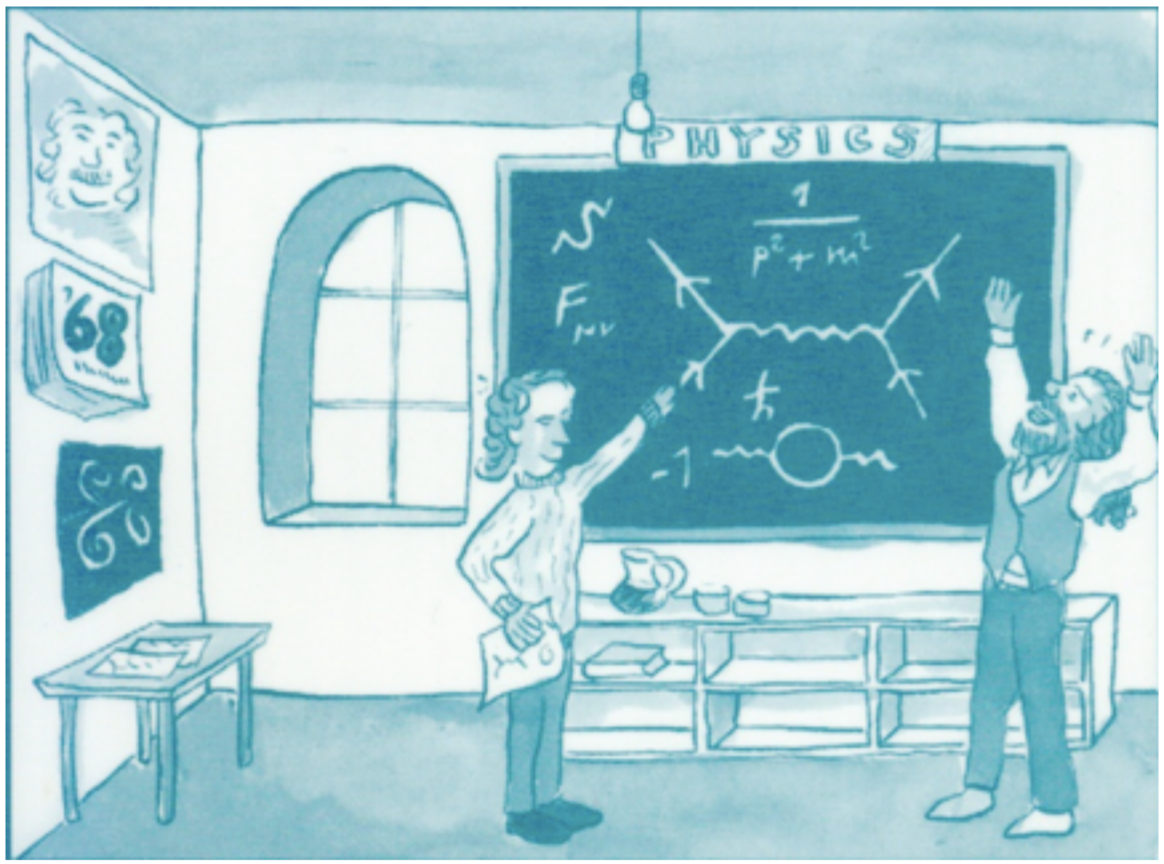
PIMS Undergraduate Workshop on Supersymmetry

Geometrization of Adinkras

Charles F. Doran
University of Alberta


PHYSICS
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$F_{\mu\nu}$
$\frac{1}{p^2+m^2}$
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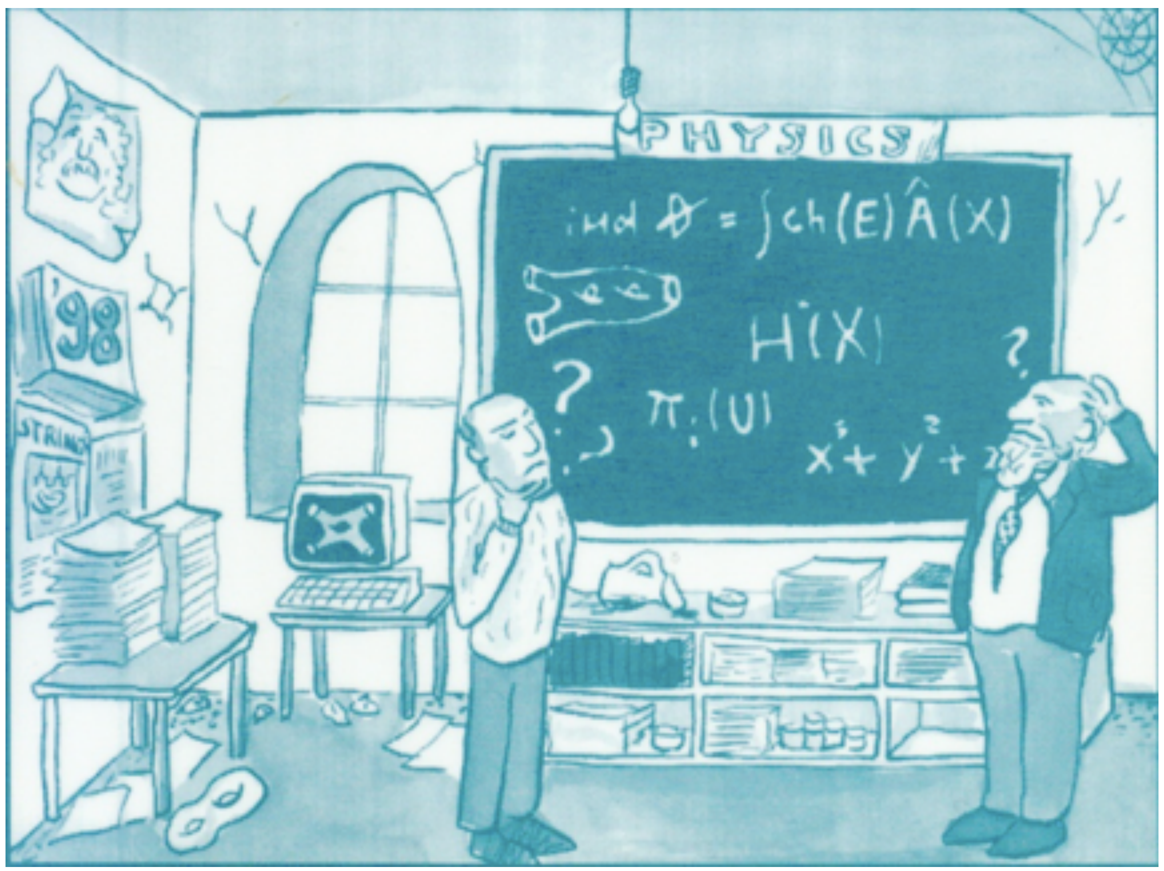
MATHEMATICS
68
$\mathrm{ind}\,\mathcal{D} = \int ch(E)\,\hat{A}(X)$
$H(X)$
$\pi_1(U)$



PHYSICS
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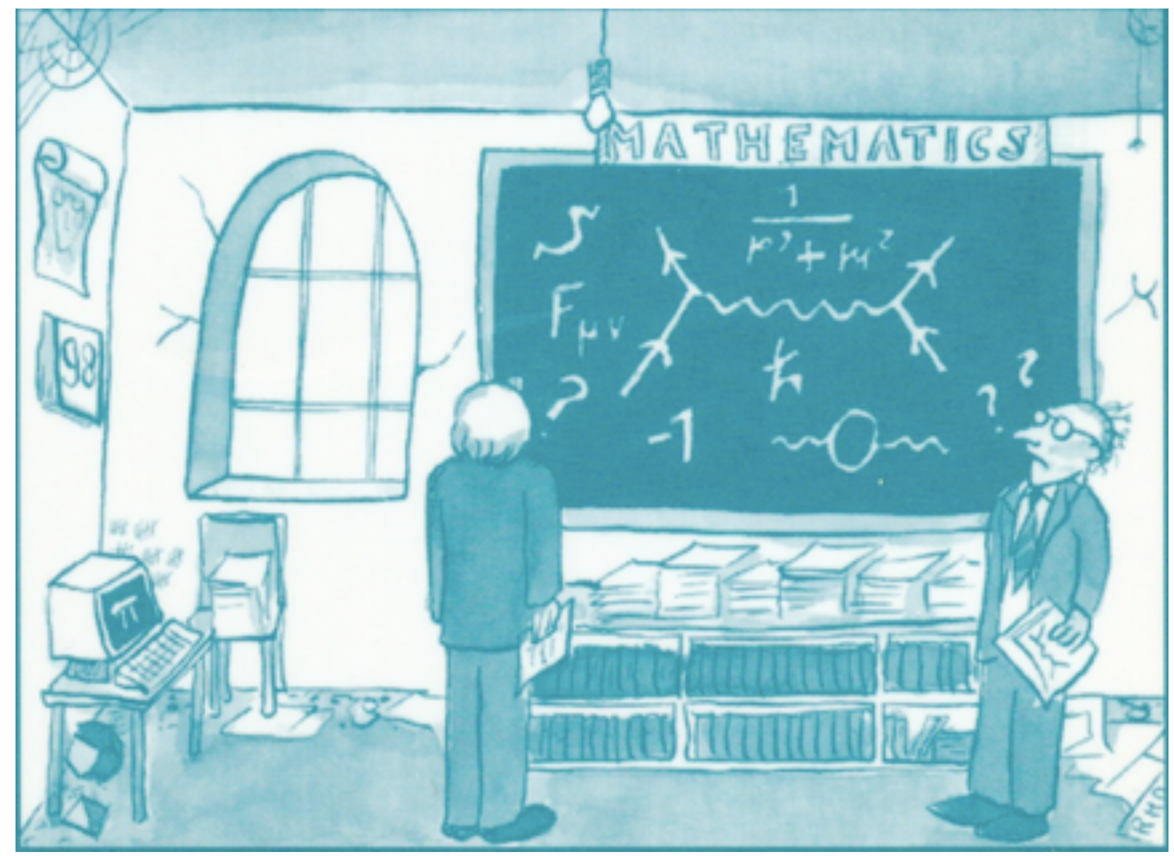

MATHEMATICS
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-1

The $(1|N)$ Superalgebra

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1-dimensional Minkowski space

time-like direction τ

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N real supersymmetry generators Q_I

supersymmetry generators commute with ∂_τ

$$\{Q_I, Q_J\} = 2i\delta_{IJ}\partial_\tau$$

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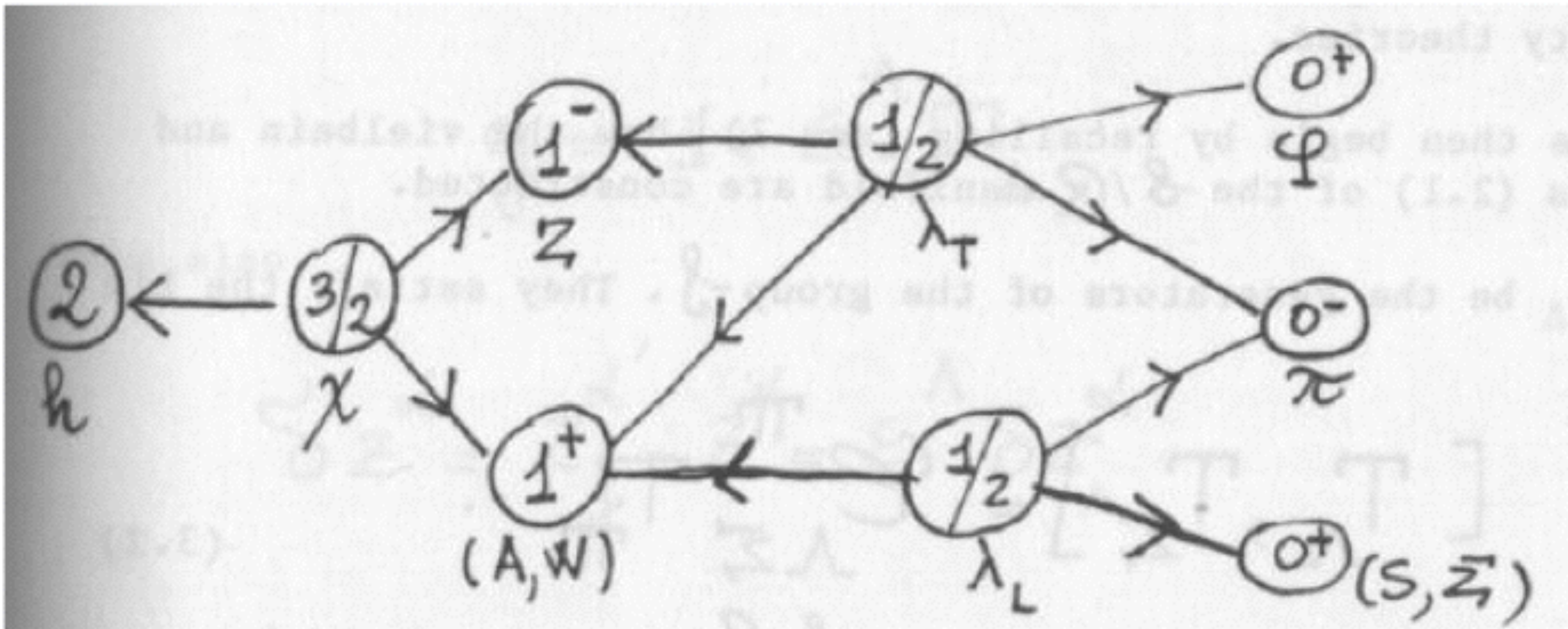
Real, finite-dimensional, linear representations of the $(1|N)$ superalgebra

spanned by a basis of real bosonic component fields $\phi_1(\tau), \ldots, \phi_m(\tau)$
fermionic component fields $\psi_1(\tau), \ldots, \psi_l(\tau)$



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Z
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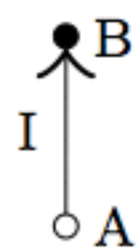
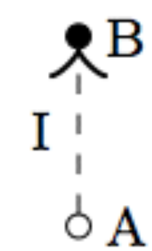
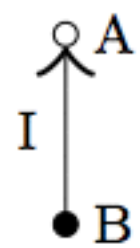
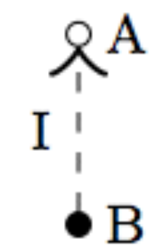
0⁺
φ
0⁻
π
0⁺
(S,Σ)

Adinkras

Action of Q_I	Adinkra	Action of Q_I	Adinkra
$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$		$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$	
$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} i\dot{\psi}_B \\ \phi_A \end{bmatrix}$		$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -i\dot{\psi}_B \\ -\phi_A \end{bmatrix}$	

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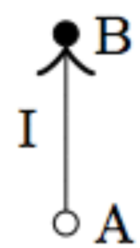
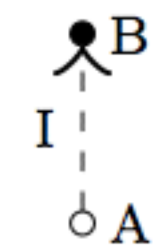
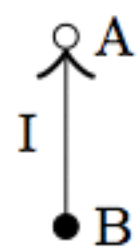
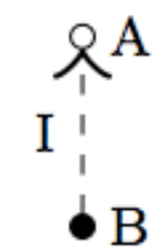
$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$		$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$	
$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} i\dot{\psi}_B \\ \phi_A \end{bmatrix}$		$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -i\dot{\psi}_B \\ -\phi_A \end{bmatrix}$	

Bipartite graph

Edges both *oriented* and *dashed*

Adinkra Axiomatics

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Let A be a finite connected simple graph such that

1. The graph A is N-regular and bipartite

$$V(A) = \text{bosons (white)} + \text{fermions (black)}$$

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III. For any pair of distinct colors, the edges of these colors form a disjoint union of 2-color 4-cycles

The graph A satisfying I, II, and III is called a chromotopology

Standard example: The Hamming Cube

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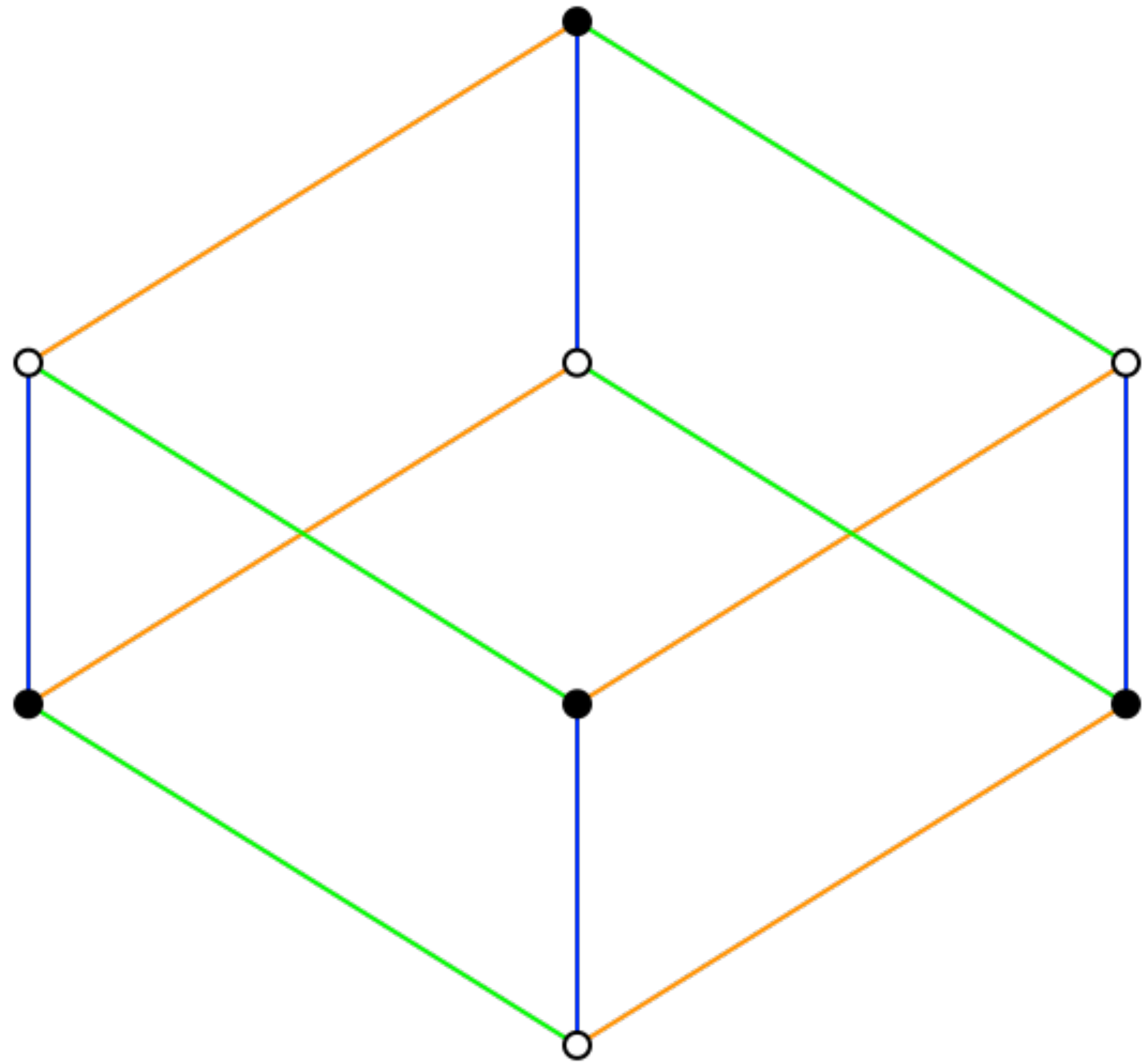

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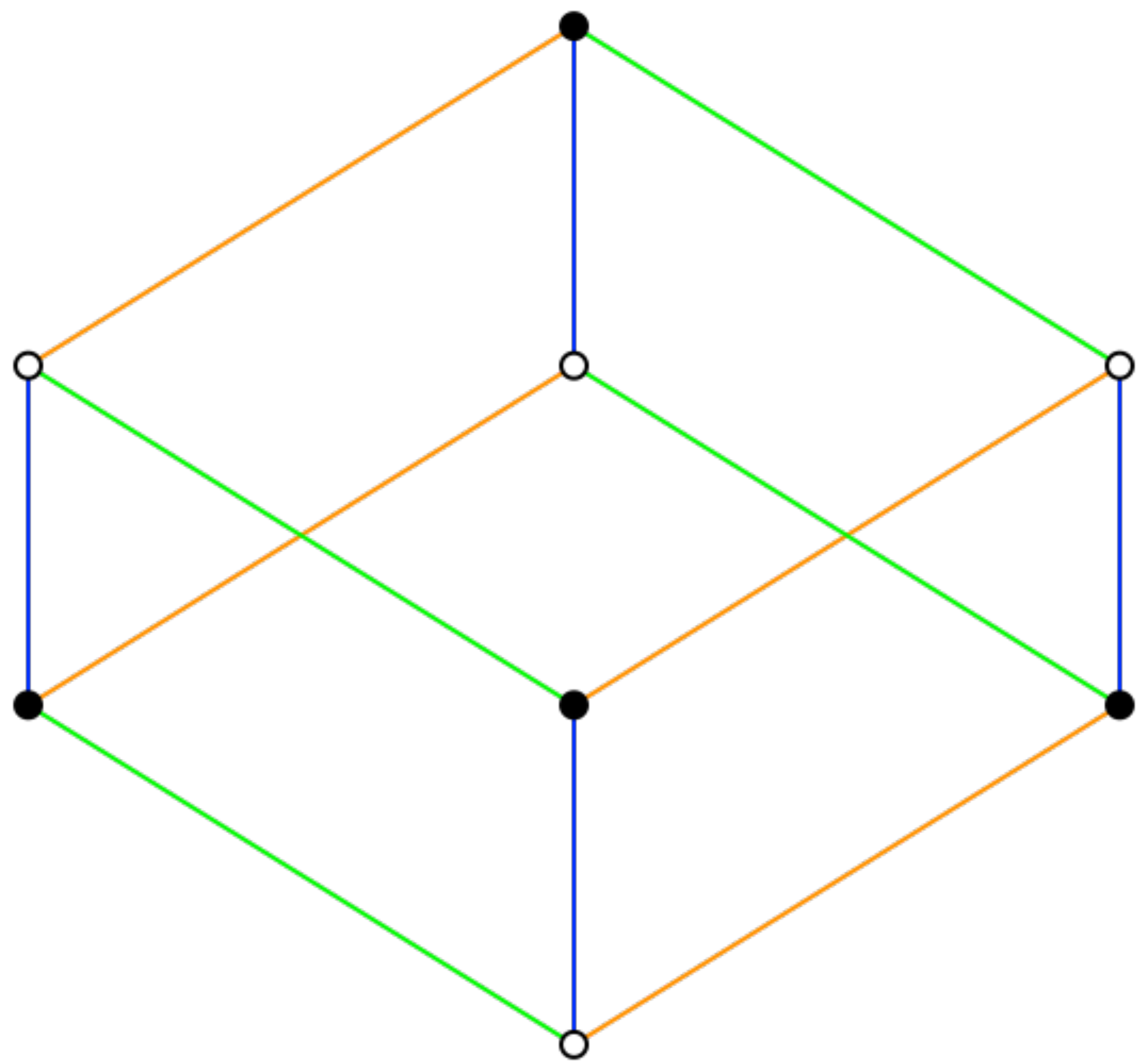
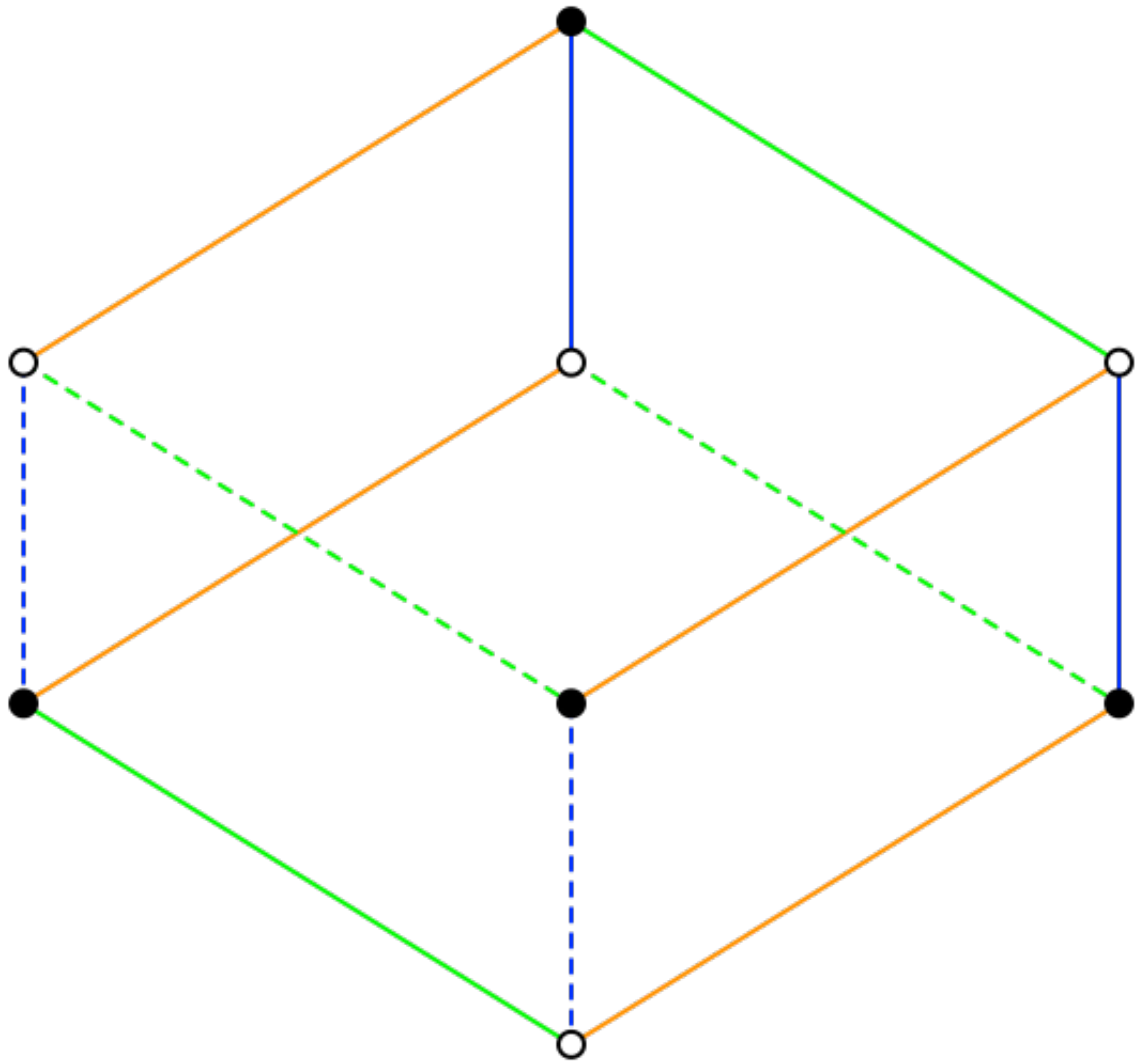


IV. Orientation/Height Assignment

V. Odd Dashing



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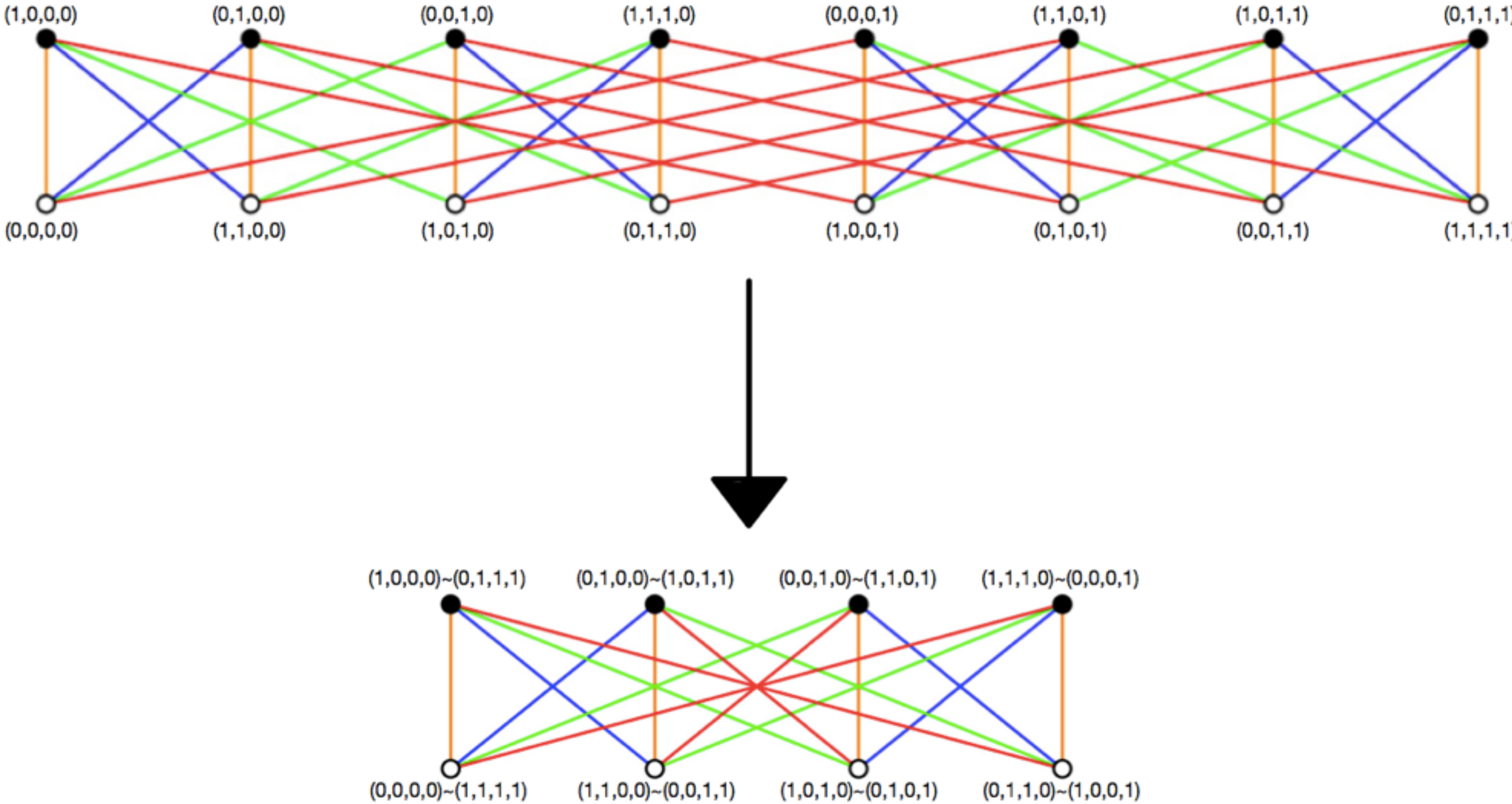
Vector subspaces are known as binary linear error-correcting codes

Idea: Achieve Adinkra chromotopologies by quotienting the Hamming cube by certain codes C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R-symmetry of the superalgebra induces permutation equivalence of the doubly-even codes

So we need to classify doubly-even codes!

N \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	N
4	1																4
5	0																5
6	0	1															6
7	0	0	1														7
8	1	1	1	1													8
9	0	0	0	0													9
10	0	1	1	1													10
11	0	0	1	1													11
12	1	2	3	4	2												12
13	0	0	1	1	2												13
14	0	2	4	6	5	4											14
15	0	0	3	6	6	4	2										15
16	1	3	8	18	21	15	7	2									16
17	0	0	2	7	14	11	5	1									17
18	0	3	9	27	44	45	21	6									18
19	0	0	6	22	52	62	40	10									19
20	1	4	17	64	149	212	156	65	10								20
21	0	0	6	36	144	276	263	114	28								21
22	0	4	20	104	373	852	971	542	149	25							22
23	0	0	12	89	475	1489	2346	1622	527	94	11						23
24	1	6	34	220	1157	4317	8584	7686	2996	620	83	9					24
25	0	0	12	148	1364	7890	23521	28001	12329	2234	215	13					25
26	0	5	39	359	3115	22278	92354	156577	87488	17233	1520	81					26
27	0	0	22	321	4397	49390	336279	908397	713640	154603	11749	422					27
28	1	7	61	701	9492	137980	1452663	6528300	8130903	2375318	189600	6877	151				28
29	0	0	24	557	13421	330979	6166802	?	?	?	4634293	126161	1789				29
30	0	7	72	1173	27638	921270	?	?	?	?	?	?	68804	731			30
31	0	0	41	1106	?	?	?	?	?	?	?	?	?	24766	210		31
32	1	9	106	?	?	?	?	?	?	?	?	?	?	?	7479	85	32
N \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	N

What about odd dashings?

Formalism of cubical complexes

Notion of equivalence under a series of vertex switches

Up to equivalence, the difference between two odd dashings on the same Adinkra chromotopology C corresponds to a class in the first cohomology group of the quotient of the Hamming cube by C

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Very similar to case of spin structures on an orientable manifold M

Second Stiefel-Whitney class of M is the obstruction to the existence of a spin structure on M

Difference between two spin structures on M defines an element of the binary-valued first cohomology group of M

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This results in an orientable compact surface X

$$g = 1 + 2^{N-k-3}(N-4)$$

Topology of Surfaces

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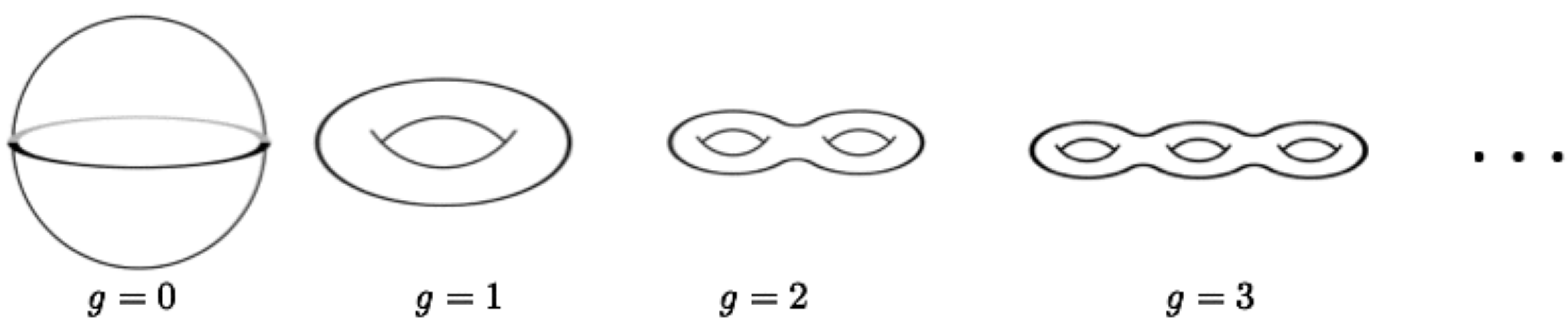


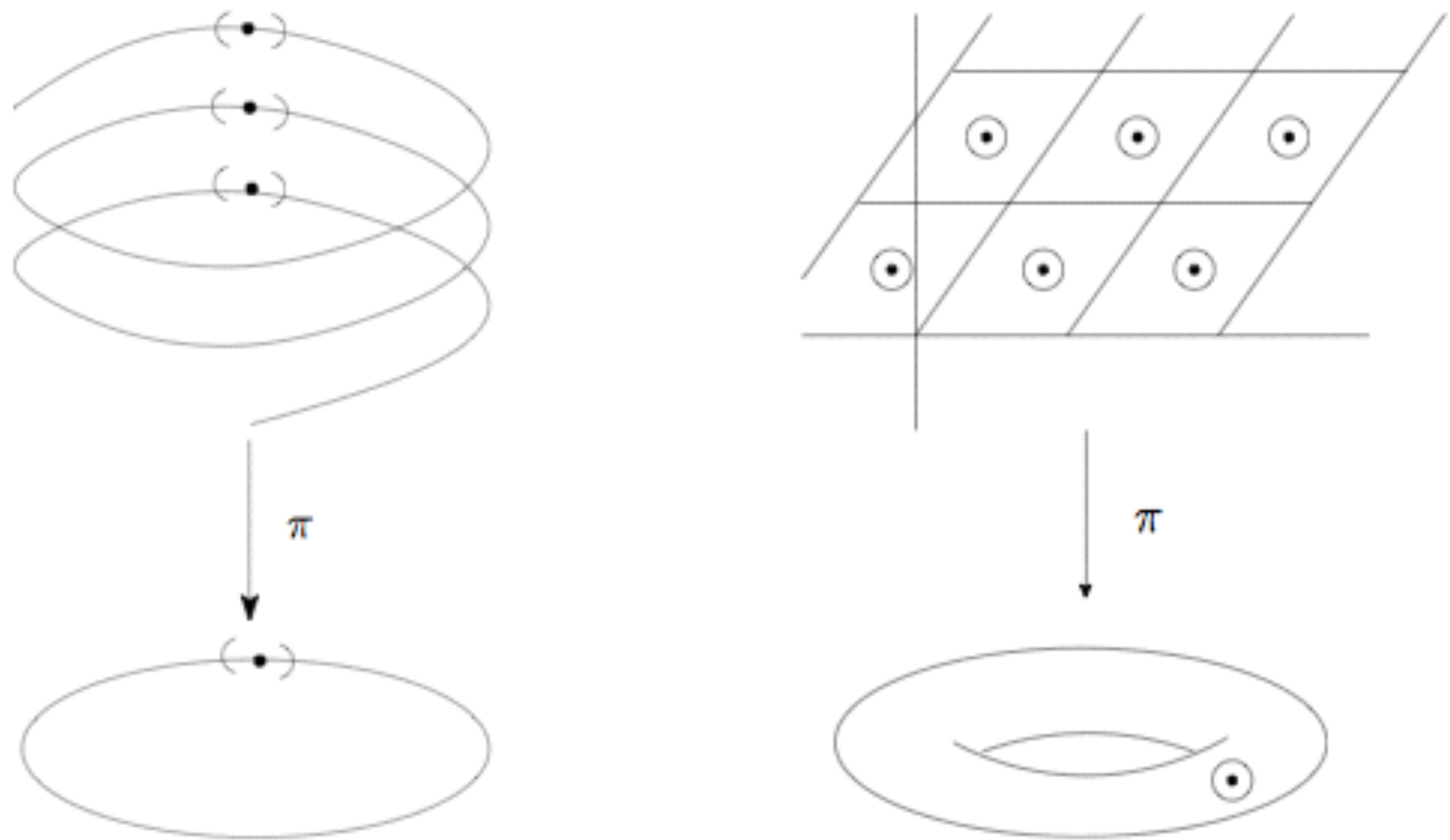
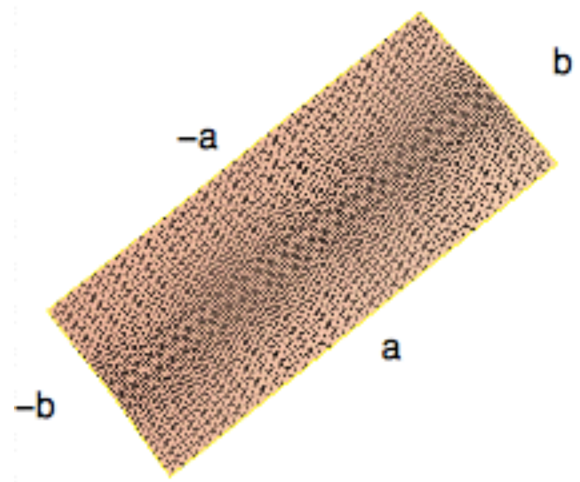
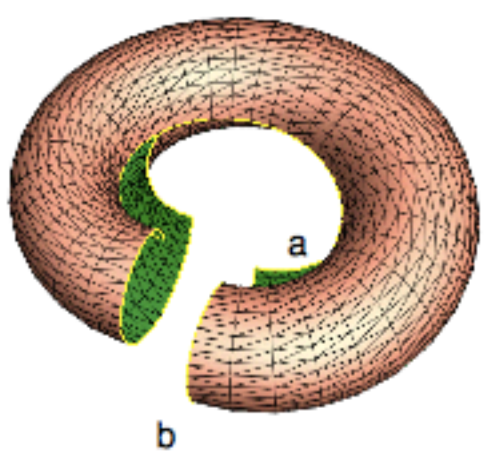


FIGURE 2.1. A cartoon of two spaces, S^1 and T^2, with their universal covers $\mathbb{R}$ and $\mathbb{C}$ and projection maps $\pi : \mathbb{R} \to S^1$ and $\pi : \mathbb{C} \to T^2$.


b
–a
a
–b


a
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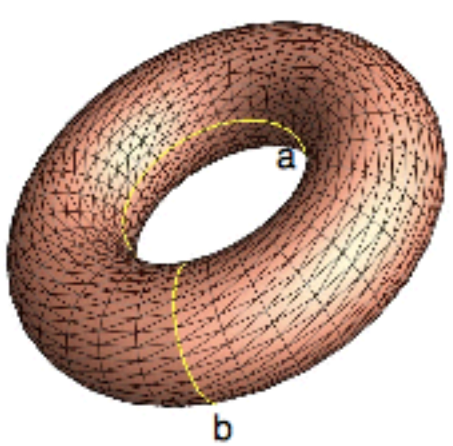
a
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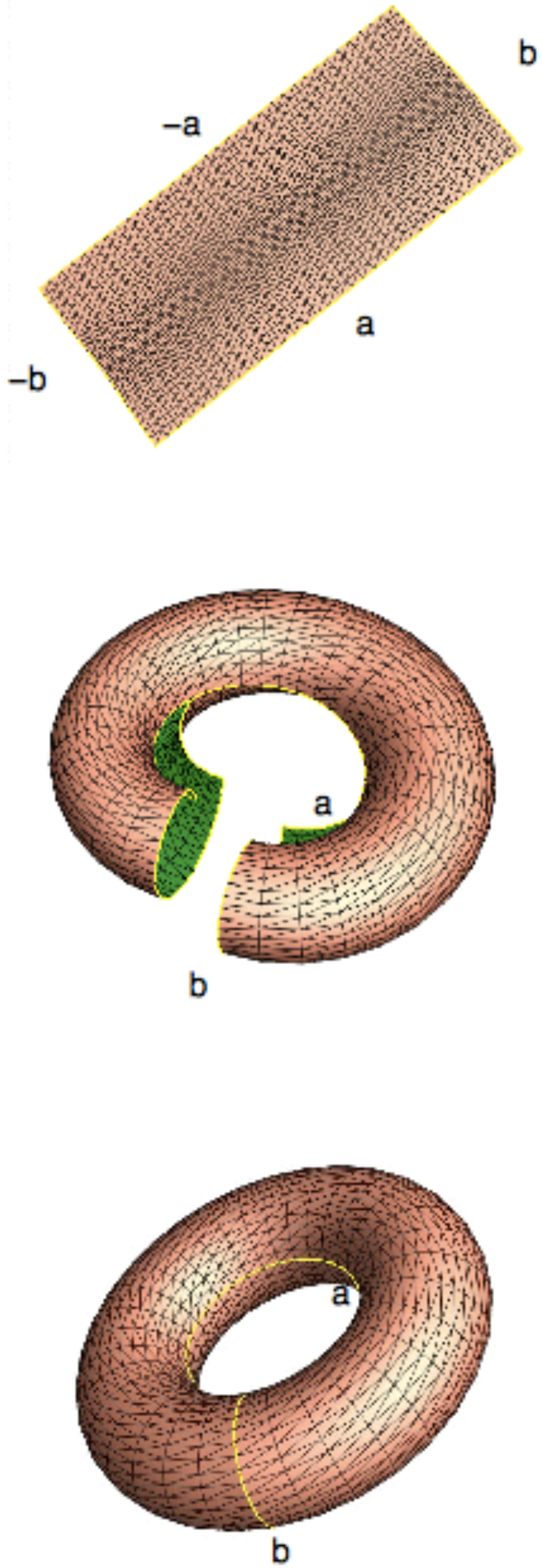

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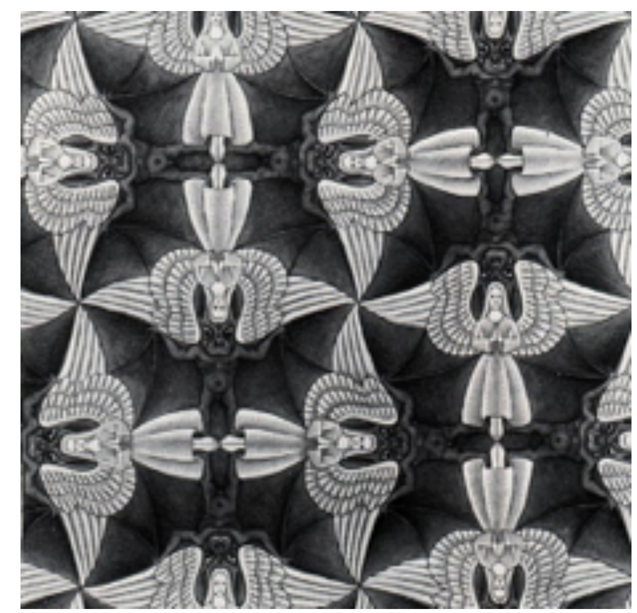

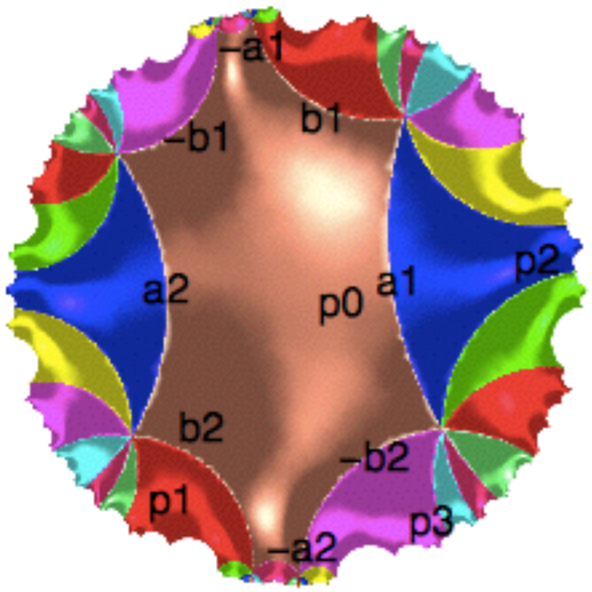
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a1
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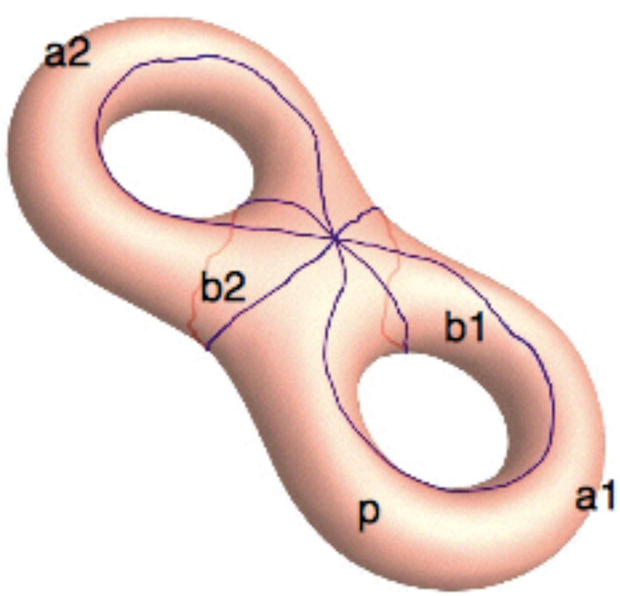
p3
-a2


a2
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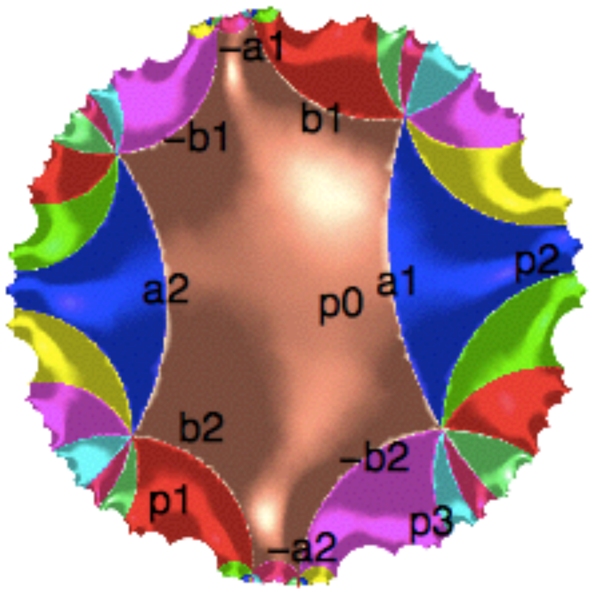
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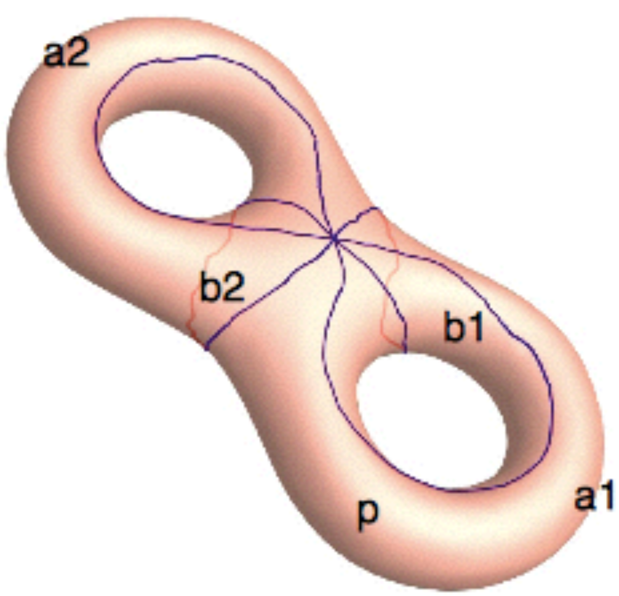
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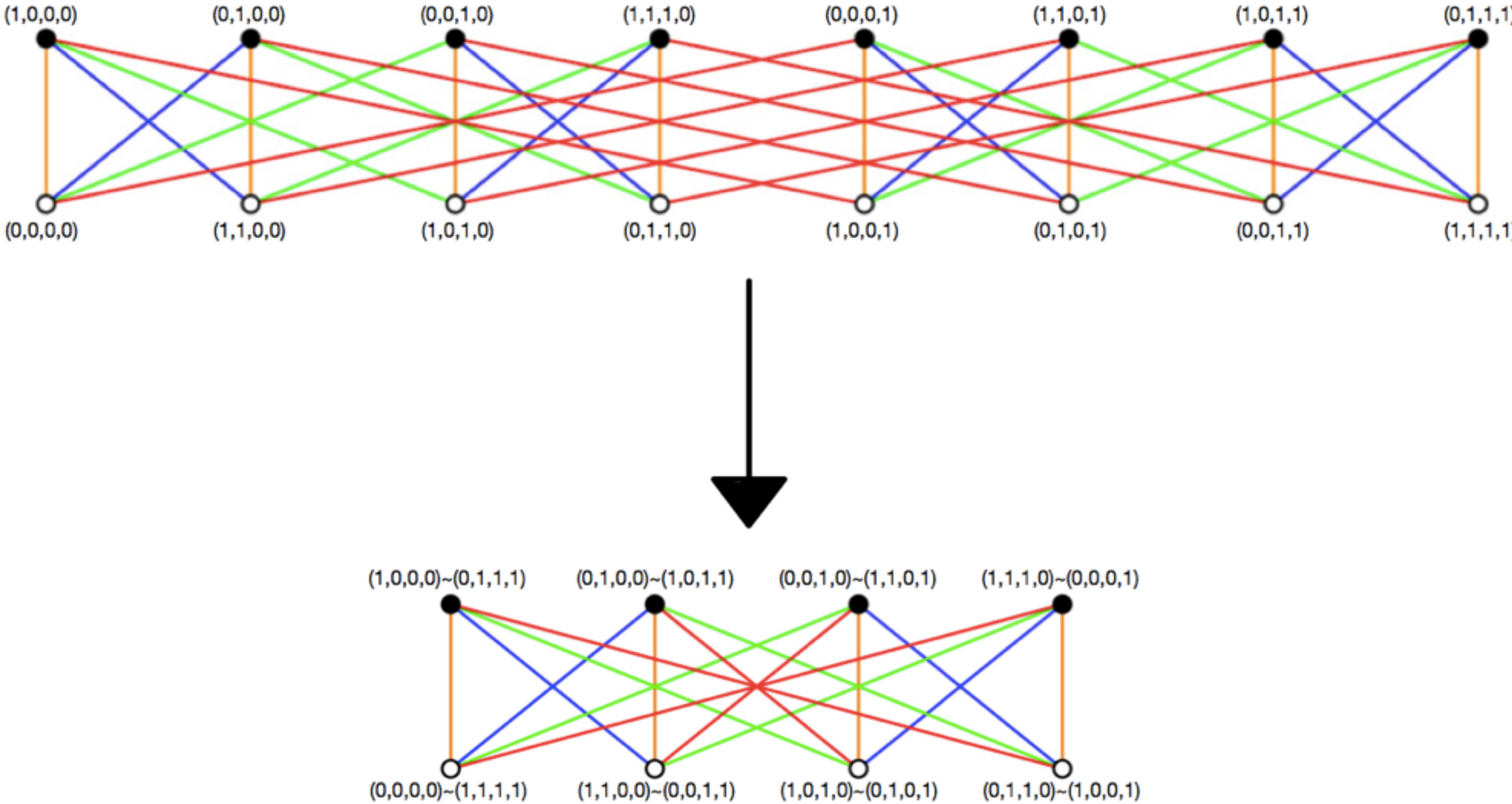

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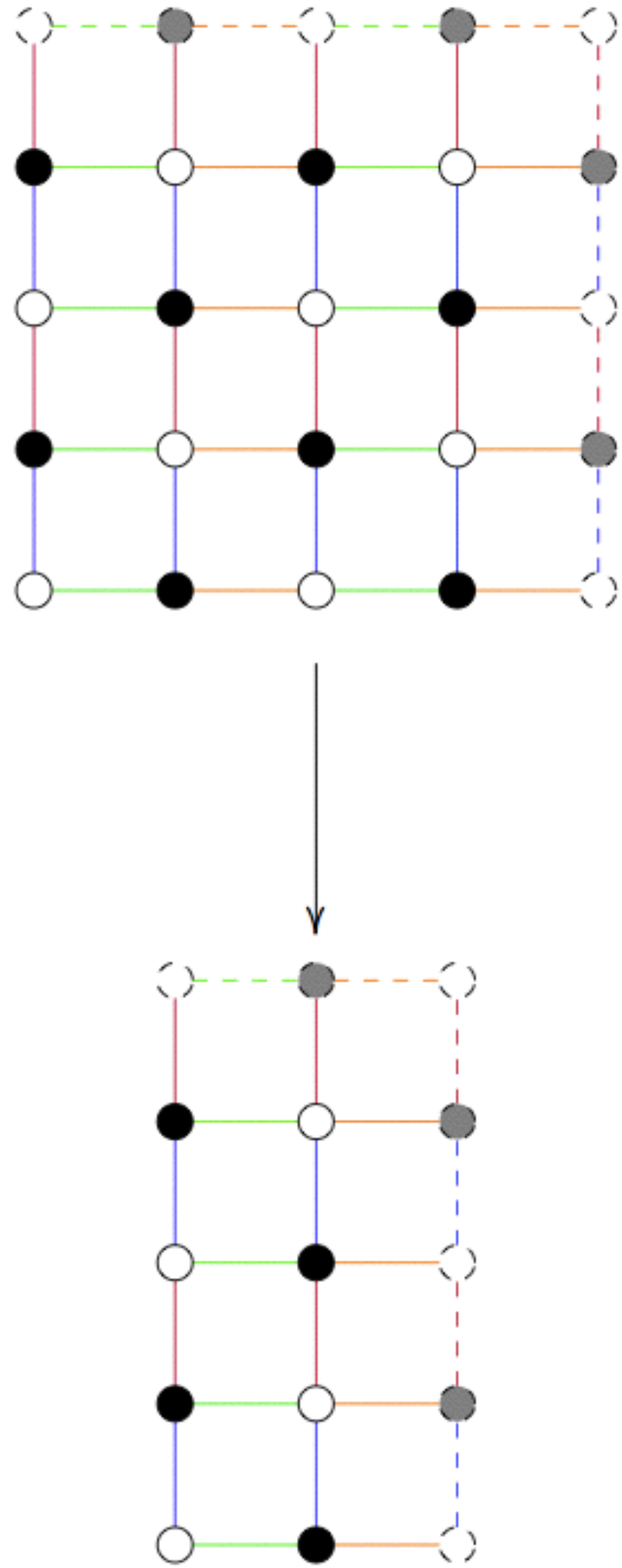

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So chromotopologies give natural Riemann surfaces … what about chromotopologies together with odd dashings?

The answer comes from the dimer models of statistical mechanics

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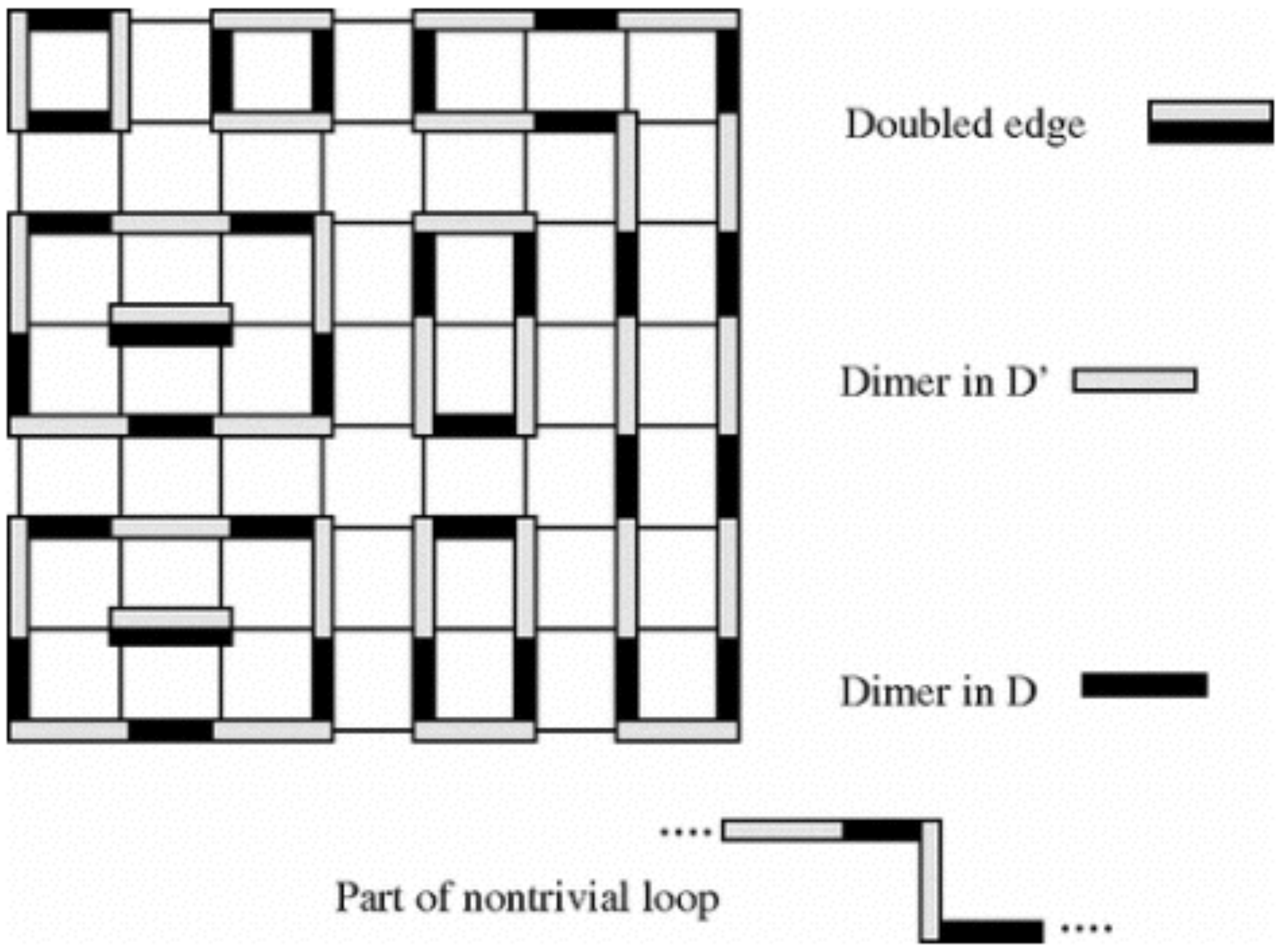

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This was essentially the same notion of equivalence we used for odd dashings. Can we relate odd dashings to Kastelyn orientations?

The bipartite orientation on an Adinkra PLUS the odd dashing makes a Kastelyn orientation.

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Cimasoni and Reshetikhin have shown that equivalence classes of Kastelyn orientations correspond to spin structures on X.

Thus we find that our chromotopologies with odd dashings correspond to very special Riemann surfaces with spin structures.

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The orientation/height assignment admits two interrelated topological/geometric interpretations

As a discretized Morse function on X

What about the orientation/height assignment?

The orientation/height assignment admits two interrelated topological/geometric interpretations

As a discretized Morse function on X

As a divisor on the Riemann surface