Unnecessarily Complicated Research Title

Counting Dashings, Cayley Graphs and Codes

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Introduction

An adinkra is a graph with the following additional properties

1. Bipartite, colored black (bosons) and white (fermions)

2. N-regular, N-edge colored

   - Kőnig's theorem guarantees a N-edge coloring a N-regular, bipartite graph

3. Quadrilateral

   - If we delete edges of all but two colors, the remaining graph is a disjoint union of 4-cycles

4. Oddly Dashed

   - We choose some edges of the adinkra to be dashed.
   - The adinkra must be *oddly dashed*–every two colored cycle must have an odd number of dashed edges

5. height assignment

   - We assign a height to each vertex. We do not allow bosons and fermions to occupy the same height
   - This is equivalent to directing the edges of the graphs, such that if we travel along a cycle counterclockwise, we meet equal number or edges which are directed with us and edges which are directed against us.

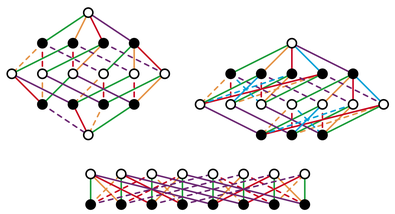

Fig 1: some examples of adinkras.

We obtain a *chromotopology* by relaxing some of these conditions.

Relaxing the Quadrilateral Condition

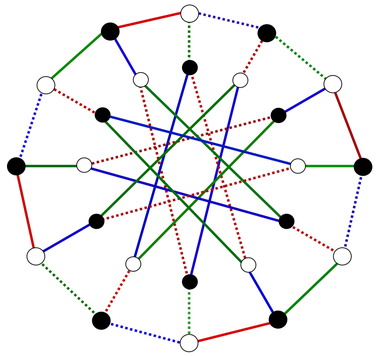

Fig. 2: A possible dashing for the Cayley graph from the generators $(1,2),(1,3),(1,4)$.

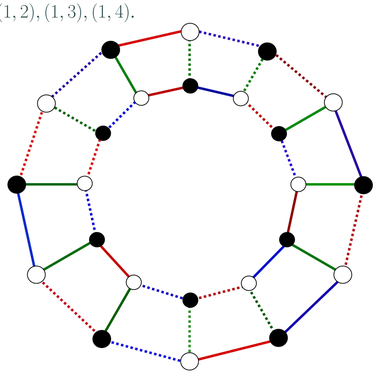

Fig. 3: A possible dashing for the Cayley graph from the generators $(1,2),(1,3),(4,5)$.

Using sub-groups of the permutative groups, it is possible to construct Adinkra like objects with a relaxed quadrilateral condition. If the chosen generators of the sub-group are involutions, then the corresponding Cayley graph will be N-regular colored. These graphs will then reduce to N-gons, where N is even, when all but 2 colors are omitted. Some of these graphs can be oddly dotted and some cannot. The method described below gives the number of possible dashings for any given Adinkra with the relaxed quadrilateral condition.

Counting Dashings

For any colored graph you can construct a matrix, called the 2-colored cycle matrix, where each 2-color cycle corresponds to a row and each edge corresponds to a column, assigning a 0 if the edge is not part of the cycle and a 1 if that edge is in the cycle. The 2-color cycle matrix when multiplied mod-2 by a binary vector, called the dashing vector where 0 corresponds to a solid edge and 1 corresponds to a dashed edge, tells us whether or not a 2-color cycle has a odd number of dashings given the dashing vector. Only if the resulting vector is entirely made of ones does the graph have odd dashing for all 2-color cycles.

Using modular linear algebra we can determine the dimension, and therefore the number of different odd-dashing arrangements we can put on a graph. Say that A is an $M \times N$ binary matrix. We know that a solution to $Ax = b$ exists if and only if b is in the image of A. Once we determine that a solution exists, using the rank nullity theorem, $|null(A)| = M - rank(A)$. We also know that the dimension of the solution is equal to the the nullity.

If A is a MxN 2-color cycle matrix and b, the vector of all ones, is in its image, at least one odd-dashing arrangement exists, and the solution set of $Ax = b$ is of dimension $M - rank(A)$. As there are only two options for each dimension, as all matrices and vectors are over $\mathbb{Z}_2$, this means there are $2^{|null(A)|} = 2^{M-|rank(A)|}$ solutions, and therefore that many odd-dashing arrangements.

Adinkras From Codes

For any N, the N dimensional hypercube is an example of an N colored adinkra. We can obtain all other N colored adinkras as quotients of the N colored adinkra by doubly even codes.

Definition 1 (linear code)**.** *A linear code is a set of codewords, binary vectors of length n, which is closed under addition.*

Definition 2. *A code is doubly even if the weight–that is, the number of ones–of each codeword is divisible by 4.*

We will study the set of doubly even codes of a given weight N by associating it to the following graph $G = V, E$.

- Let V be the set of doubly even codewords of length N, excepting the identity.
- For each pair $u, v \in V$, let $u, v \in V$ if $u + v \in V$. If $u + v \in V$, $u + v$ is also a doubly even codeword. In this case, we call u and v compatible codewords.

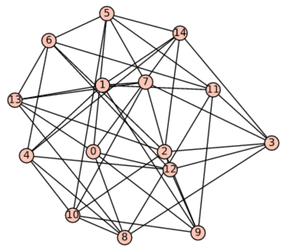

Fig 4: The graph of doubly even codewords of length 6.

Note that a doubly even linear code must be a clique in G: if the code indicated by $U \subset V$ is a doubly even linear code, and $u, v \in U$, then we must have $\{u, v\} \in E$ because the sum of all pairs of codewords in a doubly even code must be doubly even.

However, this condition is not sufficient. For instance, $\{111100000000, 000011110000, 000000001111\}$, is a clique, since all codewords are compatible, however, it is not a doubly even linear code, since the sum 111100000000+000011110000 is not included in the code.

To produce a necessary and sufficient condition, we augment the graph by turning it into a simplicial complex C. (For our purposes, a simplicial complex is a generalization of of a graph in which there can be "edges" on more than two vertices.) For $x, y, z \in V$, let $\{x, y, z\} \in C$ if $x + y = z$. (This is well defined because the code is binary: $x + y = z$ implies $x + z = y$ and $y + z = x$.) If $U \subset V$ is a doubly even code, this simplicial complex will have some nice properties: since each pair of compatible elements has a unique sum in the code, each edge (or 1-simplex) is contained within exactly one 2-simplex. We formalize this condition:

Definition 3 (playground)**.** *A simplicial complex is a playground if all facets are either vertices or 2-simplices and all intersections of 2-simplices are either empty or a single vertex.*

Fig 5. While not formally a playground, this structure inspired this definition.

Definition 4 (playground clique)**.** *A simplicial complex is a playground clique if it is a playground and its set of edges is a clique.*

Theorem 1. *A set of codes $U \subset V$ is a doubly even linear code if and only if the (induced) subcomplex on U is a playground clique.*

This follows from the definition of a playground clique. The clique condition holds if and if all elements are compatible–that is, that all sums are doubly even. The playground condition holds if and only if U is closed under addition.

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