Spin operators and matrices

Spin operators

Spin vectors are usually represented in terms of their Hermitian cartesian component operators

$$\hat{\boldsymbol{S}} = \begin{pmatrix} \hat{S}_\mathrm{x} \\ \hat{S}_\mathrm{y} \\ \hat{S}_\mathrm{x} \end{pmatrix}$$

Sometimes, the non-Hermitian ladder operators

$$\hat{S}_+ = \hat{S}_\mathrm{x} + \mathrm{i}\hat{S}_\mathrm{y} \qquad \hat{S}_- = \hat{S}_\mathrm{x} - \mathrm{i}\hat{S}_\mathrm{y}$$

are used. The cartesian operators are then given by

$$\hat{S}_\mathrm{x} = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right) \qquad \hat{S}_\mathrm{y} = \frac{1}{2\mathrm{i}}\left(\hat{S}_+ - \hat{S}_-\right)$$

Some common commutators are

$$\left[\hat{S}_\mathrm{x}, \hat{S}_\mathrm{y}\right] = \mathrm{i}\hat{S}_\mathrm{z} \qquad \left[\hat{S}_\mathrm{y}, \hat{S}_\mathrm{z}\right] = \mathrm{i}\hat{S}_\mathrm{x} \qquad \left[\hat{S}_\mathrm{z}, \hat{S}_\mathrm{x}\right] = \mathrm{i}\hat{S}_\mathrm{y}$$

and

$$\left[\hat{S}_\mathrm{z}, \hat{S}_+\right] = \hat{S}_+ \qquad \left[\hat{S}_\mathrm{z}, \hat{S}_-\right] = -\hat{S}_- \qquad \left[\hat{S}_+, \hat{S}_-\right] = 2\hat{S}_\mathrm{z}$$

Spin matrices - General

For a spin S the cartesian and ladder operators are square matrices of dimension 2S+1. They are always represented in the Zeeman basis with states $|S,m\rangle$ (m=-S,...,S), in short $|m\rangle$, that satisfy

$$\begin{aligned}
\langle m'|\hat{S}_\mathrm{x}|m\rangle &= (\delta_{m',m+1} + \delta_{m'+1,m})\frac{1}{2}\sqrt{S(S+1) - m'm} \\
\langle m'|\hat{S}_\mathrm{y}|m\rangle &= (\delta_{m',m+1} - \delta_{m'+1,m})\frac{1}{2\mathrm{i}}\sqrt{S(S+1) - m'm} \\
\langle m'|\hat{S}_\mathrm{z}|m\rangle &= \delta_{m',m}m \\
\langle m'|\hat{S}_+|m\rangle &= \delta_{m',m+1}\sqrt{S(S+1) - m'm} \\
\langle m'|\hat{S}_-|m\rangle &= \delta_{m'+1,m}\sqrt{S(S+1) - m'm} \\
\langle m'|\hat{\boldsymbol{S}}^2|m\rangle &= \delta_{m',m}S(S+1)
\end{aligned}$$

Spin matrices - Explicit matrices

For S=1/2

$$S_\mathrm{x} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_\mathrm{y} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad S_\mathrm{z} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The $|+1/2\rangle$ state is commonly denoted as α, the $|-1/2\rangle$ state as β.

For S=1

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad S_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For S=3/2

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \qquad S_y = \frac{1}{2i} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

$$S_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

For S=2

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2i} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$$S_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

For S=5/2

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & \sqrt{8} & 0 & 0 & 0 \\ 0 & \sqrt{8} & 0 & \sqrt{9} & 0 & 0 \\ 0 & 0 & \sqrt{9} & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & \sqrt{8} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix} \qquad S_y = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ -\sqrt{5} & 0 & \sqrt{8} & 0 & 0 & 0 \\ 0 & -\sqrt{8} & 0 & \sqrt{9} & 0 & 0 \\ 0 & 0 & -\sqrt{9} & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & -\sqrt{8} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & -\sqrt{5} & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 5/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5/2 \end{pmatrix}$$

$$S_+ = \begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix}$$
