Generalizing Adinkra-like Structures onto Graphs of Symmetry Groups

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Introduction

In order to generalize properties of traditional adinkras to Cayley graphs of symmetry groups, we made the following observations about both existing adinkras and potential graphs of symmetric groups:

- Group generators must have order 2.
  - If a graph has triangles (and thus, generators of order 3), it is not bipartite, and the directions of edges matter.
  - Generators of order 3 also destroy N-regularity.
- Choosing a generating set of transpositions implies that the graph will be bipartite. A_n and its coset in S_n form a partition corresponding to bipartiteness of the Cayley graph of S_n.
- Height assignment can always be induced from the bipartiteness of a graph. Alternatively, height assignment can be induced from each element's distance from the identity.
- Generalization of odd dashings:
  - All 2-color faces have an odd number of dashed edges.
  - The counts of dashed edges in different paths on 2-color faces have opposite parity (e.g. for a hexagonal face, $RGR = -GRG$).

Adjacent Transposition Generating Set

For a general symmetry group S_n choose a generating set with a rainbow such that adjacent elements in the rainbow do not commute and non-adjacent elements do commute. Adjacent elements produce subgroups that are 6-cycles while non-adjacent elements produce 4-cycles. Thus such generating sets and rainbow produce a Cayley graph composed of hexagonal and quadrilateral faces. One such generating set for S_4 is $A = \{(12), (23), (34)\}$.

Traditional N-cubic adinkras can be considered as subgraphs of these Cayley graphs generated by non-adjacent generating elements (disjoint transpositions). These subgroups are isomorphic to $\mathbb{Z}_2^n$. This implies that we can choose $k \geq 2N$ such that S_k contains a subset that is an adinkra with N-coloring. Adinkras with N colors and less than 2^N vertices can also be created in similar manner by introducing compositions of preexisting disjoint generating elements as generating elements themselves.

Considering the case of S_4 odd dashings do exist such that each 2-color face has an odd number of dashings, depicted to the right. Together with a height assignment, this gives us a generalized adinkra.

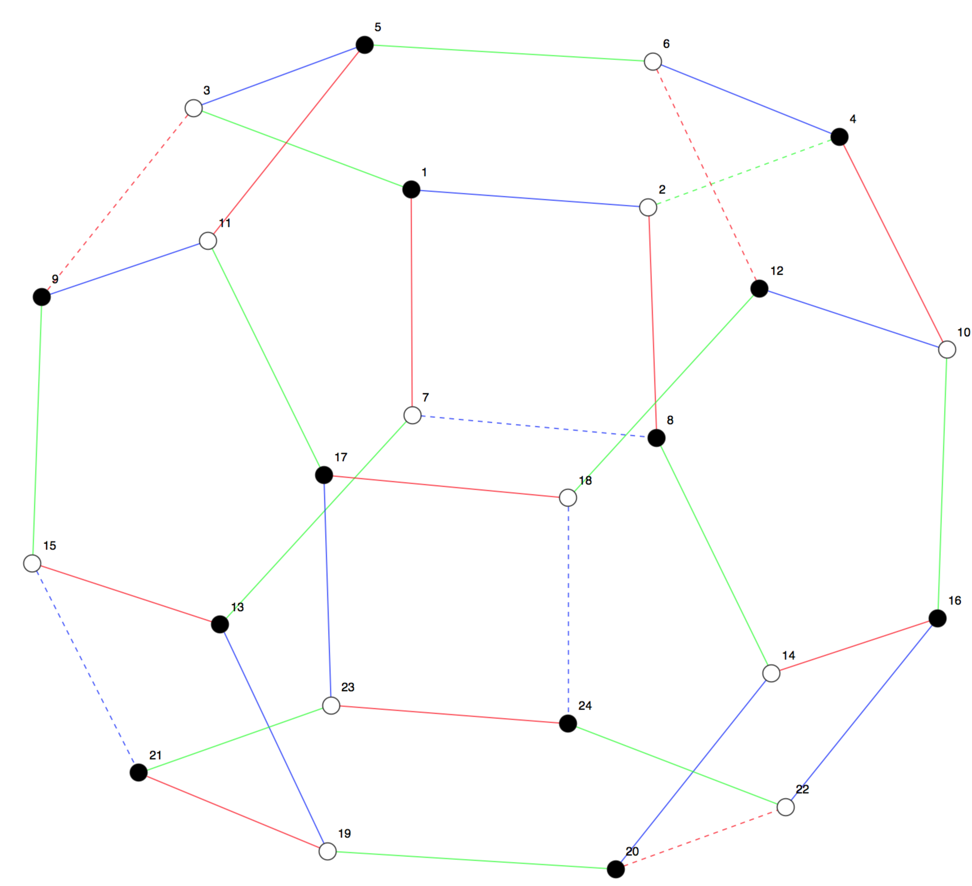

Figure 1: S_4 from the adjacent generating set.

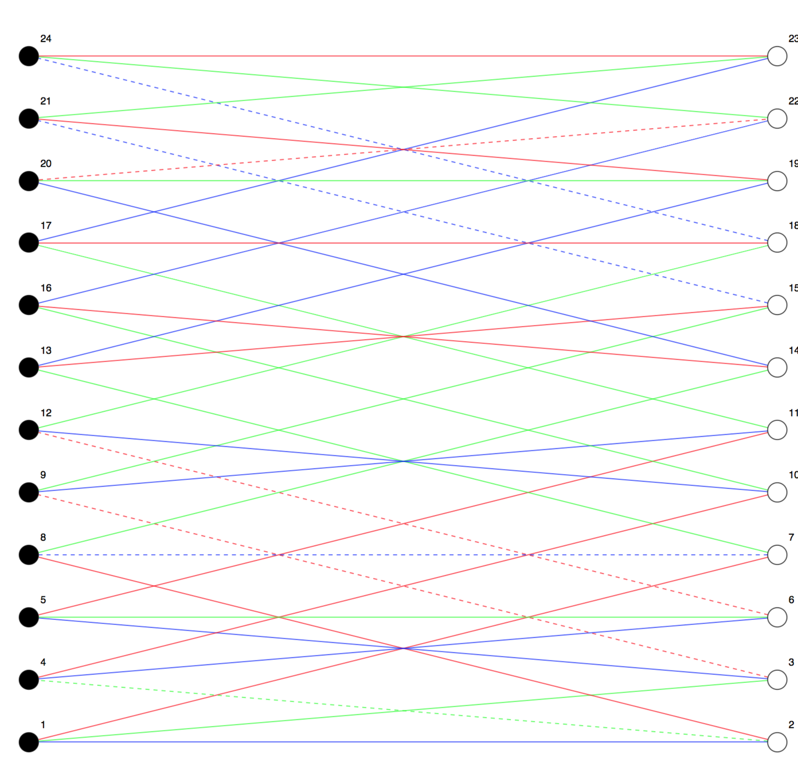

Figure 2: Figure 1 as a valise.

1-Transposition Generating Set

We now employ a new generating set: $\{(12), (13), \ldots, (1n)\}$. None of the generators commute, so all pairs of colors give us disjoint hexagons. By choosing a dashing so that each hexagon will have an odd count of dashed edges, we can find algebraic relations between operators that act similarly to the D-operators on adinkras. An odd dashing is possible on S_4, as exemplified on the right. We conjecture that this is the case for S_n. We observe these relations of "hexagonal" operators for all a, b, c:

$$D_aD_bD_a + D_bD_aD_b = 2i\delta_{ab}\frac{d}{dt}D_a \tag{1}$$

$$(D_aD_b)^3 = (-1)^{\delta_{ab}} i\frac{d^3}{dt^3} \tag{2}$$

$$(D_aD_bD_c)^4 = -\frac{d^6}{dt^6} \tag{3}$$

$$(D_aD_bD_c)^2 - (D_cD_bD_a)^2 = 0. \tag{4}$$

Equations 3 and 4 are true for the dashing we chose. We note that they rely on the fact that if we alternate between three colors, we get a cycle of length 12. For our dashing, these loops have an even number of dashed lines, but there may be some other dashing for which this is not the case.

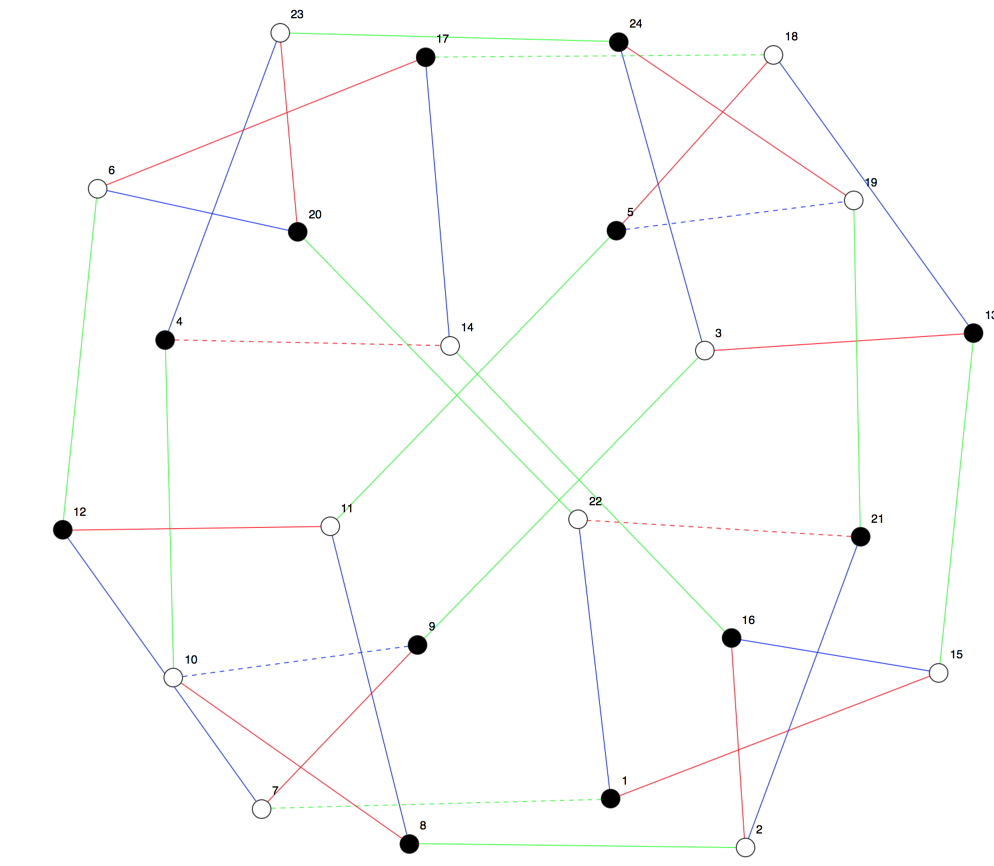

Figure 3: S_4 from the 1-transposition generating set.

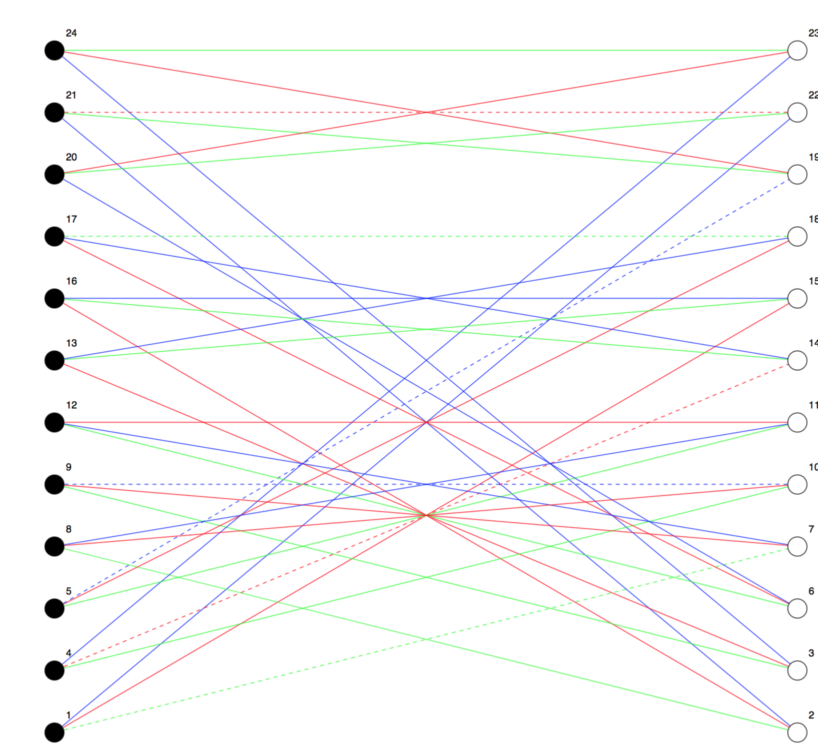

Figure 4: Figure 3 as a valise.

Conclusion

The work presented here shows the interesting results of our attempts to generalize the idea of an adinkra to larger symmetry groups. Cayley graphs of the symmetry groups S_n quickly become highly complex, and studying their properties through subgroups and subgraphs may give us insight into the structures that develop as n increases. Some questions for further exploration include the following:

- What dashing algorithms exist for each choice of generating set?
- What physical meanings and algebraic structures may be associated with these generalized adinkras?
- What equivalence classes for dashed chromotopologies of these generalized adinkras exist?

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