

2. Symmetry: A Review of 3D Rotations

We begin our introduction by considering not supersymmetry but instead by considering a more familiar symmetry, i.e. rotational symmetry in three dimensions. To this end, we may introduce a ‘generator’ denoted by \mathbf{L}_3 . This is an operator that is define by its actions on the coordinates (x, y, z)

$$\mathbf{L}_3 x = -i y \quad , \quad \mathbf{L}_3 y = i x \quad , \quad \mathbf{L}_3 z = 0 \quad . \quad (1)$$

and by the fact that it is a derivation. This means that when the operator (1) acts on a product of coordinates, it obeys a rule similar to a derivative,

i.e.

$$\begin{aligned}\mathbf{L}_3[xy] &= [\mathbf{L}_3(x)]y + x[\mathbf{L}_3(y)] \\ &= -i[y^2 - x^2] .\end{aligned}\tag{2}$$

Now we wish to prove that this operator generates a rotation about the third direction. Let γ be an angle, we may consider the object defined by

$$\mathcal{R}_3(\gamma) \equiv \exp[-i\gamma\mathbf{L}_3] ,\tag{3}$$

and evaluate its effect on (x, y, z) . A rotated set of coordinates (x', y', z') may be obtained by application of $\mathcal{R}_3(\gamma)$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathcal{R}_3(\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathcal{R}_3(\gamma)x \\ \mathcal{R}_3(\gamma)y \\ \mathcal{R}_3(\gamma)z \end{pmatrix} .\tag{4}$$

We can evaluate each of these using the definitions above

$$\begin{aligned}\mathcal{R}_3(\gamma)x &= \sum_{n=1}^{\infty} \frac{1}{n!} (-i\gamma\mathbf{L}_3)^n x = \left[1 - i\gamma\mathbf{L}_3 + \frac{1}{2}(-i\gamma\mathbf{L}_3)^2 + \dots \right] x \\ &= \left[x - \gamma y - \frac{1}{2}(\gamma)^2 x + \frac{1}{3!}(\gamma)^3 y + \dots \right] \\ &= \left[x \cos\gamma - y \sin\gamma \right] ,\end{aligned}\tag{5}$$

$$\begin{aligned}
\mathcal{R}_3(\gamma) y &= \sum_{n=1}^{\infty} \frac{1}{n!} (-i\gamma \mathbf{L}_3)^n y = \left[1 - i\gamma \mathbf{L}_3 + \frac{1}{2} (-i\gamma \mathbf{L}_3)^2 + \dots \right] y \\
&= \left[y + \gamma x - \frac{1}{2}(\gamma)^2 y - \frac{1}{3!}(\gamma)^3 x + \dots \right] \\
&= \left[x \sin \gamma + y \cos \gamma \right] ,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathcal{R}_3(\gamma) z &= \sum_{n=1}^{\infty} \frac{1}{n!} (-i\gamma \mathbf{L}_3)^n z = \left[1 - i\gamma \mathbf{L}_3 + \frac{1}{2} (-i\gamma \mathbf{L}_3)^2 + \dots \right] z \\
&= \left[z + 0 + \dots \right] = z .
\end{aligned} \tag{7}$$

So that

$$\mathcal{R}_3(\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \cos \gamma - y \sin \gamma \\ x \sin \gamma + y \cos \gamma \\ z \end{pmatrix} . \tag{8}$$

For small values of γ , (4) implies

$$\begin{aligned}\Delta_{\mathcal{R}_3}(\gamma)x &= x' - x = -\gamma y \quad , \\ \Delta_{\mathcal{R}_3}(\gamma)y &= y' - y = \gamma x \quad , \\ \Delta_{\mathcal{R}_3}(\gamma)z &= z' - z = 0 \quad .\end{aligned}\tag{9}$$

and the notation is to indicate a change Δ due to the rotation \mathcal{R}_3 thru the angle γ . In the limit where γ goes to zero we have $\Delta_{\mathcal{R}_3} = \delta_{\mathcal{R}_3}$ and these can be re-written as

$$\delta_{\mathcal{R}_3}(\gamma)x = -\gamma y \quad , \quad \delta_{\mathcal{R}_3}(\gamma)y = \gamma x \quad , \quad \delta_{\mathcal{R}_3}(\gamma)z = 0 \quad ,\tag{10}$$

and a direct calculation reveals that acting on (x, y, z)

$$\delta_{\mathcal{R}_3}(\gamma) = -i\gamma\mathbf{L}_3 \quad .\tag{11}$$

It is clearly possible to define two other similar operators

$$\begin{aligned}\mathbf{L}_1x &= 0 \quad , \quad \mathbf{L}_1y = iz \quad , \quad \mathbf{L}_1z = -iy \quad , \\ \mathbf{L}_2x &= -iz \quad , \quad \mathbf{L}_2y = 0 \quad , \quad \mathbf{L}_2z = ix \quad .\end{aligned}\tag{12}$$

Given the definitions of the operators in (1) and (12) there are interesting calculations that can be made.

$$\begin{pmatrix} \mathbf{L}_1 \mathbf{L}_2 x \\ \mathbf{L}_1 \mathbf{L}_2 y \\ \mathbf{L}_1 \mathbf{L}_2 z \end{pmatrix} = \begin{pmatrix} -i \mathbf{L}_1 z \\ \mathbf{L}_1 0 \\ i \mathbf{L}_1 x \end{pmatrix} = \begin{pmatrix} -y \\ 0 \\ 0 \end{pmatrix} , \quad (13)$$

$$\begin{pmatrix} \mathbf{L}_2 \mathbf{L}_1 x \\ \mathbf{L}_2 \mathbf{L}_1 y \\ \mathbf{L}_2 \mathbf{L}_1 z \end{pmatrix} = \begin{pmatrix} \mathbf{L}_2 0 \\ i \mathbf{L}_2 z \\ -i \mathbf{L}_2 y \end{pmatrix} = \begin{pmatrix} 0 \\ -x \\ 0 \end{pmatrix} , \quad (14)$$

$$\begin{pmatrix} [\mathbf{L}_1 , \mathbf{L}_2] x \\ [\mathbf{L}_1 , \mathbf{L}_2] y \\ [\mathbf{L}_1 , \mathbf{L}_2] z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} . \quad (15)$$

In writing this, we have introduced the standard notation of the commutator

$$[\mathcal{A} , \mathcal{B}] \equiv \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} , \quad (16)$$

for any two quantities \mathcal{A} and \mathcal{B} .

From equation (1) it follows that

$$\begin{pmatrix} \mathbf{L}_3 x \\ \mathbf{L}_3 y \\ \mathbf{L}_3 z \end{pmatrix} = \begin{pmatrix} i y \\ -i x \\ 0 \end{pmatrix} , \quad (17)$$

and on comparing (15) and (17) it is apparent that

$$\begin{pmatrix} [\mathbf{L}_1 , \mathbf{L}_2] x \\ [\mathbf{L}_1 , \mathbf{L}_2] y \\ [\mathbf{L}_1 , \mathbf{L}_2] z \end{pmatrix} = \begin{pmatrix} i \mathbf{L}_3 x \\ i \mathbf{L}_3 y \\ i \mathbf{L}_3 z \end{pmatrix} . \quad (18)$$

Since each component of this has the same general form we write

$$[\mathbf{L}_1 , \mathbf{L}_2] = i \mathbf{L}_3 . \quad (19)$$

If the calculation above is repeated in all possible ways it leads to the familiar result

$$[\mathbf{L}_i , \mathbf{L}_j] = i \epsilon_{i j k} \mathbf{L}_k . \quad (20)$$

For the 3D rotation generators, it can be shown that

$$[\mathbf{L}_i , [\mathbf{L}_j , \mathbf{L}_k]] + [\mathbf{L}_j , [\mathbf{L}_k , \mathbf{L}_i]] + [\mathbf{L}_k , [\mathbf{L}_i , \mathbf{L}_j]] = 0 . \quad (21)$$

The operators \mathbf{L}_1 and \mathbf{L}_2 lead to $\mathcal{R}_1(\alpha)$, $\mathcal{R}_2(\beta)$, defined by

$$\mathcal{R}_1(\alpha) \equiv \exp[-i\alpha \mathbf{L}_1] , \quad \mathcal{R}_2(\beta) \equiv \exp[-i\beta \mathbf{L}_2] . \quad (22)$$

and the most general rotation in 3 dimensions is defined by

$$\mathcal{R}(\alpha, \beta, \gamma) = \mathcal{R}_1(\alpha) \mathcal{R}_2(\beta) \mathcal{R}_3(\gamma) \quad . \quad (23)$$

An alternate but equivalent way to write the general rotation is in the form

$$\tilde{\mathcal{R}}(\alpha, \beta, \gamma) = \exp\left[-i(\alpha \mathbf{L}_1 + \beta \mathbf{L}_2 + \gamma \mathbf{L}_3)\right] = \exp\left[-i\alpha_i \mathbf{L}_i\right] \quad . \quad (24)$$

2.1. *Rotational Symmetry & ‘The Noether Method’*

Any quantity $\mathcal{L}(x, y, z)$ that is ‘invariant’ under a rotation satisfies

$$\begin{aligned} \mathcal{L}(x', y', z') &= \mathcal{L}(x, y, z) \quad , \\ \mathcal{R}(\alpha, \beta, \gamma) \mathcal{L}(x, y, z) &= \mathcal{L}(x, y, z) \quad , \end{aligned} \quad (25)$$

and if the angles are infinitesimals the left hand side can be written as

$$\left[1 + i(\alpha \mathbf{L}_1 + \beta \mathbf{L}_2 + \gamma \mathbf{L}_3)\right] \mathcal{L}(x, y, z) = \mathcal{L}(x, y, z) \quad , \quad (26)$$

or equivalently this can be written as

$$\delta_{\mathcal{R}}(\alpha^i) \mathcal{L}(x, y, z) = i \left(\alpha \mathbf{L}_1 + \beta \mathbf{L}_2 + \gamma \mathbf{L}_3 \right) \mathcal{L}(x, y, z) = 0 \quad , \quad (27)$$

where

$$\delta_{\mathcal{R}}(\alpha^i) = i \alpha_i \mathbf{L}_i \quad . \quad (28)$$

Since the angles are independent, this is actually equivalent to three independent conditions

$$\mathbf{L}_1 \mathcal{L}(x, y, z) = 0 \quad , \quad \mathbf{L}_2 \mathcal{L}(x, y, z) = 0 \quad , \quad \mathbf{L}_3 \mathcal{L}(x, y, z) = 0 \quad , \quad (29)$$

which can be more simply written as $\mathbf{L}_i \mathcal{L} = 0$. Any quantity that satisfies (29) is said possess rotational symmetry. In a similar manner any quantity that satisfies only

$$\mathbf{L}_3 \mathcal{L}(x, y, z) = 0 \quad , \quad (30)$$

possess rotational symmetry about the z -axis (or third direction).

Symmetries are very useful properties. For example, imagine there is some system that possesses a symmetry about the z -axis, and has an energy \mathcal{E}_t that is known to depend on both x and y . Further imagine a ‘standard measurement’ of this quantity is only be made when $y = 0$ and yields the x -dependence

$$\mathcal{E}_t(x, y = 0) = A_0 x^4 \equiv \mathcal{E}_{SM}(x) \quad , \quad (31)$$

Since the total energy \mathcal{E}_t is a function of both both x and y it can be written as

$$\mathcal{E}_t(x, y) = \mathcal{E}_{SM}(x) + \mathcal{E}_{smSM}(x, y) \quad , \quad (32)$$

where the function \mathcal{E}_{smSM} ('symmetry-modified standard measurement') must satisfy $\mathcal{E}_{smSM}(x, y = 0) = 0$.

Since \mathcal{E}_t possesses a symmetry with respect to \mathbf{L}_3 it must be the case that $\mathbf{L}_3 \mathcal{E}_t = 0$. So that

$$\begin{aligned} 0 &= \mathbf{L}_3 \left[\mathcal{E}_{SM}(x) + \mathcal{E}_{smSM}(x, y) \right] \\ &= \mathbf{L}_3 \left[\mathcal{E}_{SM}(x) \right] + \mathbf{L}_3 \left[\mathcal{E}_{smSM}(x, y) \right] \\ &= i 4 A_0 x^3 y + \mathbf{L}_3 \left[\mathcal{E}_{smSM}(x, y) \right] \quad . \end{aligned} \quad (33)$$

To find the explicit form of \mathcal{E}_{smSM} an expansion in terms of powers of y can be utilized.

$$\mathcal{E}_{smSM}(x, y) = \sum_{n=1}^{\infty} y^n f_n(x) \quad . \quad (34)$$

When this expansion is substituted into (32.) it leads to

$$\begin{aligned}
0 &= 4 A_0 x^3 y + \sum_{n=1}^{\infty} \left[y^{n+1} \frac{d f_n}{d x} - n y^{n-1} x f_n(x) \right] , \\
0 &= [4 A_0 x^3 - 2 x f_2(x)] y - x f_1(x) \\
&\quad + \sum_{n=1}^{\infty} y^{n+1} \left[\frac{d f_n}{d x} - (n+2) x f_{n+2}(x) \right] ,
\end{aligned} \tag{35}$$

i.e. a series of equations for the unknown coefficient functions $f_n(x)$. Upon separation into various powers of y this system yields

$$\begin{aligned}
y^0 : \quad x f_1(x) &= 0 \rightarrow f_1(x) = 0 , \\
y^1 : \quad 4 A_0 x^3 - 2 x f_2(x) &= 0 \rightarrow f_2(x) = 2 A_0 x^2 , \\
y^{n+1} \ (n \geq 1) : \quad f_{n+2}(x) &= \left(\frac{1}{n+2} \right) \left(\frac{1}{x} \right) \frac{d f_n}{d x} .
\end{aligned} \tag{36}$$

The complete solution to this set of equations is given by

$$f_2(x) = 2 A_0 x^2 , \quad f_4(x) = A_0 , \tag{37}$$

where all other coefficients function vanish. In supersymmetrical theories, the steps discussed above are often called “the Noether Method” and are commonly used especially in the case of supergravity theories¹⁸ .