^2}{Q^3}\Omega_0 - g_2''(t)\int\frac{xz^2}{Q^2}\Omega_0 - g_3''(t)\int\frac{z^3}{Q^2}\Omega_0\end{aligned}$$

We may use Equation 3.1 together with a Groebner basis computation to rewrite $\frac{d^2}{dt^2}\int\frac{\Omega_0}{Q}$ as a sum of integrals of expressions with Q^2 in the denominator:

$$\begin{aligned}\frac{d^2}{dt^2}\int\frac{\Omega_0}{Q} &= 2(g_2'(t))^2\int\frac{\alpha_1 xz^2+\beta_1 z^3}{Q^2}\Omega_0 + 4g_2'(t)g_3'(t)\int\frac{\alpha_2 xz^2+\beta_2 z^3}{Q^2}\Omega_0\\ &+ 2(g_3'(t))^2\int\frac{\alpha_3 xz^2+\beta_3 z^3}{Q^2}\Omega_0 - g_2''(t)\int\frac{xz^2}{Q^2}\Omega_0 - g_3''(t)\int\frac{z^3}{Q^2}\Omega_0\end{aligned} \tag{3.9}$$

Here the α_j and β_j are rational functions in g_2 and g_3. Note that we have expressed $\frac{d^2}{dt^2}\int\frac{\Omega_0}{Q}$ entirely in terms of $\int\frac{xz^2}{Q^2}\Omega_0 = -\frac{\partial}{\partial g_2}\int\frac{\Omega_0}{Q}$ and $\int\frac{z^3}{Q^2}\Omega_0 = -\frac{\partial}{\partial g_3}\int\frac{\Omega_0}{Q}$. Since $\frac{d}{dt}\int\frac{\Omega_0}{Q}$ is also written in terms of $\int\frac{xz^2}{Q^2}\Omega_0$ and $\int\frac{z^3}{Q^2}\Omega_0$, we might hope to relate $\frac{d}{dt}\int\frac{\Omega_0}{Q}$ and $\frac{d^2}{dt^2}\int\frac{\Omega_0}{Q}$. If such a relationship is to exist for an arbitrary choice of $g_2(t)$ and $g_3(t)$, $\int\frac{xz^2}{Q^2}\Omega_0$ and $\int\frac{z^3}{Q^2}\Omega_0$ cannot be independent. In fact, they are not: $xz^2 \cong \frac{-3g_3}{2g_2}z^3 \bmod J(Q)$, so applying Equation 3.1 we find that

$$\int\frac{xz^2}{Q^2}\Omega_0 = \frac{-3g_3}{2g_2}\int\frac{z^3}{Q^2}\Omega_0 + \frac{1}{4g_2}\int\frac{\Omega_0}{Q}. \tag{3.10}$$

Combining Equations 3.6, 3.9, and 3.10, and setting $\Delta = g_2^3 - 27g_3^2$, we obtain the Picard-Fuchs differential equation for a one-parameter family of elliptic curves in Weierstrass form:

$$A_2\frac{d^2}{dt^2}\int\frac{\Omega_0}{Q} + A_1\frac{d}{dt}\int\frac{\Omega_0}{Q} + A_0\int\frac{\Omega_0}{Q} = 0 \tag{3.11}$$

where

$$A_2 = 16\Delta(3g_2'g_3 - 2g_2g_3') \tag{3.12}$$
$$A_1 = 16(9g_2^2g_3(g_2')^2 - (7g_2^3 + 135g_3^2)g_2'g_3' + 108g_2g_3(g_3')^2 + \Delta(-3g_3g_2'' + 2g_2g_3''))$$
$$A_0 = 21g_2g_3(g_2')^3 - 18g_2^2(g_2')^2g_3' + 8g_3'(15g_2(g_3')^2 - \Delta g_2'') - 4g_2'(27g_3(g_3')^2 - 2\Delta g_3'')$$

If we make the substitution $j = g_2^3/\Delta$, then Equation 3.11 reduces to the standard Picard-Fuchs equation for a one-parameter family of elliptic curves in Weierstrass form, described for example in [SH85]:

$$\frac{d^2}{dt^2}\int\frac{dx}{y} + B_1\frac{d}{dt}\int\frac{dx}{y} + B_0\int\frac{dx}{y} = 0 \tag{3.13}$$

where

$$B_1 = \frac{g_3'}{g_3} - \frac{g_2'}{g_2} + \frac{j'}{j} - \frac{j''}{j'} \tag{3.14}$$

$$B_0 = \frac{(j')^2}{144j(j-1)} + \frac{\Delta'}{12\Delta}\left(B_1 + \frac{\Delta''}{\Delta'} - \frac{13\Delta'}{12\Delta}\right)$$

3.3 A Polarized Family

Let $\mathrm{M} = H \oplus E_8 \oplus E_8$ be the unique unimodular lattice of signature $(1, 17)$.

Inose constructed a two-parameter family $X(a, b)$ of M-polarized K3 surfaces by taking minimal resolutions of the projective quartics in $\mathbb{P}^3$ described by the equation: [Ino78]

$$y^2zw - 4x^3z + 3axzw^2 - \frac{1}{2}(z^2w^2 + w^4) + bzw^3 = 0 \tag{3.15}$$

Here $a, b \in \mathbf{C}$.

Clingher and Doran classified all M-polarized K3 surfaces (X, i) in [CD07], using a *normal form* based on Inose's family.

Theorem 3.3.1. [CD07] Let (X, i) be an M-polarized K3 surface. Then, there exists a triple $(a, b, d) \in \mathbf{C}^3$, with $d \neq 0$, such that (X, i) is isomorphic to the minimal resolution of the quartic surface

$$\mathrm{Q}(a, b, d)\colon y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4) = 0. \tag{3.16}$$

Distinct quartics in Equation 3.16 may yield isomorphic polarized K3 surfaces.

Theorem 3.3.2. [CD07] Two quartics $\mathrm{Q}(a_1, b_1, d_1)$ and $\mathrm{Q}(a_2, b_2, d_2)$ determine isomorphic M-polarized K3 surfaces as their minimal resolutions if and only if:

$$(a_2, b_2, d_2) = (\lambda^2 a_1, \lambda^3 b_1, \lambda^6 d_1)$$

for some parameter $\lambda \in \mathbf{C}^*$.

Thus, we obtain a coarse moduli space for M-polarized K3 surfaces in the form of the open variety:

$$\mathcal{M}_{\mathrm{M}} = \{[a, b, d] \in \mathbf{WP}(2, 3, 6) : d \neq 0\} \tag{3.17}$$

We define the *fundamental $\mathcal{W}$-invariants* $(\mathcal{W}_1, \mathcal{W}_2)$ by:

$$\mathcal{W}_1 \;=\; \frac{a^3}{d}, \;\; \mathcal{W}_2 \;=\; \frac{b^2}{d}.$$

The polarized Hodge structure of an M-polarized K3 surface is identical to the polarized Hodge structure of an abelian surface A realized as the Cartesian product of two elliptic curves E_1 and E_2. This duality of Hodge structures is the consequence of a geometric relationship between X and A induced by a symplectic automorphism on X of order 2.

Theorem 3.3.3. [CD07] Let (X, i) be an M-polarized K3 surface. Then:

1. X possesses a canonical symplectic involution β.

2. The minimal resolution Y of X/β has a canonical structure as the Kummer surface of an abelian surface $\mathrm{A} = \mathrm{E}_1 \times \mathrm{E}_2$, where E_1 and E_2 are elliptic curves.

3. This construction induces a canonical Hodge isomorphism between the M-polarized Hodge structure of X and the natural H-polarized Hodge structure of the abelian surface A.

The M-polarized K3 surfaces are completely classified by two modular invariants σ and π in $\mathbf{C}$, much in the same way as elliptic curves over $\mathbf{C}$ are classified by the j-invariant. In the context of the duality map described in Theorem 3.3.3, the two invariants are the standard symmetric functions on the invariants of j_1 and j_2 of the dual elliptic curves E_1 and E_2 :

$$\sigma = j_1 + j_2 \tag{3.18}$$

$$\pi = j_1 \cdot j_2 \tag{3.19}$$

Theorem 3.3.4. [CD07] The $\mathcal{W}$-invariants of an M-polarized K3 surface (X, i) are linked to the periods of X by the formulas

$$\mathcal{W}_1 = \pi \tag{3.20}$$

$$\mathcal{W}_2 = \pi - \sigma + 1. \tag{3.21}$$

Since the rank-18 lattice M is unimodular, the only possible lattices M' satisfying $\mathrm{M} \subset \mathrm{M}'$ and M' of rank 19 are those for which $\mathrm{M}' = \mathrm{M}_n = \mathrm{M} \oplus \langle -2n \rangle = \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \langle -2n \rangle$ for some $n \in \mathbf{N}$. An M_1-polarized K3 surface corresponds to a pair of isomorphic elliptic curves. The M_n-polarized K3 surfaces for $n > 1$ correspond to products of elliptic curves with an n-isogeny between them; the extra algebraic cycle on each K3 surface corresponds to the graph of the n-isogeny on the product of the two elliptic curves. Determining the subloci of $\mathbf{WP}(2, 3, 6)$ on which these enhancements occur thus reduces to the problem of finding relations between the j-invariants of pairs of elliptic curves. This, of course, is a classical problem with a rich history. We give one possible solution in Theorem 3.4.1.

3.4 Picard-Fuchs Equations for the M*-polarized Family*

Let us set $a = 1$ in Equation 3.16 and consider the resulting polynomial $Q = y^2zw - 4x^3z + 3xzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4)$. (We have simply reduced to the affine patch $a \neq 0$ of the parameter space $\mathbf{WP}(2,3,6)$.) Applying the Griffiths–Dwork technique to $\int \frac{\Omega_0}{Q}$ yields a pair of second-order Picard–Fuchs equations:

$$\frac{\partial^2}{\partial b^2}\int\frac{\Omega_0}{Q} - 4(d\frac{\partial^2}{\partial d^2}\int\frac{\Omega_0}{Q} + \frac{\partial}{\partial d}\int\frac{\Omega_0}{Q}) = 0 \quad (3.22)$$

$$(-1+b^2+d)\frac{\partial^2}{\partial b^2}\int\frac{\Omega_0}{Q} + 2b\frac{\partial}{\partial b}\int\frac{\Omega_0}{Q} + 4bd\frac{\partial^2}{\partial bd}\int\frac{\Omega_0}{Q} \quad (3.23)$$
$$+2d\frac{\partial}{\partial d}\int\frac{\Omega_0}{Q} + \frac{5}{36}\int\frac{\Omega_0}{Q} = 0$$

We can use the relationship between b, d and the j-invariants of elliptic curves from Theorem 3.3.4 to write $b^2 = \frac{(j_1-1)(j_2-1)}{j_1j_2}$ and $d = \frac{1}{j_1j_2}$. Here j_1 and j_2 are the j-invariants of the two elliptic curves E_1 and E_2 whose product corresponds to $X(1,b,d)$. Let E_i have affine Weierstrass model

$$y^2 = 4x^3 - g_2^{(i)}x - g_3^{(i)}$$

for $i = 1, 2$. Then we can rewrite Equations 3.22 and 3.23 in terms of j_1 and j_2. The resulting system decouples: that is, no mixed partials appear. By taking appropriate linear combinations of the resulting equations, we reduce the system to

$$0 = \quad 72j_1\left((2j_1-1)F^{(1,0)}(j_1,j_2) + 2(j_1-1)j_1F^{(2,0)}(j_1,j_2)\right) - 5F(j_1,j_2)$$
$$0 = \quad 72j_2\left((2j_2-1)F^{(0,1)}(j_1,j_2) + 2(j_2-1)j_2F^{(0,2)}(j_1,j_2)\right) - 5F(j_1,j_2)$$

where $F^{(i,j)}(j_1,j_2) = \frac{\partial^{i+j}F}{\partial j_1^i \partial j_2^j}$.

To solve this system, one need merely solve each ODE separately, then take products of the solutions. Each of these ODEs is a Picard-Fuchs differential equation satisfied by the periods of the form $\omega^{(i)} = \left(g_2^{(i)}\right)^{1/4} \frac{dx}{y}$. Thus periods satisfying the Picard-Fuchs system arising via Griffiths-Dwork are simply products of periods of $\omega^{(1)}$ and $\omega^{(2)}$ (c.f. [LY96, Theorem 1.1]).

Now, consider a one-parameter family $\mathcal{F}$ of M-polarized K3 surfaces obtained by treating b and d as functions of a single parameter t. We may use the Griffiths–Dwork technique to analyze this family, just as we computed the Picard–Fuchs equation for a one-parameter family of elliptic curves in Section 3.2.2. The result is generically a fourth-order ODE, which we do not reproduce in full here. The Picard–Fuchs equation for $\mathcal{F}$ will reduce to a third-order ODE precisely when $\mathcal{F}$ is an M_n-polarized family.

Let $j_1(t)$, $j_2(t)$ be two functions of a complex variable t such that $j_1(t) + j_2(t)$ and $j_1(t)j_2(t)$ are rational functions of t. In this case $b^2(t) = \frac{(j_1(t)-1)(j_2(t)-1)}{j_1(t)j_2(t)}$ and $d(t) = \frac{1}{j_1(t)j_2(t)}$ are also rational functions of t, and we may write the Picard–Fuchs equation for $\mathcal{F}$ in terms of $j_1(t)$ and $j_2(t)$. The coefficient $r_4(t)$ of $\frac{\mathrm{d}^4}{\mathrm{d}t^4} \int \frac{\Omega_0}{Q}$ in the Picard–Fuchs ODE then becomes

$$144((j_1(t)-1)(j_2(t)-1))^3(j_1(t)j_2(t))^4(j_1(t)-j_2(t))^7(j_1'(t)j_2'(t))^2\left(\Box(j_2(t)) - \Box(j_1(t))\right)$$

where

$$\Box(j(t)) = j'(t)^2\frac{36j(t)^2 - 41j(t) + 32}{144(j(t)-1)^2 j(t)^2} + \frac{1}{2}\{j(t), t\}$$

and

$$\{j(t), t\} = \frac{2j'(t)j'''(t) - 3j''(t)^2}{2j'(t)^2}$$

is the Schwarzian derivative.

If $j_1(t)$ and $j_2(t)$ are both nonconstant, then $r_4(t)$ will vanish if and only if either $j_1(t) = j_2(t)$ (in which case the family of K3 surfaces is M_1-polarized) or $\Box(j_1(t)) = \Box(j_2(t))$. This observation motivates the following theorem.

Theorem 3.4.1. [CDLW07] A one-parameter family $\mathcal{F}$ of M-polarized K3 surfaces generically has Picard-Fuchs equation of rank 4. The following are equivalent:

- The Picard-Fuchs equation of $\mathcal{F}$ drops to rank 3
- $\mathcal{F}$ is polarized by the enhanced lattice $\mathrm{M}_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$
- The corresponding pairs of elliptic curves $\mathrm{E}_1(t)$ and $\mathrm{E}_2(t)$ are n-isogenous
- The j-invariants of $\mathrm{E}_1(t)$ and $\mathrm{E}_2(t)$ satisfy $\square(j_1(t)) = \square(j_2(t))$

Proof. The Picard-Fuchs ODE for $\mathcal{F}$, suitably normalized, is the tensor product of the Picard-Fuchs ODEs of the two pencils of elliptic curves over $\mathbf{P}^1_t$ with functional invariants $j_1(t), j_2(t)$ respectively. If these second-order ODEs $L_1 = 0, L_2 = 0$ are in projective normal form

$$L_1 = \frac{d^2 f}{dt^2} + p_2(t) f$$
$$L_2 = \frac{d^2 g}{dt^2} + q_2(t) g$$

then $p_2(t) = \square(j_1(t))$ and $q_2(t) = \square(j_2(t))$. Their tensor product is

$$0 = H^{(4)}(t) + \frac{q_2'(t) - p_2'(t)}{p_2(t) - q_2(t)} H'''(t) + 2(p_2(t) + q_2(t)) H''(t) +$$
$$\frac{p_2(t)\left(p_2'(t) + 5q_2'(t)\right) - q_2(t)\left(5p_2'(t) + q_2'(t)\right)}{p_2(t) - q_2(t)} H'(t) +$$
$$\left((p_2(t) - q_2(t))^2 + p_2''(t) + q_2''(t) + \frac{q_2'(t)^2 - p_2'(t)^2}{p_2(t) - q_2(t)} \right) H(t)$$

According to [Fan00], this fourth-order equation factors as a third-order equation times a first-order equation if and only if $p_2(t) = q_2(t)$, that is, if and only if $\square(j_1(t)) = \square(j_2(t))$. On the other hand, the Picard-Fuchs equation of $\mathcal{F}$ has third order if and only if $\mathcal{F}$ is M_n-polarized, and this occurs if and only if the two pencils of elliptic curves are fiberwise n-isogenous. $\square$

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