1 Reducing 4D, $\mathcal{N} = 1$ SUSY To 1D, $N = 4$ SUSY

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \mathcal{T}_\mu \frac{\partial}{\partial t} + \mathcal{X}_\mu \frac{\partial}{\partial x} + \mathcal{Y}_\mu \frac{\partial}{\partial y} + \mathcal{Z}_\mu \frac{\partial}{\partial z} \quad ,$$

with $\mathcal{T}_\mu$, $\mathcal{X}_\mu$, $\mathcal{Y}_\mu$, and $\mathcal{Z}_\mu$

$$\mathcal{T}_\mu = (1, 0, 0, 0) \quad , \quad \mathcal{X}_\mu = (0, 1, 0, 0) \quad ,$$
$$\mathcal{Y}_\mu = (0, 0, 1, 0) \quad , \quad \mathcal{Z}_\mu = (0, 0, 0, 1) \quad .$$

$$\frac{\partial}{\partial t} = \cos\alpha \frac{\partial}{\partial \tau} \quad , \qquad \frac{\partial}{\partial x} = \sin\alpha \, \sin\beta \, \cos\gamma \frac{\partial}{\partial \tau} \quad ,$$
$$\frac{\partial}{\partial y} = \sin\alpha \, \sin\beta \, \sin\gamma \frac{\partial}{\partial \tau} \quad , \qquad \frac{\partial}{\partial z} = \sin\alpha \, \cos\beta \frac{\partial}{\partial \tau} \quad ,$$

$$\partial_\mu = [\cos\alpha \, \mathcal{T}_\mu + \sin\alpha \sin\beta \cos\gamma \, \mathcal{X}_\mu + \sin\alpha \sin\beta \sin\gamma \, \mathcal{Y}_\mu + \sin\alpha \cos\beta \, \mathcal{Z}_\mu] \frac{\partial}{\partial \tau} \; .$$

Four-dimensional $\mathcal{N} = 1$ theory, the anti-commutator algebra

$$\{ \mathrm{Q}_a \, , \, \mathrm{Q}_b \} = i \, 2 \, (\gamma^\mu)_{a\,b} \, \partial_\mu \quad ,$$

Substitute $\partial_\mu = \ell_\mu \partial_\tau$

$$\{ \mathrm{Q}_a \, , \, \mathrm{Q}_b \} = i \, 2 \, (\gamma^\mu)_{a\,b} \, \ell_\mu \, \partial_\tau = i \, 2 \, (\gamma \cdot \ell)_{a\,b} \, \partial_\tau \quad .$$

2 Reduction of the Chiral Multiplet

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu A \partial^\mu A \; - \; \tfrac{1}{2}\partial_\mu B \partial^\mu B \; + \; \tfrac{1}{2} i (\gamma^\mu)^{bc} \psi_b \partial_\mu \psi_c \; + \; \tfrac{1}{2} F^2 + \tfrac{1}{2} G^2$$

$$
\begin{aligned}
\mathrm{D}_a A &= \psi_a \quad , \quad \mathrm{D}_a B = i\,(\gamma^5)_a{}^b \psi_b \quad , \\
\mathrm{D}_a \psi_b &= i\,(\gamma^\mu)_{ab}\,\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\,\partial_\mu B - i C_{ab} F + (\gamma^5)_{ab} G \ , \\
\mathrm{D}_a F &= (\gamma^\mu)_a{}^b\,\partial_\mu \psi_b \quad , \quad \mathrm{D}_a G = i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu \psi_b \quad ,
\end{aligned}
$$

$\partial_\mu = \ell_\mu \partial_\tau$

$$
\begin{aligned}
\mathrm{D}_a A &= \psi_a \quad , \quad \mathrm{D}_a B = i\,(\gamma^5)_a{}^b \psi_b \quad , \\
\mathrm{D}_a \psi_b &= i\,(\gamma \cdot \ell)_{ab}\,\partial_\tau A - (\gamma^5\gamma \cdot \ell)_{ab}\,\partial_\tau B - i C_{ab} F + (\gamma^5)_{ab} G \ , \\
\mathrm{D}_a F &= (\gamma \cdot \ell)_a{}^b\,\partial_\tau \psi_b \quad , \quad \mathrm{D}_a G = i\,(\gamma^5\gamma \cdot \ell)_a{}^b\,\partial_\tau \psi_b \quad ,
\end{aligned}
$$

$$
\mathcal{L} = -\,\tfrac{1}{2}\ell_\mu \ell^\mu \left[\ (\partial_\tau A)^2 \ + \ (\partial_\tau B)^2 \ \right] \ + \ \tfrac{1}{2} i\,(\gamma \cdot \ell)^{bc}\,\psi_b \partial_\tau \psi_c \ + \ \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \quad .
$$

$$
\mathcal{L} = \tfrac{1}{2}\cos(2\alpha) \left[\ (\partial_\tau A)^2 \ + \ (\partial_\tau B)^2 \ \right] \ + \ \tfrac{1}{2} i\,(\gamma \cdot \ell)^{bc}\,\psi_b \partial_\tau \psi_c \ + \ \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \ .
$$

$F \to \partial_\tau F$ and $G \to \partial_\tau G$

$$
\begin{aligned}
\mathrm{D}_a A &= \psi_a \quad , \quad \mathrm{D}_a B = i\,(\gamma^5)_a{}^b \psi_b \quad , \\
\mathrm{D}_a \psi_b &= i\,(\gamma \cdot \ell)_{ab}\,\partial_\tau A - (\gamma^5\gamma \cdot \ell)_{ab}\,\partial_\tau B - i C_{ab} \partial_\tau F + (\gamma^5)_{ab} \partial_\tau G \ , \\
\mathrm{D}_a F &= (\gamma \cdot \ell)_a{}^b\,\psi_b \quad , \quad \mathrm{D}_a G = i\,(\gamma^5\gamma \cdot \ell)_a{}^b\,\psi_b \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L} = -\,\tfrac{1}{2}\ell_\mu \ell^\mu \left[\ (\partial_\tau A)^2 \ + \ (\partial_\tau B)^2 \ \right] \ &+ \ i\,\tfrac{1}{2}(\gamma \cdot \ell)^{bc}\,\psi_b \partial_\tau \psi_c \\
&+ \ \tfrac{1}{2}\left[\ (\partial_\tau F)^2 \ + \ (\partial_\tau G)^2 \ \right] \quad .
\end{aligned}
$$

Define

$$
\mathrm{D}_{\mathrm{I}} \ = \ \begin{bmatrix} D_{a=1} \\ D_{a=2} \\ D_{a=3} \\ D_{a=4} \end{bmatrix} \ , \ \Phi_i \ = \ \begin{bmatrix} A \\ B \\ F \\ G \end{bmatrix} \ , \ \Psi_{\hat{k}} \ = \ -\,i \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \ ,
$$

$$\mathrm{D_I}\Phi_i \;=\; i\,(\mathrm{L_I})_{i\hat{k}}\Psi_{\hat{k}} \qquad \text{and} \qquad \mathrm{D_I}\Psi_{\hat{k}} \;=\; (\mathrm{R_I})_{\hat{k}i}\partial_\tau\Phi_i \quad ,$$

$$\mathcal{L} \;=\; \tfrac{1}{2}\,\delta^{i\,j}\;(\partial_\tau\Phi_i)\,(\partial_\tau\Phi_j) \;-\; i\,\tfrac{1}{2}\,\delta^{\hat{k}\,\hat{l}}\;\Psi_{\hat{k}}\,\partial_\tau\Psi_{\hat{l}} \quad ,$$

where

$$\delta^{\hat{k}\,\hat{l}} \;=\; -\;(\gamma\cdot\ell)^{\hat{k}\,\hat{l}}$$

3 Temporal Axis Reductions

This notation is simply a nice way to represent our matrices, most easily defined through example:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \;\equiv\; \big(\bar{2}\,\bar{4}\,1\,\bar{3}\big) \;=\; (11)_b(1243)\,. \tag{3.1}$$

In the expression $(ijkl)$, i represents the column in which the non-zero entry of the first row sits; j represents the same but for the second row, and so on. A bar over the index signifies that the element in that spot should be -1 instead of $+1$.

Chiral Multiplet

Reduction for $\ell = \mathcal{T}$ gives the following L and R matrices

$$\mathrm{L}_1 \;=\; (1\,\bar{4}\,2\,\bar{3}) \;\;,\;\; \mathrm{L}_2 \;=\; (2\,3\,\bar{1}\,\bar{4}) \;\;,\;\; \mathrm{L}_3 \;=\; (3\,\bar{2}\,\bar{4}\,1) \;\;,\;\; \mathrm{L}_4 \;=\; (4\,1\,3\,2) \;\;,$$
$$\mathrm{R}_1 \;=\; (1\,3\,\bar{4}\,\bar{2}) \;\;,\;\; \mathrm{R}_2 \;=\; (\bar{3}\,1\,2\,\bar{4}) \;\;,\;\; \mathrm{R}_3 \;=\; (4\,\bar{2}\,1\,\bar{3}) \;\;,\;\; \mathrm{R}_4 \;=\; (2\,4\,3\,1) \;\;.$$

or equivalently

$$\mathrm{L}_1 \;=\; (10)_b(243) \;\;,\;\; \mathrm{L}_2 \;=\; (12)_b(123) \;\;,\;\; \mathrm{L}_3 \;=\; (6)_b(134) \;\;,\;\; \mathrm{L}_4 \;=\; (0)_b(142) \;\;,$$
$$\mathrm{R}_1 \;=\; (12)_b(234) \;\;,\;\; \mathrm{R}_2 \;=\; (9)_b(132) \;\;,\;\; \mathrm{R}_3 \;=\; (10)_b(143) \;\;,\;\; \mathrm{R}_4 \;=\; (0)_b(124) \;\;.$$