Representations of quivers and Lie algebras

Day 1: Quiver Representations and Root Systems

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Reference

- Main source:
  - Kirillov, *Quiver Representations and Quiver Varieties*
- The constituent pieces:
  - Derksen and Weyman, *An Introduction to Quiver Representations*
  - Humphreys, *Introduction to Lie Algebras and Representation Theory*

Roadmap

- Gabriel's theorem and the necessary context of quiver representations and root systems
- The proof of Gabriel's theorem: varieties of representations, reflection functors
- Strengthening the connection to Lie algebras with the Ringel-Hall algebra

General conventions

- k is a field.
- Unless specified otherwise (and it will be!), k is algebraically closed of characteristic 0.
- All vector spaces are assumed to be finite-dimensional k-vector spaces.

Why quiver representations?

A fundamental theorem of linear algebra:

Theorem

Let $f : V \to W$ be a linear map between vector spaces. Then we can choose bases of V and W such that f is given by a matrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Once more with diagrams

Theorem

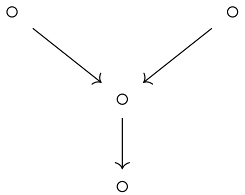
Let $f : V \to W$ be a linear map between vector spaces. Then there exist isomorphisms $\varphi : V \to k^n$, $\psi : W \to k^m$, and a map $k^n \to k^m$ given by a matrix M of the special form above, such that the diagram

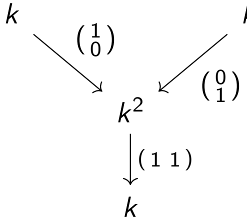
$$\begin{CD} V @>{f}>> W \\ @V{\varphi}VV @VV{\psi}V \\ k^n @>{M}>> k^m \end{CD}$$

commutes.

Quiver representations

- A **quiver** Q is a directed graph, with vertices Q_0 and edges Q_1.
- A **representation** V of a quiver Q consists of:
  - for every vertex $x \in Q_0$, a vector space $V(x)$;
  - for every edge $\alpha \in Q_1$, a linear map $V(\alpha)$ between the spaces at its endpoints.





- A **morphism of representations** $h : V_1 \to V_2$ consists of maps $h_x : V_1(x) \to V_2(x)$ for each vertex x which commute with the maps associated to the arrows.

Once more with jargon

Theorem

Let $f : V \to W$ be a linear map between vector spaces. Then there exist isomorphisms $\varphi : V \to k^n$, $\psi : W \to k^m$, and a map $k^n \to k^m$ given by a matrix M of the special form above, such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ k^n & \xrightarrow{M} & k^m \end{array}$$

commutes.

Theorem

Every representation of the quiver $\circ \longrightarrow \circ$ is isomorphic to one of the form $k^n \xrightarrow{M} k^m$ where M has the special form from above.

Once more with reductionism

- The **direct sum** of two representations V, W of a quiver is defined by

$$(V \oplus W)(x) := V(x) \oplus W(x), x \in Q_0$$
$$(V \oplus W)(\alpha) := V(\alpha) \oplus W(\alpha), \alpha \in Q_1$$

- At each edge, put together the maps in a block diagonal matrix:

$$\begin{array}{c} k^2 \xrightarrow{(1\ 1)} k \xrightarrow{2} k \\ \oplus \\ k \xrightarrow{-1} k \xrightarrow{\binom{3}{2}} k^2 \end{array} \quad \cong \quad k^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 2 \end{pmatrix}} k^3$$

- A representation which is not isomorphic to a nontrivial direct sum is **indecomposable**. A decomposition may not be easy to see:

$$k^2 \xrightarrow{(1\ 1)} k \xrightarrow{2} k \quad \cong \quad \begin{array}{c} \text{span}\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \xrightarrow{0} 0 \xrightarrow{0} 0 \\ \oplus \\ \text{span}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \xrightarrow{2} k \xrightarrow{2} k \end{array}$$

Once more with reductionism

Theorem (Krull-Schmidt)

Every representation of a quiver admits a unique decomposition into indecomposables.

- Our "special form from above" is a block diagonal matrix, and pulling it apart, we get

Theorem

The indecomposable representations of $\circ \longrightarrow \circ$ *are, up to isomorphism,*

$$k \to 0$$

$$0 \to k$$

$$k \xrightarrow{1} k$$

Motivating question #1

Question

What are the different ways a diagram of vector spaces and linear maps can behave, up to changes of basis?

Question

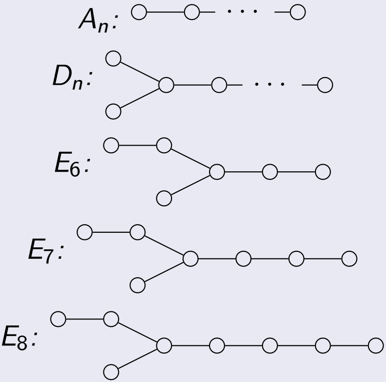
What are the indecomposable representations of a quiver up to isomorphism?

- Say a quiver is **finite type** if it has finitely many indecomposable representations.

Gabriel's theorem

Theorem (Gabriel)

A quiver is finite type if and only if its underlying undirected graph is one of these:



In this case, the indecomposable representations correspond bijectively to positive roots.

Root systems

- Let V be a vector space over $\mathbb{R}$ with inner product $\langle , \rangle$.
- For any vector $\alpha \in V$, define the **reflection along** α, $s_\alpha : V \to V$, by

$$s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

Definition

A **(finite, crystallographic) root system** in V is a finite collection of nonzero vectors Φ (called **roots**) such that:

(1) For each $\alpha \in \Phi$, Φ contains $-\alpha$, but no other multiple of α.

(2) For $\alpha, \beta \in \Phi$, $s_\alpha(\beta) \in \Phi$.

(3) For $\alpha, \beta \in \Phi$, $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

Simply laced root systems

Definition

A **(finite, crystallographic) root system** in V is a finite collection of nonzero vectors Φ (called **roots**) such that:

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(3) For $\alpha, \beta \in \Phi$, $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

- In this course, we'll only consider the case of:

Definition

A **simply laced root system** is one in which all roots have the same length (without loss of generality, $\sqrt{2}$).

Example: the A_3 system

- Let $V = \{c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 \in \mathbb{R}^4 \mid c_1 + c_2 + c_3 + c_4 = 0\}$ with inner product induced by the dot product.
- Consider

$$\Phi = \{e_i - e_j \mid 1 \leq i, j \leq 4\} \subset V$$

- Note

$$\langle e_i - e_j, e_i - e_j \rangle = 2.$$

and for any $\alpha, \beta \in \Phi$

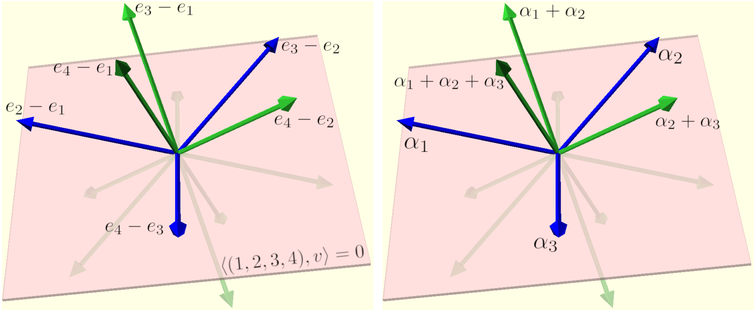
$$\langle \beta, \alpha \rangle \in \{-2, -1, 0, 1, 2\}$$

Exercise

Show that the reflection $s_{e_i - e_j}$ acts by transposing e_i and e_j (so, in particular, it preserves Φ)

Basic properties of root systems

- Cutting a root system with a hyperplane divides it into **positive** and **negative** roots.



- The dim V positive roots closest to the hyperplane are the **simple** roots.
- The simple roots form a basis, and any positive root is a nonnegative linear combination.

The Cartan matrix and Dynkin diagrams

Definition

Given a root system Φ with simple roots $\alpha_1, \ldots, \alpha_n$, its **Cartan matrix** C is defined by

$$C_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \overset{\text{(simply laced)}}{=} \langle \alpha_i, \alpha_j \rangle$$

- For example, in type A_3:

$$\alpha_i = e_{i+1} - e_i$$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & i = j \\ -1 & i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Proposition

The Cartan matrix is independent of the choice of simple roots (up to permuting rows and columns) and completely determines the root system.

The Cartan matrix and Dynkin diagrams

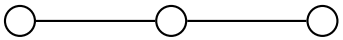
- The diagonal entries of C are all $2\frac{\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2$.
- The off-diagonal entries are nonpositive integers (the simple roots are never at acute angles)
- In the simply laced case, they can only be 0 or -1.

Definition

The **Dynkin diagram** of a simply laced root system is an undirected graph with:

- vertices indexed by the simple roots;
- an edge $i \sim j$ when $C_{ij} = -1$.

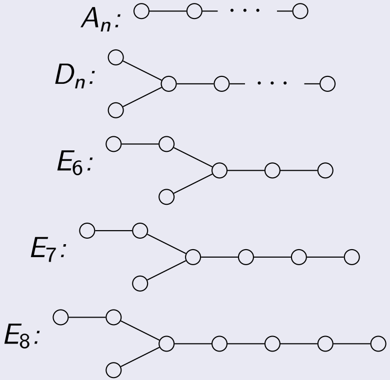
- For example, in type A_3:



The classification of (finite, simply laced) root systems

Theorem

The finite simply laced root systems have the following Dynkin diagrams:



The classification of (finite, simply laced) root systems

- Outside the simply-laced case, there are also families B_n and C_n and exceptional cases F_4, G_2.

Exercise

If you haven't seen the proof of this classification before, look one up!

- Key ingredient of proof: the Cartan matrix

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle$$

represents $\langle , \rangle$ in the basis $\alpha_1, \ldots, \alpha_n$, so it is positive definite.

Back to Gabriel's theorem

Theorem (Gabriel)

A quiver is finite type if and only if its underlying undirected graph is a Dynkin diagram. In this case, the indecomposable representations correspond bijectively to positive roots.

Definition

The **dimension vector** of a quiver representation is the tuple

$$(\dim V(x))_{x \in Q_0}$$

- Because vertices of the Dynkin diagram correspond to simple roots, such a tuple gives a positive linear combination of simple roots.
- The theorem states that, for indecomposables, the dimension vectors are exactly the positive roots (and the correspondence is 1-to-1).

Example: back to A_3

$$\circ \to \circ \to \circ$$

indecomposables	positive roots
$k \xrightarrow{0} 0 \xrightarrow{0} 0$	α_1
$0 \xrightarrow{0} k \xrightarrow{0} 0$	α_2
$0 \xrightarrow{0} 0 \xrightarrow{0} k$	α_3
$k \xrightarrow{1} k \xrightarrow{0} 0$	$\alpha_1 + \alpha_2$
$0 \xrightarrow{0} k \xrightarrow{1} k$	$\alpha_2 + \alpha_3$
$k \xrightarrow{1} k \xrightarrow{1} k$	$\alpha_1 + \alpha_2 + \alpha_3$

Exercises

- For next time:

Exercise

Suppose $\Phi \subset V$ has simple roots $\alpha_1, \ldots, \alpha_n$. Give formulas for the operations $\langle , \rangle$ and s_{α_i} in the basis $\{\alpha_i\}$, in terms of the Dynkin diagram.

- For extra practice:

Exercise

Show that the collection $\{e_i - e_j \mid 1 \leq i, j \leq n\}$ is a simply laced root system.

Exercise

Look at a proof of the classification of root systems if you haven't seen one before.

Next time...

- First hints of a geometric approach to quiver representations!
- The proof of Gabriel's theorem!
- The funhouse of reflection functors!