2. Symmetry: A Review of 3D Rotations

We begin our introduction by considering not supersymmetry but instead by considering a more familiar symmetry, i.e. rotational symmetry in three dimensions. To this end, we may introduce a 'generator' denoted by $\mathbf{L}_3$. This is an operator that is define by its actions on the coordinates $(\, x,\, y,\, z\,)$

$$\mathbf{L}_3 x \;=\; -\, i\, y \quad , \quad \mathbf{L}_3 y \;=\; i x \quad , \quad \mathbf{L}_3 z \;=\; 0 \quad . \tag{1}$$

and by the fact that it is a derivation. This means that when the operator (1) acts on a product of coordinates, it obeys a rule similar to a derivative, i.e.

$$\begin{aligned}\mathbf{L}_3\Big[x\, y \Big] \;&=\; \Big[\mathbf{L}_3\Big(x \Big) \Big]\, y \;+\; x \Big[\mathbf{L}_3\Big(y \Big) \Big] \\ &=\; -\, i\, [\, y^2 \;-\; x^2 \,] \quad . \end{aligned} \tag{2}$$

Now we wish to prove that this operator generates a rotation about the third direction. Let γ be an angle, we may consider the object defined by

$$\mathcal{R}_3(\gamma) \;\equiv\; exp\Big[-\, i\gamma\, \mathbf{L}_3 \Big] \quad , \tag{3}$$

and evaluate its effect on $(x,\, y,\, z)$. A rotated set of coordinates $(x',\, y',\, z')$ may be obtained by application of $\mathcal{R}_3(\gamma)$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \;=\; \mathcal{R}_3(\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; \begin{pmatrix} \mathcal{R}_3(\gamma)\, x \\ \mathcal{R}_3(\gamma)\, y \\ \mathcal{R}_3(\gamma)\, z \end{pmatrix} \quad . \tag{4}$$

We can evaluate each of these using the definitions above

$$\begin{aligned}\mathcal{R}_3(\gamma)\, x &= \sum_{n=1}^{\infty} \frac{1}{n!} (\, -\, i\gamma \mathbf{L}_3 \,)^n x \;=\; \Big[\; 1 \;-\; i\gamma \mathbf{L}_3 \;+\; \tfrac{1}{2}\, (\, -\, i\gamma\, \mathbf{L}_3 \,)^2 \;+\; \ldots \;\Big]\, x \\ &= \Big[\; x \;-\; \gamma\, y \;-\; \tfrac{1}{2} (\gamma)^2\, x \;+\; \tfrac{1}{3!} (\gamma)^3\, y \;+\; \ldots \;\Big] \\ &= \Big[\; x\, cos\gamma \;-\; y\, sin\gamma \;\Big] \quad , \end{aligned} \tag{5}$$

$$
\begin{aligned}
\mathcal{R}_3(\gamma)\, y &= \sum_{n=1}^{\infty} \frac{1}{n!}(-i\gamma\,\mathbf{L}_3\,)^n\, y \;=\; \Big[\, 1 \,-\, i\gamma\mathbf{L}_3 \,+\, \tfrac{1}{2}\,(-i\gamma\mathbf{L}_3\,)^2 \,+\, \ldots \,\Big]\, y \\
&= \Big[\;\, y \,+\, \gamma\, x \,-\, \tfrac{1}{2}(\gamma)^2\, y \,-\, \tfrac{1}{3!}(\gamma)^3\, x \,+\, \ldots \,\Big] \\
&= \Big[\, x\, sin\gamma \,+\, y\, cos\gamma \,\Big] \quad ,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\mathcal{R}_3(\gamma)\, z &= \sum_{n=1}^{\infty} \frac{1}{n!}(-i\gamma\mathbf{L}_3\,)^n\, z \;=\; \Big[\, 1 \,-\, i\gamma\,\mathbf{L}_3 \,+\, \tfrac{1}{2}\,(-i\gamma\,\mathbf{L}_3\,)^2 \,+\, \ldots \,\Big]\, z \\
&= \Big[\, z + 0 + \ldots \,\Big] = z \quad .
\end{aligned} \tag{7}
$$

So that

$$
\mathcal{R}_3(\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; \begin{pmatrix} x\,\cos\gamma \;-\; y\, sin\gamma \\ x\, sin\gamma \;+\; y\, cos\gamma \\ z \end{pmatrix} \quad . \tag{8}
$$

For small values of γ, (4) implies

$$\begin{aligned} \Delta_{\mathcal{R}_3}(\gamma)x &= x' - x = -\gamma y \quad , \\ \Delta_{\mathcal{R}_3}(\gamma)y &= y' - y = \gamma x \quad , \\ \Delta_{\mathcal{R}_3}(\gamma)z &= z' - z = 0 \quad . \end{aligned} \tag{9}$$

and the notation is to indicate a change Δ due to the rotation $\mathcal{R}_3$ thru the angle γ. In the limit where γ goes to zero we have $\Delta_{\mathcal{R}_3} = \delta_{\mathcal{R}_3}$ and these can be re-written as

$$\delta_{\mathcal{R}_3}(\gamma)x = -\gamma y \quad , \quad \delta_{\mathcal{R}_3}(\gamma)y = \gamma x \quad , \quad \delta_{\mathcal{R}_3}(\gamma)z = 0 \quad , \tag{10}$$

and a direct calculation reveals that acting on (x, y, z)

$$\delta_{\mathcal{R}_3}(\gamma) = -i\gamma \mathbf{L}_3 \quad . \tag{11}$$

It is clearly possible to define two other similar operators

$$\begin{aligned} \mathbf{L}_1 x &= 0 \quad , \quad \mathbf{L}_1 y = i\, z \quad , \quad \mathbf{L}_1 z = -\, i\, y \quad , \\ \mathbf{L}_2 x &= -\, i\, z \quad , \quad \mathbf{L}_2 y = 0 \quad , \quad \mathbf{L}_2 z = i\, x \quad . \end{aligned} \tag{12}$$

Given the definitions of the operators in (1) and (12) there are interesting calculations that can be made.

$$\begin{pmatrix} \mathbf{L}_1\,\mathbf{L}_2\,x \\ \mathbf{L}_1\,\mathbf{L}_2\,y \\ \mathbf{L}_1\,\mathbf{L}_2\,z \end{pmatrix} = \begin{pmatrix} -\,i\mathbf{L}_1\,z \\ \mathbf{L}_1\,0 \\ i\,\mathbf{L}_1\,x \end{pmatrix} = \begin{pmatrix} -\,y \\ 0 \\ 0 \end{pmatrix} \quad , \tag{13}$$

$$\begin{pmatrix} \mathbf{L}_2\,\mathbf{L}_1\,x \\ \mathbf{L}_2\,\mathbf{L}_1\,y \\ \mathbf{L}_2\,\mathbf{L}_1\,z \end{pmatrix} = \begin{pmatrix} \mathbf{L}_2\,0 \\ i\,\mathbf{L}_2\,z \\ -\,i\,\mathbf{L}_2\,y \end{pmatrix} = \begin{pmatrix} 0 \\ -\,x \\ 0 \end{pmatrix} \quad , \tag{14}$$

$$\begin{pmatrix} \left[\mathbf{L}_1\ ,\ \mathbf{L}_2\right]x \\ \\ \left[\mathbf{L}_1\ ,\ \mathbf{L}_2\right]y \\ \\ \left[\mathbf{L}_1\ ,\ \mathbf{L}_2\right]z \end{pmatrix} = \begin{pmatrix} -\,y \\ \\ x \\ \\ 0 \end{pmatrix} \quad . \tag{15}$$

In writing this, we have introduced the standard notation of the commutator

$$\left[\mathcal{A}\ ,\ \mathcal{B}\right] \equiv \mathcal{A}\mathcal{B} - \mathcal{B}\,\mathcal{A} \quad , \tag{16}$$

for any two quantities $\mathcal{A}$ and $\mathcal{B}$.

From equation (1) it follows that

$$\begin{pmatrix} \mathbf{L}_3\, x \\ \mathbf{L}_3\, y \\ \mathbf{L}_3\, z \end{pmatrix} = \begin{pmatrix} i\, y \\ -\, i\, x \\ 0 \end{pmatrix} \quad , \tag{17}$$

and on comparing (15) and (17) it is apparent that

$$\begin{pmatrix} \left[\mathbf{L}_1 \ , \ \mathbf{L}_2\right] x \\ \\ \left[\mathbf{L}_1 \ , \ \mathbf{L}_2\right] y \\ \\ \left[\mathbf{L}_1 \ , \ \mathbf{L}_2\right] z \end{pmatrix} = \begin{pmatrix} i\,\mathbf{L}_3\, x \\ \\ i\,\mathbf{L}_3\, y \\ \\ i\,\mathbf{L}_3\, z \end{pmatrix} \quad . \tag{18}$$

Since each component of this has the same general form we write

$$\Big[\mathbf{L}_1 \ , \ \mathbf{L}_2\Big] = i\,\mathbf{L}_3 \quad . \tag{19}$$

If the calculation above is repeated in all possible ways it leads to the familiar result

$$\Big[\mathbf{L}_i \ , \ \mathbf{L}_j\Big] = i\,\epsilon_{i\,j\,k}\,\mathbf{L}_k \quad . \tag{20}$$

For the 3D rotation generators, it can be shown that

$$\Big[\mathbf{L}_i\,,\,\Big[\mathbf{L}_j\,,\,\mathbf{L}_k\Big]\Big] + \Big[\mathbf{L}_j\,,\,\Big[\mathbf{L}_k\,,\,\mathbf{L}_i\Big]\Big] + \Big[\mathbf{L}_k\,,\,\Big[\mathbf{L}_i\,,\,\mathbf{L}_j\Big]\Big] = 0 \quad . \tag{21}$$

The operators $\mathbf{L}_1$ and $\mathbf{L}_2$ lead to $\mathcal{R}_1(\alpha)$, $\mathcal{R}_2(\beta)$, defined by

$$\mathcal{R}_1(\alpha) \equiv exp\Big[-i\alpha\,\mathbf{L}_1\Big] \quad , \quad \mathcal{R}_2(\beta) \equiv exp\Big[-i\beta\,\mathbf{L}_2\Big] \quad . \tag{22}$$

and the most general rotation in 3 dimensions is defined by

$$\mathcal{R}(\alpha,\,\beta,\gamma) \;=\; \mathcal{R}_1(\alpha)\,\mathcal{R}_2(\beta)\,\mathcal{R}_3(\gamma) \quad . \tag{23}$$

An alternate but equivalent way to write the general rotation is in the form

$$\widetilde{\mathcal{R}}(\alpha,\,\beta,\gamma) \;=\; exp\Big[-i(\,\alpha\,\mathbf{L}_1 + \beta\,\mathbf{L}_2 + \gamma\,\mathbf{L}_3\,)\Big] \;=\; exp\Big[-i\alpha_i\,\mathbf{L}_i\Big] \quad . \tag{24}$$

2.1. *Rotational Symmetry & 'The Noether Method'*

Any quantity $\mathcal{L}(\,x,\,y,\,z\,)$ that is 'invariant' under a rotation satisfies

$$\begin{aligned} \mathcal{L}(\,x',\,y',\,z'\,) \;&=\; \mathcal{L}(\,x,\,y,\,z\,) \quad , \\ \mathcal{R}(\alpha,\,\beta,\gamma)\,\mathcal{L}(\,x,\,y,\,z\,) \;&=\; \mathcal{L}(\,x,\,y,\,z\,) \quad , \end{aligned} \tag{25}$$

and if the angles are infinitesimals the left hand side can be written as

$$\Big[1 \;+\; i\Big(\alpha\,\mathbf{L}_1 \;+\; \beta\,\mathbf{L}_2 \;+\; \gamma\,\mathbf{L}_3\Big)\Big]\,\mathcal{L}(\,x,\,y,\,z\,) \;=\; \mathcal{L}(\,x,\,y,\,z\,) \quad , \tag{26}$$

or equivalently this can be written as

$$\delta_{\mathcal{R}}(\alpha^i)\,\mathcal{L}(\,x,\,y,\,z\,)\;=\;i\Big(\alpha\,\mathbf{L}_1\;+\;\beta\,\mathbf{L}_2\;+\;\gamma\,\mathbf{L}_3\Big)\,\mathcal{L}(\,x,\,y,\,z\,)\;=\;0\quad, \tag{27}$$

where

$$\delta_{\mathcal{R}}(\alpha^i)\;=\;i\,\alpha_i\,\mathbf{L}_i\quad. \tag{28}$$

Since the angles are independent, this is actually equivalent to three independent conditions

$$\mathbf{L}_1\,\mathcal{L}(\,x,\,y,\,z\,)\;=\;0\quad,\quad\mathbf{L}_2\,\mathcal{L}(\,x,\,y,\,z\,)\;=\;0\quad,\quad\mathbf{L}_3\,\mathcal{L}(\,x,\,y,\,z\,)\;=\;0\quad, \tag{29}$$

which can be more simply written as $\mathbf{L}_i\,\mathcal{L}\;=\;0$. Any quantity that satisfies (29) is said possess rotational symmetry. In a similar manner any quantity that satisfies only

$$\mathbf{L}_3\,\mathcal{L}(\,x,\,y,\,z\,)\;=\;0\quad, \tag{30}$$

possess rotational symmetry about the z-axis (or third direction).

Symmetries are very useful properties. For example, imagine there is some system that possesses a symmetry about the z-axis, and has an energy $\mathcal{E}_t$ that is known to depend on both x and y. Further imagine a 'standard measurement' of this quantity is only be made when $y = 0$ and yields the x-dependence

$$\mathcal{E}_t(x,\, y=0) \;=\; \mathrm{A}_0\, x^4 \;\equiv\; \mathcal{E}_{SM}(x) \quad , \tag{31}$$

Since the total energy $\mathcal{E}_t$ is a function of both both x and y it can be written as

$$\mathcal{E}_t(x,\, y) \;=\; \mathcal{E}_{SM}(x) \;+\; \mathcal{E}_{smSM}(x,\, y) \quad , \tag{32}$$

where the function $\mathcal{E}_{smSM}$ ('symmetry-modified standard measurement') must satisfy $\mathcal{E}_{smSM}(x,\, y=0) \,=\, 0$.

Since $\mathcal{E}_t$ possesses a symmetry with respect to $\mathbf{L}_3$ it must be the case that $\mathbf{L}_3\, \mathcal{E}_t \;=\; 0$. So that

$$\begin{aligned} 0 \;&=\; \mathbf{L}_3 \Big[\, \mathcal{E}_{SM}(x) \;+\; \mathcal{E}_{smSM}(x,\, y) \,\Big] \\ &=\; \mathbf{L}_3 \Big[\, \mathcal{E}_{SM}(x) \,\Big] \;+\; \mathbf{L}_3 \Big[\, \mathcal{E}_{smSM}(x,\, y) \,\Big] \\ &=\; i\, 4\, A_0\, x^3\, y \;+\; \mathbf{L}_3 \Big[\, \mathcal{E}_{smSM}(x,\, y) \,\Big] \quad . \end{aligned} \tag{33}$$

To find the explicit form of $\mathcal{E}_{smSM}$ an expansion in terms of powers of y can be utitilzed.

$$\mathcal{E}_{smSM}(x,\, y) \;=\; \sum_{n=1}^{\infty} y^n\, f_n(x) \quad . \tag{34}$$

When this expansion is substituted into (32.) it leads to

$$0 \;=\; 4\,A_0\,x^3\,y \;+\; \sum_{n=1}^{\infty} \Big[\, y^{n+1}\,\frac{d\,f_n}{dx} \;-\; n\,y^{n-1}\,x\,f_n(x) \,\Big] \quad ,$$

$$\begin{aligned} 0 \;&=\; [\,4\,A_0\,x^3 \;-\; 2\,x\,f_2(x)\,]\,y \;-\; x\,f_1(x) \\ &\quad + \sum_{n=1}^{\infty} y^{n+1}\Big[\,\frac{d\,f_n}{dx} \;-\; (n+2)\,x\,f_{n+2}(x)\,\Big] \quad , \end{aligned} \tag{35}$$

i.e. a series of equations for the unknown coefficient functions $f_n(x)$. Upon separation into various powers of y this system yields

$$\begin{aligned} &y^0 : \quad x\,f_1(x) \;=\; 0 \;\;\to\;\; f_1(x) \;=\; 0 \quad , \\ &y^1 : \quad 4\,A_0\,x^3 \;-\; 2x\,f_2(x) \;=\; 0 \;\to\;\; f_2(x) \;=\; 2\,A_0\,x^2 \quad , \\ &y^{n+1}\;(\,n \ge 1\,) : \quad f_{n+2}(x) \;=\; (\frac{1}{n\,+\,2})\,(\frac{1}{x})\,\frac{d\,f_n}{dx} \quad . \end{aligned} \tag{36}$$

The complete solution to this set of equations is given by

$$f_2(x) \;=\; 2\,A_0\,x^2 \;\;,\;\; f_4(x) \;=\; A_0 \quad , \tag{37}$$

where all other coefficients function vanish. In supersymmetrical theories, the steps discussed above are often called “the Noether Method” and are commonly used especially in the case of supergravity theories[18] .