MINICOURSE ON HODGE THEORY

Abstract. This is a brief (and very biased) introduction to Hodge theory.

Contents

CAUTION: This note is meant to be super informal and imprecise, which will contain jumps on arguments, errors, misinterpretations, etc.

References: [Voi02], [SS22], [Sch73], [CKS86], [Che21], [BJ17]

1. Hodge theory for compact Kähler manifolds

1.1. **Hodge decomposition.** Let X be a complex manifold. Then we can consider three different vector spaces coming from topology, algebraic geometry, and analysis.

- Singular cohomology: $H^k(X, \mathbb{C})$
- Algebraic de Rham cohomology $H^k_{dR}(X) = \mathbb{H}(X, \Omega^\bullet_X)$
- Space of harmonic forms $\mathcal{H}^k(X)$.

The three vector spaces are isomorphic in the following way.

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{\partial} \Omega^1_X \to \ldots \to \Omega^n_X$$

is an exact complex by $\overline{\partial}$-Poincaré lemma. This gives an isomorphism

$$H^k(X, \mathbb{C}) \simeq H^k_{dR}(X).$$

Fix a Riemannian metric on X. Then we can measure k-forms and we have a volume form. The adjoint operator d^* of d is defined to satisfy the following equation

$$\int_X \langle d\alpha, \beta\rangle \operatorname{vol} = \int_X \langle \alpha, d^*\beta\rangle \operatorname{vol}.$$

Definition 1.1. A k-form is d-harmonic if

$$\Delta_d \alpha = (dd^* + d^*d)\alpha = 0$$

Note that a k-form is harmonic if and only if $d\alpha = d^*\alpha = 0$ since we have

$$\langle \Delta_d \alpha, \alpha\rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2.$$

We can check that the map $\mathcal{H}^k \to H^k(X, \mathbb{C})$ given by $\alpha \mapsto [\alpha]$ is an isomorphism via orthogonal projection.

Remark 1.2. Until now, we did not use anything about the existence of the Kähler form ω on X. The upshot is that the Kähler form ω gives an additional structure on the two vector spaces $\mathcal{H}^k$ and H^k_{dR}.

The key ingredient for the Hodge decomposition is the following relation

$$\Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$$

which we will assume this. If a k-form u is d-harmonic, we can deduce that if we express $u = \sum_{p+q=k} u^{p,q}$ as (p,q)-forms, then each $u^{p,q}$ are also d-harmonic, and the same time $\overline{\partial}$-harmonic. Hence, we have

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

where $\mathcal{H}^{p,q}$ are $\overline{\partial}$-harmonic (p,q)-forms given that X admits a Kähler form ω. Also, using the $\overline{\partial}$-Poincaré lemma again, we have an exactness of the following complex

$$0 \to \Omega^p_X \to \cdots \to \mathcal{A}^{p,q}_X \to \cdots \to \mathcal{A}^{p,n}_X$$

and we have the following isomorphism

$$\mathcal{H}^{p,q} \simeq H^q(X, \Omega^p_X).$$

Once, we have this we get the equality

$$h^k = \sum_{p+q=k} h^{p,q}$$

where $h^{p,q}$ are the Hodge numbers.

On the algebraic side, the algebraic de Rham complex $\Omega^\bullet_X$ has a filtration by truncation (sometimes called the Frölicher filtration) given by

$$F^p\Omega^\bullet_X = \Omega^{\geq p}_X = [\cdots 0 \to \Omega^p_X \to \cdots \to \Omega^n_X].$$

Then we have an associated spectral sequence whose E_1 page is

$$E_1^{p,q} = H^q(X, \Omega^p_X) \implies H^{p+q}_{dR}(X).$$

However, we have an equality between the dimensions so we have the degeneration of this spectral sequence at page 1.

Note that the spectral sequence gives a filtration F^pH^k that depends only on the holomorphic data. The successive quotients $F^pH^k/F^{p+1}H^k$ are isomorphic to $H^{p,q}$, but when we try to express the cohomology groups as

$$H^k = \bigoplus H^{p,q},$$

the decomposition involves some non-holomorphic input because we have to introduce harmonic forms. This fact will be very important on studying families of manifolds (variation of Hodge structures).

1.2. **Kähler identities.** Let $\mathcal{A}_X = \bigoplus \mathcal{A}_X^{p,q}$ be the space of all differential forms. Then wedging by ω gives an operator

$$L = \omega \wedge \bullet : \mathcal{A}_X \to \mathcal{A}_X$$

which is nilpotent. Then we have the adjoint operator Λ which decreases the degree by 2. Then we have an operator H which acts as $(n-k)$ on each degree. The upshot is that H, L and Λ makes $\mathcal{A}_X$ as a representation of $\mathfrak{sl}_2(\mathbb{C})$. Also, since L is a closed form, we have the commutativity relation $[L, \partial] = [L, \overline{\partial}] = 0$ and $[\partial^*, L] = -i\overline{\partial}$ and $[\overline{\partial}^*, L] = i\partial$. Playing with these relations, we get the identity

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d.$$

1.3. **Lefschetz decomposition.** If we fix a Kähler form ω, this gives an additional structure on the cohomology $H^k(X, \mathbb{C})$. This is because the Lefschetz operator L acts on the total cohomology

$$H^\bullet(X, \mathbb{C}) = \bigoplus_k H^k(X, \mathbb{C})$$

and we can upgrade this into a representation of $\mathfrak{sl}_2(\mathbb{C})$. Hence, we have primitive parts of the cohomology and also the hard Lefschetz.

1.4. **Hodge structures.** Note that $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ by universal coefficient theorem and therefore, the conjugation makes sense on $H^k(X, \mathbb{C})$. On the otherhand, we have the decomposition $H^k = \oplus H^{p,q}$. Note that for a given harmonic form u of type (p, q), the conjugate $\bar{u}$ is of type (q, p) which gives an isomorphism

$$H^{p,q} = \overline{H^{q,p}}.$$

This gives a rich structure of the cohomology group $H^k(X, \mathbb{Z})$ and it is definitely meaningful to give an abstract definition for it.

Definition 1.3. A $\mathbb{Z}$-Hodge structure of weight k is a free abelian group $V_{\mathbb{Z}}$ with a decreasing filtration F^p on $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$ satisfying the following property.

$$F^pV \cap \overline{F^{q+1}V} = 0$$

for $p + q = k$. Once we have this, we recover the usual Hodge decomposition $H^{p,q} = F^pV \cap \overline{F^qV}$.

The Hodge structure often comes with a polarization. If we have a Kähler form ω, then we have a bilinear pairing on $H^k(X, \mathbb{C})$ given by

$$\alpha, \beta \mapsto \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Note that this pairing is skew-symmetric if k is odd, and symmetric if k is even. Therefore, we have an induced hermitian form

$$H(\alpha, \beta) = i^k Q(\alpha, \bar{\beta}).$$

This polarization satisfies the following property.

- Each component $H^{p,q}$ and the Lefschetz decomposition is mutually orthogonal with respect to H.
- On the primitive component $H^k(X)_{prim}$, we have

  $$i^{p-q-k}(-1)^{\frac{k(k-1)}{2}} H > 0.$$

  This means that the signs of the definiteness of H alternates when p varies.

2. Hodge theory for families of compact Kähler manifolds

In this section $f : X \to S$ is a proper holomorphic submersion (or smooth and proper (projective) morphism between smooth quasi-projective varieties over $\mathbb{C}$).

Now, we move on to the relative situation and consider families of Kähler manifolds. Namely, we have a proper (projective) holomorphic submersion $f : X \to S$ where each fibre X_s is Kähler (or projective). Then, we want to study how the singular cohomology $H^k(X_s, \mathbb{C})$ and their Hodge decomposition $H^{p,q}(X_s, \mathbb{C})$ varies in family.

2.1. **Invariance of Hodge numbers.**

First, we need a way to identify or compare the cohomology groups $H^k(X_s, \mathbb{C})$ for different $s \in S$. By Ehreshmann's theorem, these groups are all isomorphic.

Theorem 2.1 (Ehreshmann)**.** *Let $f : X \to S$ be a proper submersion. Then for each $s \in S$, there exists a neighborhood $U \ni s$ such that $f^{-1}(U) \simeq_{\mathcal{C}^\infty} U \times X_s$ over U. In other words, the isomorphism commutes with the projection to U.*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\quad\simeq\quad} & U \times X_s \\ & \searrow \quad \swarrow & \\ & U. & \end{array}$$

From this fact, we see that for $s, s' \in U$, we have the following isomorphism (identification) $H^k(X_s, \mathbb{C}) \simeq H^k(X_{s'}, \mathbb{C})$ given by

$$X_s \hookrightarrow f^{-1}(U) \hookleftarrow X_{s'}.$$

This insures that $h^k(X_s, \mathbb{C})$ is a constant function. On the other hand, if $f : X \to S$ is projective, then the semi-continuity theorem says that

$$h^{p,q}(X_s) = h^q(X_s, \Omega^p_{X_s})$$

is upper-semicontinuous. However, using this fact and the Hodge decomposition, we see that these numbers do not depend on s.

2.2. Classical Riemann-Hilbert correspondence.

Theorem 2.2 (Classical Riemann-Hilbert correspondence)**.** *We have a correspondence between local systems and vector bundle with integrable connections. This is given by*

$$\mathbb{V} \mapsto \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$$

and

$$\mathcal{V} \mapsto \mathcal{V}^{\nabla} := \ker(\mathcal{V} \to \Omega^1_X \otimes \mathcal{V}).$$

Note that local system is a purely topological data and they correspond to a representation of $\pi_1(X)$.

By Ehreshman's theorem, we see that the sheaf $R^k f_* \mathbb{C}_X$ is a local system on S, and therefore, we have an associated vector bundle $\mathcal{H}^k = R^k f_* \mathbb{C}_X \otimes_{\mathbb{C}} \mathcal{O}_S$ with an integrable connection ∇ which is often called the Gauss-Manin connection. We can consider the sub-vector bundles parametrizing the Hodge decomposition $H^{p,q}$, but they don't vary holomorphically since the Hodge decomposition involves complex conjugate. However, we will soon see that the Hodge filtration $F^p H = \oplus_{p' \geq p} H^{p',q'}$ is varies holomorphically, which means that $F^p \mathcal{H}^k$ is a holomorphic subbundle of $\mathcal{H}^k$.

2.3. Period Domain and Period Maps.

The period domain is a space parametrizing the polarizable Hodge structures. Suppose we have an integral lattice $V_{\mathbb{Z}}$ with a quadratic form $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$. We will consider the possible Hodge structures that we can give to $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Hence, the Hodge filtration is given is parametrized by $\mathcal{D}^{\vee}$ which is the flag variety with appropriate numbers. Also, we consider $\mathcal{D} \subset \mathcal{D}^{\vee}$ to be the open subset where the signature condition for Q is satisfied. We define a real group

$$G = \{g \in \mathrm{GL}(V) : Q(gv, gw) = Q(v, w)\} \subset \mathrm{GL}(V)$$

note that G acts transitively on $\mathcal{D}$ and $\mathrm{GL}(V)$ acts transitively on $\mathcal{D}^{\vee}$.

Let S be simply connected and fix a reference point s_0. Then the local system $R^k f_* \mathbb{C}_X$ can be trivialized by $\underline{H^k(X_{s_0}, \mathbb{C})}$. For a proper submersion $f : X \to S$, we denote by $b^{p,k} = \dim_{\mathbb{C}} F^p H^k(X_s, \mathbb{C})$ which is independent of s as we seen before. Then we have a map

$$\mathcal{P} : S \to D, \qquad s \mapsto (F^p H^k(X_s, \mathbb{C}))_p.$$

This is called the period map. The fact that $F^p \mathcal{H}^k$ is a holomorphic subbundle exactly corresponds to the (local) period map being holomorphic.

There are two approaches for the holomorphicity, one coming from the Cartan-Lie formula, the other which is 'purely algebraic(?)' by constructing the subbundle $F^p \mathcal{H}^k$ only using coherent data (following Katz-Oda).

2.4. Cartan-Lie formula.

The Cartan-Lie formula allows us to compute the Gauss-Manin connection. We consider Ω a smooth differential k-form such that $\Omega|_{X_s}$ is closed for each $s \in S$. Then

$$s \mapsto [\Omega|_{X_s}]$$

gives a $\mathcal{C}^\infty$-section ω on $\mathcal{H}^k$. The goal is to give an explicit way of computing $\nabla\omega$. We fix a trivialization

$$T : X \simeq X_s \times B$$

where $B \subset S$ is an open ball. Via this trivialization, we view $\phi_b = \Omega|_{X_b}$ as a k-form on X_s via the isomorphism $T|_{X_b} : X_b \xrightarrow{\simeq} X_s$. Then for a tangent vector $u \in T_sS$, we get

$$\nabla_u\omega = [d_u\phi_b].$$

Fix a coordinate system t_i of B and let

$$\Omega = \Phi + \sum dt_i \wedge \psi_i + \Omega'$$

where Ω' has two or more terms of dt_i and ψ_i does not have dt_i, and $\Phi|_{X_s\times b} = \phi_b$. Since ϕ_b is a closed form, we get

$$d\Omega = \sum_i dt_i \wedge \frac{\partial\phi_b}{\partial t_i} - \sum dt_i \wedge d\psi_i + d\Omega'$$

If we denote by v a horizontal lift of u, then we get

$$i_v d\Omega|_{X_s} = d_u\phi_b - \sum_i u_i d\psi_i$$

where $u = \sum u_i \frac{\partial}{\partial t_i}$. But since $d\psi_i$ is exact, we get

$$\nabla_u\omega = [i_v d\Omega|_{X_s}].$$

2.5. **Holomorphicity of the Period map and Griffiths transversality.** In order to show that the period map is holomorphic, it is enough to show that each component of the map to the Grassmannian is holomorphic. First, we identify the tangent space of a Grassmannian. For a vector space V and $W \in \mathrm{Gr}(V)$, the tangent space $T_{\mathrm{Gr}(V),W}$ can be canonically identified as

$$T_{\mathrm{Gr}(V),W} = \mathrm{Hom}(W, V/W).$$

If we fix a basis $w_1, \cdots, w_r$ of W and $\phi \in \mathrm{Hom}(W, V/W)$, then the tangent vector of the Grassmannian corresponds to a holomorphic arc

$$\epsilon \mapsto \mathrm{span}(w_1 + \epsilon\phi(w_1), \cdots, w_r + \epsilon\phi(w_r)).$$

Here, we have V/W on the target of ϕ since we want to parametrize the 'direction differing from W'.

Suppose that S is contractible by possibly shrinking. Then we can identify $H^k(X, \mathbb{C})$ with $H^k(X_b, \mathbb{C})$ for all $b \in S$. Consider the p-th component of the period map

$$\mathcal{P}^p : S \to \mathrm{Gr}(*, V), \qquad b \mapsto F^pH^k(X_b, \mathbb{C})$$

The ultimate goal is to show that the differential

$$d\mathcal{P}^p : T_bS \to \mathrm{Hom}(F^pH^k(X_b, \mathbb{C}), H^k(X, \mathbb{C})/H^k(X_b, \mathbb{C}))$$

is $\mathbb{C}$-linear. In other words, the complexified map vanishes on the $(0,1)$-components of $T_bS \otimes_{\mathbb{R}} \mathbb{C}$. Let $u \in T_bS$ and $\sigma \in F^pH^k(X_b, \mathbb{C})$. Then the derivative $d\mathcal{P}^p u(\sigma)$ can be computed as

$$\nabla_u\tilde{\sigma}$$

where $\tilde{\sigma}$ is a section on $\mathcal{H}^k$ such that $\tilde{\sigma}(b) = \sigma$ and $\tilde{\sigma}(b') \in F^p\mathcal{H}^k$ for all b'. We use the Cartan-Lie formula to compute this. Hence, consider $\Omega \in F^p\mathcal{A}^k_X$ such that the class of $\Omega|_{X_{b'}}$ is $\tilde{\sigma}(b')$. Then we have

$$\nabla_u \tilde{\sigma} = [i_v(d\Omega)|_{X_b}] \mod F^p H^k(X_b)$$

We see that if v is of type $(0,1)$, then $i_v(d\Omega)$ still has at least p holomorphic parts, and therefore $\nabla_u\tilde{\sigma} = 0$. Also, for $(1,0)$-parts, $i_v(d\Omega)$ still has at least $p-1$ holomorphic parts in the terms. Therefore, the image of the differential actually lies in

$$\mathrm{Hom}(F^p H^k(X_b, \mathbb{C}), F^{p-1}H^k(X_b,\mathbb{C})/F^pH^k(X_b,\mathbb{C})).$$

This phenomenon is called the Griffiths transversality. We can rephrase this in terms of connections.

Theorem 2.3. *Let $\mathcal{H}^k = R^k f_*\mathbb{C}_X \otimes_{\mathbb{C}} \mathcal{O}_S$ with the Gauss-Manin connection ∇. Then $F^p\mathcal{H}^k$ is a holomorphic subbundle of $\mathcal{H}^k$ and the image of $F^p\mathcal{H}^k$ through ∇ lies inside $\Omega^1_S \otimes F^{p-1}\mathcal{H}^k$.*

2.6. **Algebraic approach to Griffiths Transversality.** Here, we give a purely algebraic (coherent) approach to Griffiths transversality. The key idea is not to use $\mathbb{C}_X$, but use the de Rham complex $\Omega^\bullet_X$ instead. For simplicity, we assume that $\dim S = 1$. Then we have the following exact sequence

$$0 \to f^*\Omega_S \to \Omega_X \to \Omega_{X/S} \to 0.$$

Since $f^*\Omega_S$ has rank 1, by taking the wedge product, we have the short exact sequence

$$0 \to f^*\Omega_S \otimes \Omega^p_{X/S} \to \Omega^{p+1}_X \to \Omega^{p+1}_{X/S} \to 0.$$

Hence, we have a short exact sequence of complexes

$$0 \to f^*\Omega_S \otimes \Omega^{\bullet-1}_{X/S} \to \Omega^\bullet_X \to \Omega^\bullet_{X/S} \to 0$$

Note that by the (relative) Poincaré lemma, we have

$$\mathbb{C}_X \simeq_{qis} \Omega^\bullet_X \qquad \text{and} \qquad f^{-1}\mathcal{O}_B \simeq_{qis} \Omega^\bullet_{X/S}$$

By the projection formula, we have

$$R^k f_*(f^{-1}\mathcal{O}_B) = \mathcal{O}_B \otimes_{\mathbb{C}} R^k f_* \mathbb{C}_X = \mathcal{H}^k.$$

Hence, taking $\mathbf{R}f_*$ to the short exact sequence above, we get

$$\cdots \to R^k f_*\mathbb{C}_X \to \mathcal{H}^k \xrightarrow{\nabla} \Omega_S \otimes \mathcal{H}^k \to \cdots$$

Remark 2.4. It is a priori not clear that this map ∇ satisfies the Leibnitz rule

$$\nabla(fs) = df \otimes s + f\nabla s.$$

This can be obtained by carefully looking at the connecting map. Also, this map is not $\mathcal{O}_S$-linear since the differential map on $\Omega^\bullet_X$ is not $\mathcal{O}_B$-linear.

The Griffiths transversality should follow from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{p-1}\Omega^{\bullet-1}_{X/S} & \longrightarrow & F^p\Omega^\bullet_X & \longrightarrow & F^p\Omega^\bullet_{X/S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{\bullet-1}_{X/S} & \longrightarrow & \Omega^\bullet_X & \longrightarrow & \Omega^\bullet_{X/S} & \longrightarrow & 0.
\end{array}$$

2.7. **Polarization on VHS.** The bilinear pairing Q comes from the intersection pairing. Therefore, if we have flat sections σ, τ on $\mathcal{H}^k$, then $Q(\sigma, \tau)$ should be constant. This tells us that the bilinear form Q is compatible with the connection ∇. In other words, we have

$$dQ(\sigma, \tau) = Q(\nabla\sigma, \tau) + Q(\sigma, \nabla\tau).$$

Since we have a signature condition for this Q, the vector bundle $\mathcal{H}^k$ has an additional structure of an hermitian vector bundle by performing appropriate sign changes on each Hodge bundle. This serves as a key analytic ingredient for studying variation of Hodge structures.

3. Variation of Hodge structures

Now, we focus on an abstract tool for studying the vector bundle $\mathcal{H}^k = R^k f_* \mathbb{C}_X \otimes_{\mathbb{C}} \mathcal{O}_S$. We try to forget the map $f : X \to S$ and encode the Hodge theoretic information as a vector bundle on S. The key ingredients should be

(1) Local system $\mathbb{V}$ and the corresponding vector bundle $\mathcal{V}$ with a flat connection ∇. (The local system can be integral, rational, or $\mathbb{C}$) topological information
(2) A filtration F^p on $\mathcal{V}$ with the transversality condition (algebraic information)

$$\nabla(F^p\mathcal{V}) \subset \Omega^1_S \otimes F^{p-1}\mathcal{V}$$

(3) A sesquilinear pairing $Q : \mathcal{V} \times \mathcal{V} \to \mathcal{A}_S$ which is compatible with ∇ and satisfying appropriate signature conditions. analytic information

Remark 3.1. Note that from now on, we are switching to use sesquilinear pairing for Q.

3.1. **Harmonic Bundles approach.** We have Hodge decomposition for each fibre $\mathbb{V}_s$ which does not vary holomorphically. However, it is still worth to keep track on the Hodge decomposition. For this, we need a tool for going back and forth between holomorphic vector bundles and $\mathcal{C}^\infty$-bundles.

Let E be a smooth vector bundle on a complex manifold S. Then giving a structure of a holomorphic vector bundle on E is equivalent to giving a flat connection $\overline{\partial}$ of type $(0,1)$. Once we have a flat connection of type $(0,1)$, then $\mathcal{E} = \ker(\overline{\partial} : E \to \mathcal{A}^{0,1}_X \otimes_{\mathcal{A}_X} E)$ is the wanted holomorphic vector bundle. Also, we can recover the structure of a smooth vector bundle from the holomorphic vector bundle $\mathcal{E}$ by

$$E = \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{A}_S.$$

Since $\mathcal{V}$ is a holomorphic vector bundle with flat connection (of type (1,0)), we get an induced smooth vector bundle E with flat connection $E \to \mathcal{A}^1_X \otimes E$. The $(1,0)$ part of this connection should be ∇ and $(0,1)$ part should be $\overline{\partial}$. Now, E decomposes into smooth vector bundles

$$E =_{\mathcal{C}^\infty} \bigoplus_{p+q=k} E^{p,q}.$$

The sesquilinear pairing Q should respect this decomposition in the following sense:

(1) $E^{p,q}$ are mutually orthogonal with respect to Q

(2) $(-1)^q Q$ is positive definite on each $E^{p,q}$. In other words, if we define

$$h(u,v) = \sum_q (-1)^q Q(u^{p,q}, v^{p,q})$$

then h gives a hermitian metric on the vector bundle E.

Now, we see the behavior of the connection $d : E \to \mathcal{A}^1_X \otimes E$. We break d into $d' + d''$ by holomorphic and anti-holomorphic part. Since

$$F^p E = E^{p,q} \oplus E^{p+1,q-1} \oplus \cdots$$

is a holomorphic subbundle, d'' is preserves $F^p E$. Since

$$\partial Q(u,v) = Q(d'u, v) + Q(u, d''v),$$

and orthogonality, d' preserves

$$\overline{F^{k-p}E} = E^{p,q} \oplus E^{p-1,q+1} \oplus \cdots$$

However, by Griffiths transversality, we d' has only two components $d' = \nabla^{1,0} + \theta$ and d'' has two components $\overline{\partial} + \theta^*$ which sends $E^{p,q}$ to

$$\mathcal{A}^{1,0}_X \otimes E^{p,q} \oplus \mathcal{A}^{1,0}_X \otimes E^{p-1,q+1} \oplus \mathcal{A}^{0,1}_X \otimes E^{p,q} \oplus \mathcal{A}^{0,1}_X \otimes E^{p+1,q-1} \qquad \nabla^{1,0} + \theta + \overline{\partial} + \theta^*.$$

For $u \in E^{p,q}$ and $v \in E^{p-1,q+1}$, we get

$$0 = \partial Q(u,v) = Q(\theta u, v) + Q(u, \theta^* v).$$

However, the signs of Q and h alternates, so we have

$$h(\theta u, v) = h(u, \theta^* v)$$

which means that θ and θ^* are adjoint operators. This θ is called the Higgs field and plays a crucial role on studying the variation of Hodge structures.

Note that $(E^{p,q}, \overline{\partial})$ gives a structure of a holomorphic vector bundle and that is isomorphic to the holomorphic structure of $F^p\mathcal{E}/F^{p+1}\mathcal{E}$.

3.2. **Curvature formula for Hodge bundles.** For each Hodge bundle $E^{p,q}$, we have a nice formula for the Hodge bundle. Since Q and h differs by a sign for each $E^{p,q}$ and Q is compatible with d, we see that h is compatible with $\nabla^{1,0} + \overline{\partial}$. Therefore, the curvature of $E^{p,q}$ with respect to the Hodge metric can be calculated as

$$\Theta_{E^{p,q}} = (\nabla^{1,0} + \overline{\partial})^2 = -(\theta\theta^* + \theta^*\theta) \in \mathcal{A}^{1,1}_X(\mathrm{Hom}(E^{p,q}, E^{p,q})).$$

The last equality follows from type comparison and $d^2 = 0$.

Let's expand this in local coordinates as $\theta = \sum \theta_j dz^j$. Then we have

$$\Theta_{E^{p,q}} = \sum_{j,k} (-\theta_j \theta_k^* + \theta_k^* \theta_j) dz^j \wedge d\overline{z}^k.$$

Note that the first term contributes negatively to the curvature and the second term positively contributes to the curvature.

3.3. **Positivity of the lowest piece.** At this point, it might seem meaningless to just study variation of Hodge structures. As a slight detour, we give one application which is crucially used in birational geometry.

Theorem 3.2 (Positivity of $f_*\omega_{X/S}$)**.** *Let $f : X \to S$ be a smooth projective morphism between smooth projective varieties. Then $f_*\omega_{X/S}$ is a nef vector bundle.*

Proof. Let n be the relative dimension of f. Then we see that $f_*\omega_{X/S}$ is the lowest piece of the vector bundle $\mathcal{H}^n = (R^n f_*\mathbb{C}_X)_{prim} \otimes_{\mathbb{C}} \mathcal{O}_S$. We denote it by $F^n\mathcal{H}^n$. For simplicity, we assume that $f_*\omega_{X/S} = \mathcal{L}$ is a line bundle. Then it is enough to show that for every curve C, we have

$$\int_C i\Theta_{\mathcal{L}} \geq 0.$$

However,

$$i\Theta_{\mathcal{L}} = i\sum \theta_k^*\theta_j dz^j \wedge d\overline{z}^k$$

which is positive for all $v \in T_xS$. □

The problem of this theorem is that in practical situations, most of the morphisms have singular fibres. This means that the morphism $f : X \to S$ fails to be smooth. One can get around with this issue mainly in two different ways. By generic smoothness, we have an open subset $S^o \subset S$ wehre $f : X^o \to S^o$ is smooth. Hence, the coherent sheaf $f_*\omega_{X/S}$ is a generically defined variation of Hodge structure and we can use Saito's machinery on Hodge modules to attack the behaviour on the boundary. The main tool is the weak positivity of the lowest piece of an Hodge module. Another way to get around this issue is to study the asymptotic behaviour of a variation of Hodge structures towards the boundary. In some cases, we can resolve singularities and get a diagram like

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \tilde{S} & \xrightarrow{\phi} & S \end{array}$$

where ϕ is an isomorphism over S^o and $\tilde{S}\backslash S^o$ is an snc divisor. Then the situation gets simpler since it reduces to studying the asymptotic behaviour of variation of Hodge structures defined on $(\Delta^*)^l \times \Delta^{n-l}$.

The case when $l = 1$ is the result of Schmid, and when $l > 1$ is the result of Cattani-Kaplan-Schmid.

4. Hodge norm estimates for VHS on Δ^*

Now, we study Schmid's result on variation of Hodge structures over a punctured disk. We consider a variation of Hodge structure E over Δ^* with a polarization. After an appropriate switch of the sign, we have a sesquilinear pairing Q (which is different from the one that I introduced for the Hodge-Riemann bilinear pairing) on the vector bundle E such that

- $E^{p,q}$ are mutually orthogonal,
- $h(u, v) = \sum(-1)^q Q(u^{p,q}, v^{p,q})$ is positive definite.
- Q is compatible with the connection d.

The connection d decomposed into $\partial + \theta + \overline{\partial} + \theta^*$.

4.1. **The Monodromy operator.** Since Δ^* is not simply connected, we cannot define a period map. Therefore, we pull back this VHS to the upper-half plane

$$\exp : \mathbb{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\} \to \Delta^*.$$

We will use $t = e^{2\pi i z}$ for the coordinate in Δ^*. We denote by V the space of flat sections on $\exp^* E$. We have the monodromy operator $T \in GL(V)$ as follows.

$$Tv(z) = v(z-1).$$

Here, we can the expression $v(z-1)$ makes sense since the fibres E_z and E_{z-1} can be identified because E comes from the pullback of the exponential map. One other way to view this is the following. Fixing a point $z \in \mathbb{H}$ gives an isomorphism between V and E_z via evaluation. Then T is defined in a way such that the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{\simeq} & E_z \\ \downarrow{\scriptstyle T} & & \| \\ V & \xrightarrow{\simeq} & E_{z+1}. \end{array}$$

Since $Q(v, w)$ is constant for flat sections, we have the following identity

$$Q(Tv, Tw) = Q(v, w).$$

In other words, T lies in the real orthogonal group G.

Note that the data of the variation of Hodge structure on $\mathbb{H}$ with a monodromy transformation $T \in \mathrm{GL}(V)$ fully recovers the data of a variation of Hodge structures on Δ^*.

4.2. **Bound for the Higgs field and the monodromy theorem.** The key result for everything is that there is a very strong constraint for the Higgs field. For this, we will do some soft calculus. First, we notice that the derivative of $h(v, v)$ is controlled by θ. This means the following. If $v \in V$ is a flat section, then

$$\partial h(v, v) = -2h(\theta v, v).$$

We can see that $\partial - \theta + d''$ is a metric connection on E. Hence,

$$\begin{aligned} \partial h(v, v) &= h(\partial - \theta v, v) + h(v, d'' v) \\ &= h(\partial - \theta v, v) \\ &= -2h(\theta v, v). \end{aligned}$$

As a consequence, we get the following.

$$\left| \frac{\partial}{\partial z} \log h(v, v) \right| \leq 2 \|\theta_{\partial/\partial z}\|_{\mathrm{End}(E)}.$$

We also can show that $\varphi = \log h(v, v)$ is subharmonic.

$$\frac{\partial^2 \varphi}{\partial z \partial \overline{z}} = \frac{\partial}{\partial z} \left(\frac{1}{h} \frac{\partial h}{\partial \overline{z}} \right) = \frac{1}{h^2} \left(h \frac{\partial^2 h}{\partial z \partial \overline{z}} - \left| \frac{\partial h}{\partial z} \right| \right).$$

We have $\partial^2 h / \partial z \partial \overline{z} = h(Av, Av)$ where $A = \theta_{\partial/\partial z}$.

We will prove the following theorem.

Theorem 4.1. *There is a constant $C_0 > 0$ only depending on the rank of E such that*

$$\|\theta_{\partial/\partial z}\|^2 \leq \frac{C_0}{|\mathrm{Im} z|^2}.$$

Proof. The key ingredient of this theorem is in threefold.

(1) Ahlfors' lemma. If $f : \mathbb{H} \to (0, +\infty)$ is a positive smooth function such that

$$\frac{\partial^2 \log f}{\partial z \partial \overline{z}} \geq C f$$

for some $C > 0$, then we have

$$f(z) \leq \frac{1}{2C \cdot |\mathrm{Im} z|^2}, \qquad \text{for all } z \in \mathbb{H}$$

(2) If we denote by

$$A = \theta_{\partial/\partial z}, \qquad A^* = \theta^*_{\partial/\partial \overline{z}},$$

then we have

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log h_{\mathrm{End}(E)}(A, A) \geq \frac{\|[A^*, A]\|^2}{\|A\|^2}.$$

This is essentially the same computation as before, but there is a clever way to do this. We can view $\mathrm{End}(E)$ as a variation of Hodge structure of weight 0. Then $\theta^*_{\partial/\partial\overline{z}}$ actually corresponds to $[A^*, -]$ and $A = \theta_{\partial/\partial z}$ is a holomorphic section on $\mathrm{End}(E)$ such that $\partial_{\mathrm{End}(E)} A = \theta_{\mathrm{End}(E)} A = 0$.

(3) The third one is a purely linear algebra one. If A is a nilpotent operator, then

$$\binom{r+1}{3} \|[A^*, A]\|^2 \geq 2\|A\|^4$$

The idea is to diagonalize A by a strictly upper-triangular matrix, and see the diagonal entries

$$[A^*, A]_{kk} = (|a_{1,k}|^2 + \cdots + |a_{k-1,k}|^2) - (|a_{k,k+1}|^2 + \cdots + |a_{k,r}|^2).$$

□

4.3. **Monodromy Theorem.** As a cheap consequence, we have the monodromy theorem.

Theorem 4.2. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of T. Then $|\lambda| = 1$.*

Proof. Let $v \in V$ be an eigenvector such that $Tv = \lambda v$. This means that

$$h(v(z-1), v(z-1)) = h(Tv(z), Tv(z)) = h(\lambda v(z), \lambda v(z)) = |\lambda|^2 h(v(z), v(z))$$

This implies that

$$|\log |\lambda|^2| = \int_x^{x+1} |\partial_x \log h(v, v)| dx \lesssim \frac{C_0}{y^2}$$

And then we can take $y \to \infty$. □

As a consequence, we can perform the Jordan decomposition

$$T = T_s \cdot T_u = e^{2\pi i (S+N)}$$

where S is a semisimple operator with real eigenvalues, and N is a nilpotent operator. Note that we have some freedom of choice of S. Also, we know that $T_s, T_u \in G$ and

$$Q(Nu, v) + Q(u, Nv) = 0.$$

4.4. **Hodge norm estimates.** Let $v \in V$ be a flat section. Since $\mathbb{H}$ is simply connected, we have the period map

$$\Phi : \mathbb{H} \to D$$

where D is an open subset of the flag manifold. The group G acts transitively on D and the period map satisfies the following identity from the monodromy transformation

$$\Phi(z+1) = T \cdot \Phi(z).$$

The sesquilinear pairing $Q : V \times V \to \mathbb{C}$ is fixed, but for each $z \in \mathbb{H}$, we have a different h_z on V by

$$h_z(u, v) = \sum (-1)^q Q(u^{p,q}, v^{p,q}) = \langle u, v \rangle_{\Phi(z)}$$

since the decomposition $V = \bigoplus V^{p,q}_{\Phi(z)}$ changes. We are interested in the asymptotic behaviour of $\|v\|^2_{\Phi(z)}$ when $\mathrm{Im} z \to \infty$, where $v \in V$ is a flat section. The goal is to show the following.

Theorem 4.3. *Let $W_\bullet$ be an increasing filtration on V such that*

$$N : W_k V \to W_{k-2} V$$

and

$$N^k : \mathrm{gr}^W_k V \xrightarrow{\simeq} \mathrm{gr}^W_{-k} V.$$

Then for each $v \in W_k \setminus W_{k-1}$, we have

$$\|v\|^2_{\Phi(z)} \sim |\operatorname{im} z|^k.$$

Lemma 4.4. *The first lemma is as follows. If we fix a vertical strip,*

$$|\mathrm{Im} z|^{-2C_0} \|v\|^2_{\Phi(i)} \lesssim \|v\|^2_{\Phi(z)} \lesssim |\mathrm{Im} z|^{2C_0} \|v\|^2_{\Phi(i)}$$

Proof. If we let $\varphi(x+iy) = \log \|v\|^2_{\Phi(z)}$, then we have $|\partial_z \varphi| \leq C_0/y$. Therefore,

$$\varphi(x+iy) - \varphi(-1) \leq C_0 \log y$$

Therefore, we can deduce the result by taking the exponential. □

This means that $h(v, v)$ grows less than a polynomial of $|\operatorname{im} z|$ and decays less than a polynomial of $|\operatorname{im} z|$.

Lemma 4.5. *For $v \in V$ such that $Tv = \lambda v$, the norm $\|v\|^2_{\Phi(z)}$ is bounded as $\operatorname{im} z > *$.*

Proof. We consider the function $\varphi = \log h(v, v)$ which is known to be subharmonic. In one dimensional case, this corresponds to being convex ($\varphi'' > 0$). If we show that $\lim_{y\to\infty} \varphi'(y) = 0$, then we know that φ is a decreasing function and φ is bounded above. In order to use this trick, we switch to an one-dimensional situation by averaging. Then we define

$$f(y) = \int_{x=0}^{1} \varphi(x+iy)dx$$

Then we get

$$\begin{aligned} f''(y) &= \int_{x=0}^{1} \partial_{yy}\varphi(x+iy)dx \\ &\geq -\int_{x=0}^{1} \partial_{xx}\varphi(x+iy)dx \\ &= \partial_x\varphi(iy) - \partial_x\varphi(1+iy) = 0 \end{aligned}$$

since the function φ is 1-periodic due to $Tv = \lambda v$. Also,

$$|f'(y)| \leq \int_0^1 |\partial_y\varphi(x+iy)|dy \lesssim \frac{1}{y} \xrightarrow{y\to\infty} 0.$$

One more ingredient is to compare the horizontal direction but this is easy. □

Proof of Main theorem. There are mainly two ways to do this. First, we define an auxiliary filtration $M_\bullet$ in the following way

$$M_k = \{v \in V : \|v\|^2_{\Phi(z)} \lesssim |\operatorname{im} z|^k\}$$

and show that this filtration satisfies the two property that uniquely determines the weight filtration. The other method is using the comparison theorem. □

Theorem 4.6 (Comparison Theorem)**.** *If (E, d) and (E', d') are two polarized variation of Hodge structures that are isomorphic as vector bundle with integrable connection (i.e., the monodromy is the same), then the Hodge norm estimates are mutually bounded.*

Proof. The isomorphism $\phi : E \to E'$ is a flat section on $\operatorname{Hom}(E, E')$ such that $T\phi = \phi$. Therefore, $\|\phi\|$ is bounded. We can use a similar argument to ϕ^{-1} to get

$$\|v\|_E \sim \|\phi(v)\|_{E'}$$

for all $v \in V$. □

Remark 4.7. This suggests the following. On each is graded piece, the rate of growth (or decay) the Hodge norms are different, so analytically, you have to rescale each graded piece, or treat them separately in order to get a meaningful information of the limit. Hence, we have to consider the 'mixed Hodge structure' on V such that each graded piece $W_\bullet V$ has its own Hodge structure at the limit.

5. Nilpotent orbit theorem

The goal is to show the following theorem.

Theorem 5.1. *Let $\Phi : \mathbb{H} \to \mathcal{D}$ be the period map, and*

$$\Psi_S : \Delta^* \to \mathcal{D}^\vee, \qquad \Psi(e^{2\pi iz}) = e^{-2\pi i(S+N)z}\Phi(z)$$

be the untwisted period map. Then this map extends through the origin.

5.1. **Brief sketch of Deligne's canonical extension.** Suppose we have a local system on Δ^* with monodromy $T \in \mathrm{GL}(V)$. We fix the log of this matrix as $T = e^{2\pi i B}$. Then the vector bundle E with flat connection ∇ extends to a vector bundle $\tilde{E}$ with a flat logarithmic connection

$$\nabla : \tilde{E} \to \Omega^1_\Delta(\log 0) \otimes \tilde{E}$$

The precise way is a follows. Let v be a multivalued flat section. Then we define

$$\tilde{v}(z) := e^{2\pi i B z} v(z).$$

Then we see that $\tilde{v}(z+1) = \tilde{v}(z)$ by looking at the monodromy transformation. We define $\tilde{E}$ to be the locally free sheaf generated by these $\tilde{v}$. We compute how the connection behaves.

$$\nabla \tilde{v} = e^{2\pi i B z} 2\pi i B v dz = B\tilde{v}\frac{dt}{t}.$$

Hence, we can see the residue of the connection is exactly B.

If we have a variation of Hodge structures, then $T = e^{2\pi i (S+N)}$ where S is a semisimple operator with real eignevalues and N is a nilpotent operator. Therefore, if we fix a half open interval $[-\lambda, -\lambda+1)$, then we can choose $S = S_\lambda$ so the the Deligne extension $\tilde{E}$ having eigenvalues of the residue lying inside this interval.

5.2. **Consequences of the nilpotent orbit theorem.** One of the main consequences of the nilpotent orbit theorem is as follows. If we have an snc divisor $D = \sum_{i=1}^{\nu} D_i$ in a smooth manifold X. Suppose we have a variation of Hodge structures E on the complement $j : X^o = X \setminus D \to X$. Then if we pick real numbers $\alpha_i \in \mathbb{R}$ and corresponding intervals $[-\alpha_i, -\alpha_i + 1)$, then we can consider the Deligne's extension E_α as a holomorphic vector bundle on X with logarithmic connection $\nabla : E_\alpha \to \Omega^1_X(\log D) \otimes E_\alpha$.

The nilpotent orbit theorem says that moreover, the Deligne extension E_α carries a filtration $F^p E_\alpha = j_* F^p E \cap E_\alpha$ by vector bundles and the connection satisfies a similar Griffiths transversality condition

$$\nabla(F^p E_\alpha) \subset \Omega^1_X(\log D) \otimes F^{p-1} E_\alpha.$$

In dimension 1, the precise relation is as follows. We have a trivialization of $E_\alpha \simeq \Delta \times V$ as follows.

$$V \otimes_{\mathbb{C}} \mathcal{O}_\Delta \mapsto E_\alpha, \qquad v \mapsto \tilde{v}(z) = e^{2\pi i (S+N) z} v(z).$$

We consider the following subbundle of $V \times \Delta$.

$$F^p V := \{(v, t) \in V \times \Delta : v \in F^p_{\Psi_S(t)} V\}.$$

Then we can precisely see that this bundle in E_α is exactly $F^p E_\alpha$ since the untwisted period map is defined as $\Psi_S(e^{2\pi i z}) = e^{-2\pi i (S+N) z}\Phi(z)$ so the two factors cancel out.

5.3. **Brief strategy of the proof of nilpotent orbit theorem.** First, we see that $\mathrm{GL}(V)$ acts transitively on the flag manifold. Therefore, the tangent space of a flag manifold can be viewed as

$$\mathrm{End}(V)/\{\text{ endomorphism of the direction preserving the flag }\}$$

and this is precisely $F^0\operatorname{End}(V)$ where the Hodge structure on $\operatorname{End}(V)$ is defined as

$$F^t\operatorname{End}(V) = \{\phi : V \to V | \phi(F^pV) \subset F^{p+t}V\}.$$

The derivative of the untwisted period map $\Psi_S(e^{2\pi iz})e^{-2\pi iz(S+N)}\Phi(z)$ can be calculated as

$$e^{-2\pi iz(S+N)}\theta_{\partial/\partial z}e^{2\pi iz(S+N)} - 2\pi i(S+N) \mod F^0\operatorname{End}(V)_{\Psi_S(e^{2\pi iz}}.$$

Note that the first term is holomorphic if we mod out by $F^0\operatorname{End}(V)$, but not holomorphic as a function $\Delta^* \to \operatorname{End}(V)$. At the moment, we suppose that the first term is holomorphic and we call this

$$B(e^{2\pi iz}) - (S+N).$$

Then we consider the differential equation for $g : \mathbb{H} \to \mathrm{GL}(V)$

$$g'(z) = (B(e^{2\pi iz}) - (S+N))g(z)$$
$$g(i) = \mathrm{id}.$$

Then can see that $g(z)^{-1}\Psi_S(e^{2\pi iz})$ is a constant function. Since $g(z+1)g(i+1)^{-1}$ is also a solution for this differential equation, we have $g(z+1)g(i+1)^{-1} = g(z)$. Hence, we have some operator A such that

$$M(e^{2\pi iz}) = g(z)e^{2\pi izA}.$$

Then we can write

$$\Psi_S(e^{2\pi iz}) = g(z)\Psi_S(e^{-2\pi}) = M(e^{2\pi iz})e^{2\pi izA}\Psi_S(e^{-2\pi}).$$

Then we can erase $e^{2\pi izA}$ on the equation. The remaining part is to show that $M(t)$ is meromorphic at $t=0$. For this we need the following two ingredients.

(1) We have to construct a holomorphic lift $\vartheta : A(\Delta^*, \operatorname{End}(E))$ such that

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0\operatorname{End}(V).$$

Here, we want to produce a 'holomorphic object' on a vector bundle and one useful way to produce these objects is to solve the $\overline{\partial}$-equation.

(2) Have to show that $B(e^{2\pi iz})$ is holomorphic at the origin. Then some theory about ordinary differential equations will imply that M is meromorphic.

5.4. **Hörmander's L^2-estimates and application to VHS.** The goal is to show the following theorem.

Theorem 5.2. *Let $X \subset \mathbb{C}$ be a domain and E be a holomorphic vector bundle on X with a metric h. Let $d = \delta' + \delta''$ be the (unique) connection compatible to the metric and $\delta'' = \overline{\partial}$ and let $\Theta \in \mathcal{A}^{1,1}_X(X, \operatorname{End}(E))$ be the curvature. We assume that there is a positive smooth function $\rho : X \to \mathbb{R}_{>0}$ such that*

$$\int_X h(\Theta_{\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial\overline{z}}}\alpha, \alpha)d\mu \geq \int_X \rho^2 h(\alpha,\alpha)d\mu$$

which is a coercive positivity condition. The if $f \in A^0(X,E)$, then there exists a smooth section $u \in A^0(X,E)$ such that $\overline{\partial}u = fd\overline{z}$ with a norm estimate

$$\int_X h(u,u)d\mu \leq \int_X \frac{1}{\rho^2}h(f,f)d\mu.$$

Proof. We have two differential operators $E \to E$ given by $\delta'_{\partial/\partial z}$ and $\delta''_{\partial/\partial \overline{z}}$. Integration by parts give

$$\int_X h(\delta''_{\partial/\partial\overline{z}} u, \alpha) d\mu = -\int_X h(u, \delta'_{\partial/\partial z}\alpha) d\mu.$$

Hence, for existence, we only need to show that the map

$$\delta'_{\partial/\partial z}\alpha \mapsto -\int_X h(f, \alpha) d\mu$$

is bounded. Then the rest follows by Hahn-Banach. In other words, we only have to show that

$$\left| \int_X h(f, \alpha) d\mu \right|^2 \leq \left(\int_X \frac{1}{\rho^2} h(f, f) d\mu \right) \cdot \left(\int_X h(\delta'_{\partial/\partial z}\alpha, \delta'_{\partial/\partial z}\alpha) d\mu \right).$$

However, we know the inequality when the last term is replaced with $\int_X \rho^2 h(\alpha, \alpha) d\mu$ by Cauchy-Schwartz. Hence, we have to show

$$\int_X \rho^2 h(\alpha, \alpha) d\mu \leq \int_X h(\Theta_{\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}}} \alpha, \alpha) d\mu \leq_? \int_X h(\delta'_{\partial/\partial z}\alpha, \delta'_{\partial/\partial z}\alpha) d\mu.$$

Since $\Theta = \delta''\delta' + \delta'\delta''$, we get

$$h(\Theta_{\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}}} \alpha, \alpha) = h(\delta'_{\partial/\partial z}\delta''_{\partial/\partial\overline{z}}\alpha, \alpha) - h(\delta''_{\partial/\partial\overline{z}}\delta'_{\partial/\partial z}\alpha, \alpha)$$

and we can use integration by parts to get the desired inequality. □

Now, we want to apply this theorem to a graded piece $E^{p,q}$ on Δ^*. However, the problem is as follows. The curvature Θ is given by $-(\theta\theta^* + \theta^*\theta)$ and we have

$$h(\Theta_{\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial \overline{t}}} u, u) = \|\theta_{\partial/\partial t} u\|^2 - \|\theta^*_{\partial/\partial \overline{t}} u\|^2.$$

Hence, we have a positive term and a negative term. However, the key fact is that the we can precisely bound how negative the term is. So we twist the Hodge metric by $e^{-\varphi}$ where

$$e^{-\varphi} = |t|^a (-\log|t|)^b, \qquad \frac{\partial^2 \varphi}{\partial t \partial \overline{t}} = \frac{b}{4|t|^2(-\log|t|)^2}.$$

The key point is that we had $C_0 > 0$ only depending on the rank of E such that $\|\theta_{\partial/\partial\overline{t}}\|^2 \leq C_0/|t|^2(-\log|t|)^2$. Also, if we twist the metric by $e^{-\varphi}$, a standard computation says that the curvature changes by

$$\Theta^\varphi = \Theta + \partial\overline{\partial}\varphi.$$

Therefore, if $b \gg C_0$, then we get

$$h(\Theta_{\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial \overline{t}}} u, u) \geq \frac{1}{|t|^2(-\log|t|)^2} \|u\|^2$$

and we are in a good shape. Here, we will use the function

$$\rho = \frac{1}{|t|(-\log|t|)}$$

obviously. Then we can get the following theorem.

Theorem 5.3. *Let $f \in A^0(E^{p,q})$ such that*

$$\int_{\Delta^*} \|f\|^2 |t|^{a+2}(-\log|t|)^{b+2}) d\mu < +\infty.$$

Then, we get u such that $\overline{\partial} u = f\bar{t}$ such that

$$\int_{\Delta^*} \|u\|^2 |t|^a (-\log|t|)^b d\mu \leq \int_{\Delta^*} \|f\|^2 |t|^{a+2}(-\log|t|)^{b+2} d\mu.$$

5.5. **Holomorphic lift of the Higgs field.** Now, we want to construct a holomorphic section ϑ on $\mathrm{End}(E)$ such that

$$\vartheta \equiv \theta_{\partial/\partial z} = t\theta_{\partial/\partial t} \mod F^0 \,\mathrm{End}(E)$$

with a certain norm bound.

Proof. Note that the section $t\theta_{\partial/\partial t}$ is holomorphic on each graded piece. Let $f_0 = t\theta_{\partial/\partial t} \in \mathrm{End}(E)^{-1,1}$. Then

$$d''_{\mathrm{End}(E)} = \overline{\partial}_{\mathrm{End}(E)}(t\theta_{\partial/\partial t}) + \theta^*_{\mathrm{End}(E)}(t\theta_{\partial/\partial t}) = t[\theta^*_{\partial/\partial \bar{t}}, \theta_{\partial/\partial t}] d\bar{t}$$

and call $f_0 = t[\theta^*_{\partial/\partial \bar{t}}, \theta_{\partial/\partial t}]$. Then we get

$$\|f_0\|^2 \leq |t|^2 \|\theta_{\partial/\partial t}\|^2 \lesssim \frac{C_0^4}{|t|^2(-\log|t|)^4}.$$

If $a > -2$ and b is big enough, then we have

$$\int_{\Delta^*} \|f_0\|^2 |t|^{a+2}(-\log|t|)^{b+2} d\mu \leq C_0^4 \int_{\Delta^*} |t|^a (-\log|t|)^{b-2} d\mu < +\infty.$$

Therefore, we have $u_0 \in \mathrm{End}(E)^{0,0}$ such that $\overline{\partial} u_0 + f_0 d\bar{t} = 0$ with

$$\int_{\Delta^*} \|u_0\|^2 |t|^a (-\log|t|)^b d\mu \leq C_0^4 \cdots < \infty$$

We then have $\theta^*_{\mathrm{End}(E)} u_0 = [\theta^*, u_0] = f_1 d\bar{t}$. Then we see that we f_1 satisfies

$$\int_{\Delta^*} \|f_1\|^2 |t|^{a+2}(-\log|t|)^{b+2} d\mu \leq \int_{\Delta^*} \|u_0\|^2 \frac{C_0^2}{|t|^2(-\log|t|)^2} |t|^{a+2}(-\log|t|)^{b+2} d\mu < +\infty.$$

Therfore, we have u_1 such that $\overline{\partial} u_1 + f_1 d\bar{t} = 0$. We can do this procedure finitely many times and get $\vartheta = t\theta_{\partial/\partial t} + \sum u_i$ what we wanted. Actually, we have a little more. We actually have a bound on the norm

$$\int_{\Delta^*} \|\vartheta\|^2 |t|^a (-\log|t|)^b d\mu < 0.$$

$\square$

5.6. **Holomorphicity of $B(e^{2\pi iz})$ at the origin.** We pull back the section ϑ to $\mathbb{H}$ and identify E with its trivialization V. In this way, we can view ϑ as a holomorphic map

$$\vartheta : \mathbb{H} \to \mathrm{End}(V)$$

which satisfies the following monodromy transformation

$$\vartheta(z+1) = T\vartheta(z)T^{-1}.$$

Hence, we can untwist this map as

$$B(e^{2\pi iz}) = e^{-2\pi iz(S+N)}\vartheta(z)e^{2\pi iz(S+N)}.$$

One can show that $\|B(e^{2\pi iz}\|^2 \lesssim |z|^a\|\vartheta\|^2$ for an appropriate $a > -2$ which is really close to -2 (this is essentially basic linear algebra).

6. Limit mixed Hodge structures

We will not give the Schmid's version of limiting mixed Hodge structure, but rather a more geometric method of constructing. Also, this and the following section would be super expository without giving any proofs.

6.1. **Steenbrink's result.** Let $f : X \to \Delta$ be a projective holomorphic map which is smooth over Δ^* and denote X_0 the singular fibre. After taking the base change $t \mapsto t^N$, we can assume that the monodromy is unipotent, and the singular fibre is reduced and simple normal crossing (semi-stable reduction theorem).

We denote $Y = X_0 = \sum E_i$. Then we have the following diagram

Steenbrink pointed out that there is a complex on X whose cohomology computes the limit Hodge structure.

6.2. **Relative log de Rham complex.** We consider the relative log de Rham complex

$$\Omega^\bullet_{X/\Delta}(\log Y) = \Omega^\bullet_X(\log Y)/f^*\Omega^1_\Delta(\log 0) \otimes \Omega^{\bullet-1}_X(\log Y)$$

In local coordinates, we can write $t = z_0 \cdots z_k$ and we have

$$\Omega_\Delta(\log 0) = \mathcal{O}_\Delta\frac{dt}{t}, \qquad \Omega^1_X(\log Y) = \mathcal{O}_X\frac{dz_0}{z_0} \oplus \cdots \oplus \mathcal{O}_X\frac{dz_k}{z_k} \oplus \mathcal{O}_X dz_{k+1} \oplus \cdots \oplus \mathcal{O}_X dz_n.$$

Note that we have

$$f^*\frac{dt}{t} = \sum \frac{dz_i}{z_i} = \sum_{i=0}^k \xi_i$$

and this gives a relation on $\Omega_{X/\Delta}(\log Y)$. The sheaf $\mathcal{T}_{X/\Delta}(\log Y)$ is generated by $z_i\partial_i - z_0\partial_0$ and ∂_j. We have a short exact sequence

$$0 \to f^*\Omega_\Delta(\log 0) \otimes \Omega^{\bullet+n}_{X/\Delta}(\log Y) \to \Omega^{\bullet+n+1}_{X/\Delta}(\log Y) \to \Omega^{\bullet+n+1}_{X/\Delta}(\log Y) \to 0$$

By trivializing $\Omega^1_\Delta(\log 0)$, we have an endomorphism ∇ of $\Omega^{\bullet+n}_{X/\Delta}$ in the derived category $\mathcal{D}^b(X, \mathbb{C})$ (by the cone construction). After taking $\mathbf{R}f_*$, we also get

$$\nabla : \mathbf{R}f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \to \mathbf{R}f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

and we have $[\nabla, g] = tg'$ which is the Leibnitz rule. Hence, we have a logarithmic connection.

One can show that

$$\mathbf{R}^l f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

is a locally free sheaf and commutes with base change. In particular,

$$\mathbf{R}^l f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y) \otimes \mathbb{C}(p) \simeq \mathbb{H}^l(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_{X_p}).$$

On the general fibre $p \neq 0$, this gives us the cohomology $H^{n+l}(X_p, \mathbb{C})$ and on the special fibre, we have

$$\mathbb{H}^l(Y, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y).$$

The goal today is to construct a mixed Hodge structure on this vector space via $\mathcal{D}$-modules. In other words, we want to construct a $\mathcal{D}$-module $\mathcal{M}$ supported on Y such that its de Rham complex computes the cohomology.

6.3. **$\mathcal{D}$-module approach.** Since $\mathcal{T}_{X/\Delta}(\log Y) \subset \mathcal{T}_X$, we have a map

$$\mathcal{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes \mathcal{D}_X, \qquad P \mapsto \sum \xi_i \otimes D_i P.$$

This extends to a complex of right $\mathcal{D}_X$-modules

$$\Omega^{\bullet+n}_{X/\Delta}(\log Y) \otimes \mathcal{D}_X = [\mathcal{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes \mathcal{D}_X \to \cdots \to \Omega^n_{X/\Delta}(\log Y) \otimes \mathcal{D}_X][n].$$

We denote by $\tilde{\mathcal{M}}$ the cokernel of the last map. This complex also caries a filtration by order as

$$F_l(\mathit{complex}) = [F_l \mathcal{D}_X \to \cdots \to \Omega^n_{X/\Delta} \otimes F_{l+n} \mathcal{D}_X][n].$$

We can see that this complex is a filtred resolution of $(\tilde{\mathcal{M}}$. 'Locally' we can describe $\tilde{\mathcal{M}}$ as

$$\tilde{\mathcal{M}} = \mathcal{D}_X / (D_1, \cdots, D_n) \mathcal{D}_X.$$

We define $\mathcal{M} = \tilde{\mathcal{M}} / t \tilde{\mathcal{M}}$ which is a 'holonomic' $\mathcal{D}_X$-module supported on Y. One can also show that

$$\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y \otimes \mathcal{D}_Y$$

is a filtred resolution of $\mathcal{M}$ and also, $\mathrm{DR}_X(\mathcal{M}) \simeq \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$. Also, the ∇ on $\Omega^{\bullet}_{X/\Delta}(\log Y)$ induces a map

$$R : (\mathcal{M}, F_\bullet \mathcal{M}) \to (\mathcal{M}, F_{\bullet+1} \mathcal{M})$$

a nilpotent operator that is locally given by $z_0 \partial_0$. Therefore, we have a weight filtration $W_l \mathcal{M}$ that satisfies the two condition

$$R : W_l \mathcal{M} \to W_{l-2} \mathcal{M}, \qquad R^l : \mathrm{gr}^W_l \mathcal{M} \xrightarrow{\simeq} \mathrm{gr}^W_{-l} \mathcal{M}.$$

Note that this operator satisfies strictness $R^a F_b \mathcal{M} = F_{a+b} R^a \mathcal{M}$ and we define

$$F_\bullet W_l \mathcal{M} = W_l \mathcal{M} \cap F_\bullet \mathcal{M}.$$

The nice part is that we can see the primitive parts using the structure of Y. We have

$$\mathcal{P}_r = \ker(R^{r+1} : \mathrm{gr}^W_r \mathcal{M} \to \mathrm{gr}^W_{-r-2} \mathcal{M})$$

with the Lefschetz decomposition

$$\mathrm{gr}^W_r \mathcal{M} = \bigoplus_{l \geq 0, -r/2} R^l \mathcal{P}_{r+2l}$$

which behaves nicely with respect to the Hodge filtration in the following way

$$F_\bullet \operatorname{gr}_r^W \mathcal{M} = \bigoplus R^l F_{\bullet - l} \mathcal{P}_{r+2l}.$$

The upshot is that

$$\phi : (\mathcal{P}_r, F_\bullet \mathcal{P}_r) \xrightarrow{\simeq} \tau_+^{(r+1)} \omega_{\tilde{Y}^{(r+1)}}(-r)$$

where

$$\tau^{(r+1)} : \tilde{Y}^{(r+1)} = \coprod_{|J|=r+1} Y^J \to X$$

where $Y^J = \cap_{j\in J} Y_j$. This map essentially comes from the residue map

$$\operatorname{Res} : \Omega_X^{\bullet+n+1}(\log Y)|_Y \to \Omega_{Y^J}^{\bullet+n-r}(\log Y_J)$$

where $Y_J = (Y - \sum_j Y_j)|_{Y_J}$.

Remark 6.1. There is also an approach for non-semistable situation which is a bit messier but useful for many situations.

7. Degeneration of Calabi-Yau and non-Archimedean geometry

We compare this result with a non-Archimedean geometry result regarding degeneration of Calabi-Yau manifolds. Let's say we have a family of Calabi-Yau's $f : X \to \Delta^*$ with unipotent monodromy and fix an snc model $f : \mathcal{X} \to \Delta$. Then we have

$$f_* \omega_{\mathcal{X}/\Delta}(\log \mathcal{X}_0) \subset \mathbf{R}^0 f_* (\Omega^{n+\bullet}_{\mathcal{X}/\Delta}(\log \mathcal{X}_0))$$

which is a canonical determined object which does not depend on the choice of the snc compactification. We know that this is a line bundle and fix a non-vanishing section

$$\alpha \in H^0(\Delta, f_*\omega_{\mathcal{X}/\Delta}(\log \mathcal{X}_0)) = H^0(\mathcal{X}, \omega_{\mathcal{X}/\Delta}(\log \mathcal{X}_0)).$$

Then for each $t \in \Delta^*$, we get a (non-vanishing) volume form $\alpha_t \in H^0(X_t, \omega_{X_t})$ and we can view X_t as a measure space (X_t, ν_t) where

$$\nu_t = \frac{i^{n^2}}{2^n} \alpha_t \wedge \overline{\alpha_t}.$$

For simplicity, we assume that $f : \mathcal{X} \to \Delta$ is semisimple. In this case, the log-relative canonical bundle is the same as $\omega_{\mathcal{X}/\Delta}$ and there is a line bundle $\mathcal{L}$ and

$$\omega_{\mathcal{X}/\Delta} = \mathcal{L} + \sum a_i E_i$$

such that α defines a continuous metric on $\mathcal{L}$. The numbers a_i is exactly the vanishing order of α along the divisors E_i (which is either 0 or 1).

Let $\Delta(\mathcal{X})$ be the dual complex of the snc divisor $\mathcal{X}_0$ and let $\Delta(\mathcal{L})$ be the subcomplex of $\Delta(\mathcal{X})$ such that a face in $\Delta(\mathcal{X})$ is in $\Delta(\mathcal{L})$ if and only if all of the vertices E_i has $a_i = 0$. Then the result of Boucksom-Jonsson is the following.

Theorem 7.1.

$$\nu(X_t) \sim c \cdot (-\log |t|)^d$$

where $d = \dim \Delta(\mathcal{L})$ *and the rescaled measure*

$$\mu_t := \frac{\nu_t}{(-2\pi \log |t|)^d}$$

converges weakly to a reasonable limit μ_0 inside the 'hybrid space' $\mathcal{X}_0^{hyb}$.

This is a 'sheafified version' of Schmid's result in the following sense. One can deduce $\nu(X_t) \sim c \cdot (-\log|t|)^d$ from Schmid and Steenbrink's result. We briefly recall the norm estimates according to the weight filtration.

Theorem 7.2. *For $v \in W_k \setminus W_{k-1}$, we have*

$$\|v\|^2_{\Phi(z)} \sim |\operatorname{im} z|^k$$

We can express

$$\alpha = f_1\tilde{v}_1 + \cdots + f_r\tilde{v}_r$$

where f_i is a holomorphic function on Δ and

$$\tilde{v}_i = e^{2\pi i N z} v_i(z) = \sum_{k=0}^{K} (2\pi i z)^k (N^k v_i)(z).$$

where v_i is a flat section. Note that the contribution of $(2\pi i z)^k$ and $N^k v_i$ cancels out since N decreases the weight filtration by 2. Hence, we get

$$\|\tilde{v}_i\|^2 \sim (-\log|t|)^k$$

if $v_i \in W_k \setminus W_{k-1}$. Therefore, the dominating term are the v_i's such that $f_i(0) \neq 0$. And one should be able to detect which weight does α live in since there is a very explicit description of the Lefschetz decomposition in terms of the residue map which is described in terms of the dual complex.

References

[BJ17] Sébastien Boucksom and Mattias Jonsson, *Tropical and non-Archimedean limits of degenerating families of volume forms*, J. Éc. polytech. Math. **4** (2017), 87–139. MR3611100 ↑1

[Che21] Qianyu Chen, *Limits of hodge structures via holonomic d-modules*, arXiv preprint arXiv:2103.03983 (2021). ↑1

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457–535. MR840721 ↑1

[Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. MR382272 ↑1

[SS22] Claude Sabbah and Christian Schnell, *Degenerating complex variations of hodge structure in dimension one*, arXiv preprint arXiv:2206.08166 (2022). ↑1

[Voi02] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps. MR1967689 ↑1

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA

Email address: calvinyw@umich.edu