Critical Groups of Graphs with Dihedral Symmetry

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1 Introduction

We will consider the critical group of a graph Γ with an action by the dihedral group D_n. After defining a extended version of the critical group, which we denote $\mathrm{eCrit}(\Gamma)$, we will show the following result, which is similar to [2], [3]:

Theorem 1.1. *Let Γ be a circulant graph on n vertices, and note that such a graph admits a D_n-action by graph automorphisms. Let σ_1 and σ_2 be two involutions which generate D_n. Then*

$$\mathrm{eCrit}(\Gamma) = \mathrm{Fix}(\sigma_1, \mathrm{eCrit}(\Gamma)) \oplus \mathrm{Fix}(\sigma_2, \mathrm{eCrit}(\Gamma)),$$

where $\mathrm{Fix}(\sigma_j, \mathrm{eCrit}(\Gamma))$ *denotes the $\mathbb{Z}$-module of fixed points of σ_j in* $\mathrm{eCrit}(\Gamma)$.

An interesting corollary is that if n is odd, then $\mathrm{eCrit}(\Gamma)$ is the direct sum of two isomorphic submodules, because the two reflections σ_1 and σ_2 are conjugate, and thus have isomorphic fixed-point submodules.

We will prove Theorem 1.1 using the characterization of the critical group in terms of harmonic functions (see [1]), which is substantially different from the approach of [2] and [3], and in some respects simpler.

Our method also applies to more general graphs with dihedral symmerty but the statement is more complicated, so we postpone it until later. We also remark that the fixed point submodule $\mathrm{Fix}(\sigma_j, \mathrm{eCrit}(\Gamma))$ can be identified with the extended critical group of Γ/σ_j (under mild assumptions on the group action), but we will not detail the quotient graph construction in this note.

2 Critical Groups

2.1 Basic Definitions

Definition 2.1 (Graphs)**.** We consider a graph to be an undirected multi-graph with self-looping edges allowed. A graph Γ will be given by a finite set V of vertices, a finite set E of darts (oriented edges), the involution $e \mapsto \overline{e}$ that maps a dart to its reverse, and the map $e \mapsto e_+$ that maps a dart to its starting vertex. We assume $\overline{e} \neq e$. All graphs will be assumed to be connected unless otherwise stated.

Definition 2.2 (Graph Laplacian)**.** The **Laplacian** corresponding to a graph Γ is a matrix Δ with rows and columns indexed by the vertices of Γ. When Γ has no self-loops, the matrix Δ is given by

$$\Delta_{x,y} = \begin{cases} d(x) & x = y \\ -d(x,y) & x \neq y \end{cases}$$

where $d(x)$ is the number of darts exiting from x (the degree of x), and $d(x,y)$ is the number of darts from x to y. If Γ has self-loops, then the Laplacian is defined to be the Laplacian matrix of the graph obtained by deleting the self-loops.

For a graph $\Gamma = (V, E)$, and for a $\mathbb{Z}$-module M, let M^V be the collection of functions from $V \to M$. The matrix Δ defines a linear map $M^V \to M^V$, which we will denote by Δ_M.

Let $\mathbf{1} \in \mathbb{Z}^V \subset \mathbb{Q}^V$ be the vector $(1, \ldots, 1)$. Note that $\Delta_{\mathbb{Z}}(\mathbf{1}) = 0$. For a connected graph, $\ker \Delta_{\mathbb{Q}}$ and $\ker \Delta_{\mathbb{Z}}$ are both spanned by $\mathbf{1}$.

Let $\epsilon : M^V \to M$ be the map which sums the values of a function. Note that $\epsilon \circ \Delta = 0$; that is, $\Delta M^V \subset \ker \epsilon$. In fact, as just mentioned, Δ has rank $|V| - 1$, and so its image in $\mathbb{Q}^V$ must be all of the $(|V| - 1)$-dimensional space $\ker(\epsilon)$.

Definition 2.3 (Critical Group)**.** For a connected graph Γ, the **critical group** $\mathrm{Crit}(\Gamma)$ is $(\ker \epsilon_{\mathbb{Z}}) / \operatorname{im} \Delta_{\mathbb{Z}}$.

Since $\epsilon \circ \Delta = 0$, ϵ descends to the quotient $\operatorname{coker}_{\mathbb{Z}} \Delta$, and we have an exact sequence

$$0 \to \mathrm{Crit}(\Gamma) \to \operatorname{coker} \Delta_{\mathbb{Z}} \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

This sequence splits, with a section of ϵ given by choosing an arbitrary vertex x and sending 1 in $\mathbb{Z}$ back to the function δ_x which is 1 at x and 0 elsewhere. Thus $\operatorname{coker}_{\mathbb{Z}} \Delta \cong \mathbb{Z} \oplus \mathrm{Crit}(\Gamma)$.

2.2 The Critical Group and Harmonic Functions

An alternative perspective on the critical group comes from considering harmonic functions taking values in $\mathbb{Q}/\mathbb{Z}$ (or equivalently $\mathbb{R}/\mathbb{Z}$). This connection was observed earlier by [4, §2], and was further developed by the authors and their collaborators in [1].

Definition 2.4. A function $u \in M^V$ is **harmonic** if $\Delta_M u = 0$. We define

$$\begin{aligned} \mathcal{U}(\Gamma, M) &= \ker_M(\Delta) \\ \widetilde{\mathcal{U}}(\Gamma, M) &= \ker_M(\Delta)/(\text{constant functions}). \end{aligned}$$

Proposition 2.5. $\mathrm{Crit}(\Gamma) \cong \widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z})$.

Proof. We apply the Snake Lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^V & \longrightarrow & \mathbb{Q}^V & \longrightarrow & (\mathbb{Q}/\mathbb{Z})^V & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \Delta_{\mathbb{Z}}} & & \big\downarrow{\scriptstyle \Delta_{\mathbb{Q}}} & & \big\downarrow{\scriptstyle \Delta_{\mathbb{Q}/\mathbb{Z}}} & & \\
0 & \longrightarrow & \mathbb{Z}^V & \longrightarrow & \mathbb{Q}^V & \longrightarrow & (\mathbb{Q}/\mathbb{Z})^V & \longrightarrow & 0,
\end{array}$$

yielding the long exact sequence

$$\begin{array}{l}
0 \longrightarrow \ker \Delta_{\mathbb{Z}} \longrightarrow \ker \Delta_{\mathbb{Q}} \longrightarrow \ker \Delta_{\mathbb{Q}/\mathbb{Z}} \\
\longrightarrow \operatorname{coker} \Delta_{\mathbb{Z}} \xrightarrow{\epsilon} \operatorname{coker} \Delta_{\mathbb{Q}} \longrightarrow \operatorname{coker} \Delta_{\mathbb{Q}/\mathbb{Z}} \longrightarrow 0.
\end{array}$$

In other words, we have

$$\begin{array}{l}
0 \longrightarrow \mathbf{1} \cdot \mathbb{Z} \longrightarrow \mathbf{1} \cdot \mathbb{Q} \longrightarrow \mathcal{U}(\Gamma, \mathbb{Q}/\mathbb{Z}) \\
\longrightarrow \operatorname{Crit}(\Gamma) \oplus \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.
\end{array}$$

Now the image of $\mathbf{1} \cdot \mathbb{Q}$ in $\mathcal{U}(\Gamma, \mathbb{Q}/\mathbb{Z})$ is clearly the constant functions, and the kernel of the map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is $\mathbb{Z}$. Therefore, we have a short exact sequence

$$0 \to \mathcal{U}(\Gamma, \mathbb{Q}/\mathbb{Z})/(\text{constants}) \to \operatorname{Crit}(\Gamma) \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

Since the $\mathbb{Z}$ summand of the middle term must be mapped isomorphically onto the last term, we get an isomorphism

$$\widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Crit}(\Gamma).$$

□

Remark 2.6. The term "extended critical group" is suitable because the proposition above shows that $\operatorname{eCrit}(\Gamma)$ is a group extension of $\operatorname{Crit}(\Gamma)$.

Remark 2.7. In fact, the map $\operatorname{coker} \Delta_{\mathbb{Z}} = \operatorname{Crit}(\Gamma) \oplus \mathbb{Z} \to \mathbb{Z}$ produced by the Snake Lemma is the same as the map induced by ϵ.

2.3 The Extended Critical Group

In order to state and prove Theorem 1.1, it turns out to be convenient to consider an extended version of the critical group, $\operatorname{eCrit}(\Gamma)$. This $\mathbb{Z}$-module is defined by dealing with the one-dimensional degeneracy of Δ in a different way: Rather than forming $\operatorname{coker} \Delta_{\mathbb{Z}}$ and restricting to elements that correspond to $\ker \epsilon$, we will take $\operatorname{coker} \Delta_{\mathbb{Z}}$ and further quotient out by the constant vector $\mathbf{1}$ so that the resulting $\mathbb{Z}$-module is purely a torsion module. Equivalently,

Definition 2.8. Let A be the matrix formed by augmenting Δ with an additional column consisting of the vector $\mathbf{1}$. For a $\mathbb{Z}$-module M, let A_M be the corresponding $\mathbb{Z}$-module morphism $M^V \times M \to M^V$. Then we define

$$\mathrm{eCrit}(\Gamma) := \operatorname{coker} A_{\mathbb{Z}}.$$

We showed before that $\ker \Delta_M$ consists of harmonic M-valued functions, and gave a characterization of $\mathrm{Crit}(\Gamma)$ in terms of harmonic functions. The analogous results for $\mathrm{eCrit}(\Gamma)$ are as follows:

Observation 2.9. The kernel of A_M is isomorphic to the module of functions u such that $\Delta_M u$ is a constant function.

Proof. Let $(u, a) \in M^V \times M$. Then $A_M(u, a) = 0$ is equivalent to

$$\Delta u(x) = -a \text{ for all } x \in V(\Gamma).$$

Thus, u is a function with Δu constant, and conversely if u is such a function, we can take $a = -\Delta u(x)$ and obtain $(u, a) \in \ker A_M$. □

Definition 2.10. We define $\mathcal{W}(\Gamma, M) = \{u \in M^V : \Delta u \text{ is constant}\}$ and set

$$\widetilde{\mathcal{W}}(\Gamma, M) = \mathcal{W}(\Gamma, M)/(\text{constants}).$$

Proposition 2.11. $\mathrm{eCrit}(\Gamma) \cong \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$.

Proof. Similar to before, we apply the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^V \times \mathbb{Z} & \longrightarrow & \mathbb{Q}^V \times \mathbb{Q} & \longrightarrow & (\mathbb{Q}/\mathbb{Z})^V \times (\mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A_{\mathbb{Z}}} & & \downarrow{\scriptstyle A_{\mathbb{Q}}} & & \downarrow{\scriptstyle A_{\mathbb{Q}/\mathbb{Z}}} & & \\ 0 & \longrightarrow & \mathbb{Z}^V & \longrightarrow & \mathbb{Q}^V & \longrightarrow & (\mathbb{Q}/\mathbb{Z})^V & \longrightarrow & 0 \end{array}$$

We observe that $\ker A_{\mathbb{Z}}$ and $\ker A_{\mathbb{Q}}$ both consist of constant functions. Indeed, any function $u \in \mathbb{Q}^V$ with Δu constant must actually have $\Delta u = 0$ because $\sum_{x \in V} \Delta u(x) = 0$. Hence, u must be harmonic and so it is constant. We also observe that $\operatorname{coker} A_{\mathbb{Q}} = 0$ because the columns of A span $\mathbb{Q}^V$ as a $\mathbb{Q}$-vector space. Thus, our long exact sequence is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{1} \cdot \mathbb{Z} & \longrightarrow & \mathbf{1} \cdot \mathbb{Q} & \longrightarrow & \mathcal{W}(\Gamma, \mathbb{Q}/\mathbb{Z}) \\ & & & & & \swarrow & \\ \mathrm{eCrit}(\Gamma) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

It follows that $\mathrm{eCrit}(\Gamma) \cong \mathcal{W}(\Gamma, \mathbb{Q}/\mathbb{Z})/(\text{constants})$. □

We can relate these two groups together as follows.

Proposition 2.12. *Let $n = |V|$. There is a short exact sequence*

$$0 \to \mathrm{Crit}(\Gamma) \to \mathrm{eCrit}(\Gamma) \to \mathbb{Z}/n \to 0.$$

Proof. Because $\Delta_{\mathbb{Q}/\mathbb{Z}} u = 0$ implies $\Delta_{\mathbb{Q}/\mathbb{Z}} u =$ constant, we have an inclusion $\widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$.

We also have a map $\widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ which maps u to the constant value of $\Delta_{\mathbb{Q}/\mathbb{Z}} u$. But note that since $\sum_{x \in V} \Delta_{\mathbb{Q}/\mathbb{Z}} u(x) = 0$, the value of Δu must in fact lie in the submodule of $\mathbb{Q}/\mathbb{Z}$ which is isomorphic to $\mathbb{Z}/n$.

Putting this together, we have a sequence of maps

$$0 \to \widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi} \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} \mathbb{Z}/n \to 0.$$

Clearly, ϕ is injective and $\operatorname{im} \phi = \ker \psi$. To see that ψ is surjective, pick a value $a \in \mathbb{Z}/n$ (considered as a submodule of $\mathbb{Q}/\mathbb{Z}$). We can pick a function $\tilde{a} \in \mathbb{Q}^V$ such that $\tilde{a}(x) + \mathbb{Z} = a$ for all x and $\sum_{x \in V} a(x) = 0$. But then there exists $\tilde{u} \in \mathbb{Q}^V$ such that $\Delta_{\mathbb{Q}} \tilde{u} = \tilde{a}$. If u is the projection of $\tilde{u}$ into $(\mathbb{Q}/\mathbb{Z})^V$, then we have $\Delta_{\mathbb{Q}/\mathbb{Z}} u = a$, which proves surjectivity of ψ. □

Remark 2.13. An alternative way to obtain the map $\mathrm{Crit}(\Gamma) \to \mathrm{eCrit}(\Gamma)$ is by the composition

$$\mathrm{Crit}(\Gamma) \to \operatorname{coker} \Delta_{\mathbb{Z}} \to \operatorname{coker} \Delta_{\mathbb{Z}} / (\mathbb{Z} \cdot \mathbf{1}) = \operatorname{coker} A_{\mathbb{Z}}.$$

It follows from the naturality of Snake Lemma exact sequence that this map is equivalent to the inclusion $\widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$. Since we will not use this fact later in this note, we leave the details to the reader.

3 Graphs with Dihedral Symmetry

3.1 Symmetry Actions

Let G be a group which acts by automorphisms on the graph Γ. If M is a $\mathbb{Z}$-module, then G acts on M^V by

$$gu(x) = u(g^{-1}x).$$

Similarly, G acts on $M^V \times M$ by the product of the G-action on M^V with the trivial action of G on M. This means that M^V and $M^V \times M$ are modules over the group ring $\mathbb{Z}G$. Furthermore, since G acts by graph automorphisms, we have

$$g \Delta_M u = \Delta_M g u, \qquad g A_M(u, a) = A_M g(u, a),$$

and hence Δ and A are $\mathbb{Z}G$-module homomorphisms.

Therefore, the construction of $\mathrm{Crit}(\Gamma)$, $\mathrm{eCrit}(\Gamma)$, $\widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z})$, and $\widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$ can be performed in the category of $\mathbb{Z}G$-modules. The Snake Lemma argument holds in the category of $\mathbb{Z}G$-modules as well. Therefore,

Proposition 3.1. *We have*

$$\mathrm{Crit}(\Gamma) \cong \widetilde{\mathcal{U}}(\Gamma, \mathbb{Q}/\mathbb{Z})$$
$$\mathrm{eCrit}(\Gamma) \cong \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$$

as $\mathbb{Z}G$-modules. Moreover, we have a short exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathrm{Crit}(\Gamma) \to \mathrm{eCrit}(\Gamma) \to \mathbb{Z}/n \to 0,$$

where G acts trivially on $\mathbb{Z}/n$.

3.2 Dihedral Action for Circulant Graphs

We are now ready to prove Theorem 1.1. Suppose Γ is a circulant graph on n vertices. First, let us explain the assertion that D_n acts on Γ by graph automorphisms.

Observe that a circulant graph with n vertices is equivalent to a Cayley graph of $\mathbb{Z}/n$ for some choice of generators. Then we can index the vertices by $\mathbb{Z}/n$. Let us label them x_j for $j \in \mathbb{Z}/n$.

Let D_n denote the dihedral group of order $2n$, which is given by the group presentation:

$$D_n = \langle r, s : r^n = s^2 = (rs)^2 = 1\rangle.$$

Then D_n acts on our circulant graph Γ by

$$rx_j = x_{j+1}, \qquad sx_j = x_{-j}.$$

Now consider two reflections σ_1 and σ_2 which generate D_n. We want to show that (under appropriate assumptions)

$$\mathrm{eCrit}(\Gamma) = \mathrm{Fix}(\sigma_1, \mathrm{eCrit}(\Gamma)) \oplus \mathrm{Fix}(\sigma_2, \mathrm{eCrit}(\Gamma)).$$

We will first show that the two fixed point sub-$\mathbb{Z}$-modules span $\mathrm{eCrit}(\Gamma)$, and after that we will prove that their intersection is trivial.

Toward our first goal,

Lemma 3.2. *Let Y be a D_n-set and consider $\mathbb{Z}^S$ as a left $\mathbb{Z}D_n$ module. Suppose that σ_1 and σ_2 generate D_n, and that each orbit of S has either a fixed point of σ_1 or a fixed point of σ_2. Then we have*

$$\mathbb{Z}^Y = \mathrm{Fix}(\sigma_1, \mathbb{Z}^Y) + \mathrm{Fix}(\sigma_2, \mathbb{Z}^Y).$$

Proof. By considering each orbit separately, we may reduce to the case where S has only one orbit. For the duration of the proof, denote

$$N = \mathrm{Fix}(\sigma_1, \mathbb{Z}^Y) + \mathrm{Fix}(\sigma_2, \mathbb{Z}^Y).$$

Let $y_0 \in Y$ be a fixed point of one of the σ_j's. Then clearly $\delta_{y_0} \in N$.

Furthermore, we claim that if $y \in S$ and $\delta_y \in N$, then we have $\delta_{\sigma_j y} \in N$ also. This follows from the observation that

$$\delta_{\sigma_j y} = (\delta_y + \delta_{\sigma_j y}) - \delta_y,$$

and $\delta_y + \delta_{\sigma_j y} \in N$ because it is a fixed point of σ_j.

Now we have shown that $\delta_{x_0} \in N$ and $\delta_y \in N$ implies $\delta_{\sigma_j y} \in N$. Because σ_1 and σ_2 generate D_n, D_n acts transitively on S, all the basis vectors $\delta_y \in \mathbb{Z}^S$ are contained in N as desired. □

Lemma 3.3. *Let Γ be a circulant graph of size n. Let σ_1 and σ_2 be two reflections which generate D_n. Then*

$$\mathrm{eCrit}(\Gamma) = \mathrm{Fix}(\sigma_1, \mathrm{eCrit}(\Gamma)) + \mathrm{Fix}(\sigma_2, \mathrm{eCrit}(\Gamma)).$$

Proof. Observe that either σ_1 or σ_2 must have a fixed point in V (since V is the canonical D_n-set on n elements). This is immediate when n is odd, and the even case can be proved by contradiction; we leave this to the reader.

Therefore, we have

$$\mathbb{Z}^V = \mathrm{Fix}(\sigma_1, \mathbb{Z}^V) \oplus \mathrm{Fix}(\sigma_2, \mathbb{Z}^V).$$

But recall that $\mathrm{eCrit}(\Gamma)$ is the quotient of $\mathbb{Z}^V$ by $\mathrm{im}\, A_{\mathbb{Z}}$, as a $\mathbb{Z}D_n$-module. Thus, if $[f] = f + \mathrm{im}\, A_{\mathbb{Z}}$ is an element of $\mathrm{eCrit}(\Gamma)$, we may write $f = f_1 + f_2$, where $f_j \in \mathrm{Fix}(\sigma_j, \mathbb{Z}^V)$. This implies that $[f] = [f_1] + [f_2]$, where $[f_j] \in \mathrm{Fix}(\sigma_j, \mathrm{eCrit}(\Gamma))$. □

Having shown that $\mathrm{Fix}(\sigma_1, \mathrm{eCrit}(\Gamma))$ and $\mathrm{Fix}(\sigma_2, \mathrm{eCrit}(\Gamma))$ span $\mathrm{eCrit}(\Gamma)$, we now turn to the proof that they are linearly independent.

Lemma 3.4. *Suppose that n is odd and that σ_1 and σ_2 are reflections which generate D_n. Then*

$$\mathrm{Fix}(\sigma_1, \mathrm{eCrit}(\Gamma)) \cap \mathrm{Fix}(\sigma_2, \mathrm{eCrit}(\Gamma)) = 0.$$

Proof. Since σ_1 and σ_2 generate D_n, this amounts to showing that $\mathrm{Fix}(D_n, \mathrm{eCrit}(\Gamma)) = 0$. Because $\mathrm{eCrit}(\Gamma)$ and $\widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$ are isomorphic as $\mathbb{Z}D_n$-modules, it suffices to show that $\mathrm{Fix}(D_n, \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})) = 0$.

Suppose that $[u] = u + \mathbf{1} \cdot (\mathbb{Q}/\mathbb{Z})$ is a fixed point of every element of D_n in $\widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$. In particular, this implies that for the generating rotation r, we have

$$r[u] = [u], \qquad ru = u + c,$$

where c is some constant in $\mathbb{Q}/\mathbb{Z}$. Therefore, we have

$$u(x_j) = u(x_0) - jc \text{ for } j \in \mathbb{Z}/n.$$

In particular, $nc = 0$.

Also, if s is the reflection across x_0, we have $su = u + c'$ for some constant c'. But then $u(0) = su(0) = u(0) + c'$, so that $c' = 0$. Moreover,

$$u(x_0) - jc = u(x_j) = su(x_j) = u(x_0) + jc$$

for each $j \in \mathbb{Z}/n$, and hence $2c = 0$. Because n is odd and $nc = 0$, this implies that $c = 0$. Therefore, $ru = u$, which means that u is constant. Therefore, $[u] = 0$ in $\widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z})$. □

3.3 Generalization

Our method applies to more general dihedral actions. The generalization of Lemma 3.3 is straightforward, and we leave the modification of the proof to the reader:

Lemma 3.5. *Suppose that D_n acts by graph automorphisms on a graph Γ. Suppose that reflections σ_1 and σ_2 generate D_n, and suppose that every orbit of $V(\Gamma)$ has a fixed point of either σ_1 or σ_2. Then*

$$\mathrm{eCrit}(\Gamma) = \mathrm{Fix}(\sigma_1, \mathrm{eCrit}(\Gamma)) + \mathrm{Fix}(\sigma_2, \mathrm{eCrit}(\Gamma)).$$

The linear independence argument also applies to a more general situation:

Lemma 3.6. *Suppose that n is odd and D_n acts by graph automorphisms on Γ. Suppose that V has an orbit $\mathcal{O}$ which is isomorphic as a D_n-set to the canonical D_n-set with n elements (a.k.a. the vertices of the n-gon). Then $\mathrm{Fix}(D_n, \widetilde{\mathcal{W}}(\Gamma, \mathbb{Q}/\mathbb{Z}))$ consists of functions which are constant on each orbit of V.*

Remark 3.7. Under additional hypotheses on the orbits, we can compute functions which are constant on each orbit by examining the extended critical group on the quotient graph.

Proof. Let $[u]$ be a fixed point. Then $ru = u + c$ and $su = u + c'$. Applying the same argument as in Lemma 3.4 to the distinguished orbit $\mathcal{O}$, we see that $c = c' = 0$. Thus, u is fixed by D_n (not modulo constants). So u is constant on each orbit. □

4 Concluding Remarks

In the proof of Theorem 1.1, we proved that the two fixed point submodules span $\mathrm{eCrit}(\Gamma)$ by using the characterization of $\mathrm{eCrit}(\Gamma)$ as a **quotient** of $\mathbb{Z}^V$. On the other hand, we proved that they were independent using the characterization of $\mathrm{eCrit}(\Gamma)$ as **submodule** of $(\mathbb{Q}/\mathbb{Z})^V$ (modulo constants). Thus, our method relied on taking the two characterizations developed in §2 and playing them off of each other, as in [1].

References

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