Name(s): ____________________

SUPERSYMMETRY PROBLEMS: MONDAY

(1) **Symmetries.**

(a) Construct an orthonormal basis of 3-dimensional space, using for two of your vectors: $\vec{v} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$ and $\vec{w} = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$. Note: you will need to find your third vector. Check that it's an orthonormal basis.
Now describe a change of basis taking (x, y, z) in the standard basis to (x', y', z') in your new basis. Verify that the matrix that describes this satisfies the condition $A^T A = I$.

(b) Consider the set of symmetries on (x, y, z), for each t:

$$\begin{aligned} x' &= x + 2ty + 3t^2 z \\ y' &= y + 3tz \\ z' &= z \end{aligned}$$

Show that these symmetries form a group under composition (that is, doing one such symmetry using t followed by another such symmetry using s).

(c) For the symmetry in (b), find the infinitesimal description.

(d) For $O(n)$ prove that the condition that $A^T A = I$ turns into the condition that $A^T = -A$ for the Lie algebra.

(e) For $O(3)$ use the generators of the Lie algebra

$$J_X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} J_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that

$$[J_x, J_y] = J_z$$
$$[J_y, J_z] = J_x$$
$$[J_z, J_x] = J_y$$

(f) Derive Equation (21) in Jim Gates' notes: `gates-Lect1.pdf` .

(g) Suppose we have the complex matrices

$$s_x = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad s_y = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad s_z = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and show that these satisfy the same commutation relations as for J_x, J_y, and J_z in $O(3)$. Characterize the matrices spanned by s_x, s_y, and s_z. Note that this means that at least infinitesimally, $O(3)$ can be a symmetry of 2-dimensional complex ordered pairs (w, z) using these relations. These (w, z) are called *spinors*.

(2) **Electromagnetism.**

Let $\vec{E} = (E^1, E^2, E^3)$ be the electric field and $\vec{B} = (B^1, B^2, B^3)$ be the magnetic field. We will work in Lorentz-Heaviside units, where the speed of light c, the magnetic constant μ_0, and the electric constant ϵ_0 are all 1.

Let ϕ be the *electromagnetic scalar potential* and let $\vec{A} = (A^1, A^2, A^3)$ be the *electromagnetic vector potential*. Recall that $\vec{E}$ and $\vec{B}$ are given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$$
$$\vec{B} = \nabla \times \vec{A},$$

where ∇ is the operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

This problem will use the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ for raising and lowering of indices.

(a) Define the *current 4-vector* J^μ by (ρ, J^1, J^2, J^3), and define the *electromagnetic four-potential* by $A_\mu = (\phi, \vec{A})$. Expand $A_\mu J^\mu$. What is the *current one-form* J_μ?

(b) Define the *field strength tensor* $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Write $F_{\mu\nu}$ as a 4×4 matrix in terms of the components of $\vec{E}$ and $\vec{B}$.

(c) Two of Maxwell's equations for the electromagnetic field are:

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

Show that these equations are equivalent to

$$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0.$$

(d) Write $F_{\mu\nu}F^{\mu\nu}$ in terms of E and B, where $E = ||\vec{E}||$ and $B = ||\vec{B}||$.

(e) Let $\vec{J} = (J^1, J^2, J^3)$. The rest of Maxwell's equations for the electromagnetic field are:

$$\nabla \cdot \vec{E} = \rho$$
$$\frac{\partial}{\partial t}\vec{E} - \nabla \times \vec{B} = -\vec{J}$$

Show that these are equivalent to:

$$\partial_\nu F^{\mu\nu} = J^\mu.$$

(3) **Topologies and Chromotopologies.**

In *Adinkras for mathematicians*, Y. Zhang writes:

An n-dimensional adinkra *topology*, or topology for short, is a finite connected simple graph A such that A is bipartite and n-regular (every vertex has exactly n incident edges). We call the two sets in the bipartition of $V(A)$ bosons and fermions, though the actual choice is mostly arbitrary and we do not consider it part of the data. A *chromotopology* of dimension n is a topology A such that the following holds.

- The elements of $E(A)$ are colored by n colors ... such that every vertex is incident to exactly one edge of each color.

- For any distinct i and j, the edges in $E(A)$ with colors i and j form a disjoint union of 4-cycles.

(a) Draw two different chromotopologies.

(b) Construct one of your chromotopologies as a directed graph in Sage.
You may wish to consult the documentation at:
`http://www.sagemath.org/doc/prep/Quickstarts/Graphs-and-Discrete.html`
or
`http://www.sagemath.org/doc/reference/graphs/sage/graphs/digraph.html`

(c) Can you give an example of an adinkra topology that does not admit a chromotopology?

(d) When defining chromotopologies in her senior thesis at Bard College, S. Naples wrote:

> Finally, every pair of edge colors $\{a, b\}$ incident to a single vertex is part of a 4-cycle of alternating edge colors, such that the edge colors alternate $abab$.

Does Naples' condition follow from Zhang's definition of a chromotopology? Why or why not?

(e) How many chromotopologies can you construct on at most 6 vertices, and with $n \leq 4$?

(f) What do you think it should mean for two chromotopologies to be isomorphic?