Representations of quivers and Lie algebras

Day 3: Gabriel's theorem part 2, and also Lie algebras

Will Dana

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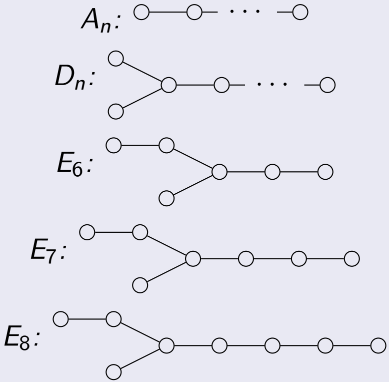
Last time: Gabriel's theorem

Theorem (Gabriel)

A quiver is finite type if and only if its underlying undirected graph is one of these:



In this case, the indecomposable representations correspond bijectively to positive roots.

Last time: parametrizing representations

- To prove finite type ⇒ Dynkin, we introduced the representation space

$$\begin{aligned}\mathrm{Rep}(Q,\alpha) &:= \{\text{representations } V \text{ of } Q \mid V(x) = k^{\alpha(x)}\} \\ &= \bigoplus_{e \in Q_1} \mathrm{Hom}(k^{\alpha(\mathrm{tail}(e))}, k^{\alpha(\mathrm{head}(e))})\end{aligned}$$

  and an action of the group

$$\mathrm{GL}_\alpha = \prod_{x \in Q_0} \mathrm{GL}_{\alpha(x)}(k);$$

  by changing bases.
- Isomorphism classes correspond to orbits.
- By counting the dimensions of these things, we showed that Q can only be finite type if its underlying undirected graph has positive definite Cartan matrix (thus is a Dynkin diagram).

Last time: reflection functors

- To show indecomposable representations of a Dynkin quiver correspond to roots, we introduced reflection functors.
- Let x be a sink of the quiver Q, and let $s_x(Q)$ be the quiver with the arrows into x reversed.
- Let V be a representation, and consider the map $\varphi_{\partial x} : \bigoplus_{y \to x} V(y) \to V(x)$.
- Then $\Phi_x^+(V)$ is the representation of $s_x(Q)$ with $\Phi_x^+(V)(x) = \ker(\varphi_{\partial x})$, mapping out to the neighboring $V(y)$ by projection.
- We define $\Phi_x^-(V)$ dually when x is a source.

Last time: properties of reflection functors

- Let S_x be the 1-dimensional representation supported at x, and let $\text{Rep}_x(Q)$ be the representations which don't have S_x as a summand.

Lemma

For $V \in \text{Rep}_x(Q)$,

$$\dim(\Phi_x^{\pm}(V)) = s_x(\dim(V))$$

where we view $\dim(V)$ as a linear combination of simple roots in the root system of Q, and s_x is the simple reflection along the simple root at x.

Lemma

The functors $\Phi_x^+ : \text{Rep}_x(Q) \to \text{Rep}_x(s_x(Q))$ and $\Phi_x^- : \text{Rep}_x(s_x(Q)) \to \text{Rep}_x(Q)$ are inverse equivalences of categories.

Reflection functors and indecomposables

- Each step we take in defining the functor preserves direct sums, thus:

Lemma

$$\Phi_x^+(V \oplus W) \cong \Phi_x^+(V) \oplus \Phi_x^+(W)$$

Corollary

If V is an indecomposable representation other than S_x, $\Phi_x^+(V)$ is an indecomposable representation of $s_x(Q)$.

Proof.

If $\Phi_x^+(V)$ decomposed as a direct sum, so would $\Phi_x^-\Phi_x^+(V) \cong V$. □

Reflection functors and indecomposables

Corollary

If V is an indecomposable representation other than S_x, $\Phi_x^+(V)$ is an indecomposable representation of $s_x(Q)$.

- There's an important parallel in the theory of root systems:

Proposition

If α is a positive root other than α_x, $s_x(\alpha)$ is a positive root.

- By starting with the simple representations S_y and applying reflection functors, we can build up indecomposables of the various orientations of a Dynkin diagram, corresponding to roots.
- The remaining questions:
  - Do all the indecomposables arise in this way?
  - Do all the positive roots appear for each orientation?

Every indecomposable corresponds to a root

- Let V be an indecomposable representation of Q.
- Our strategy: throw reflection functors at V and hope that we eventually get some S_y.
- We track what happens to the dimension vector as we apply reflections. If applying s_y causes it to have a negative coefficient, we must have applied Φ_y^+ to a representation not in Rep_y.
- But, because the representations we get are indecomposable, that can only have been S_y!
- For this to work, we'll need to apply many reflection functors in succession.

Definition

A sequence of vertices $x_1, \dots, x_m$ of Q is **adapted to** Q if x_i is a sink of $s_{x_{i-1}} \cdots s_{x_2} s_{x_1}(Q)$.

Coxeter elements

Lemma

Let Q be a quiver without directed cycles. Then all the vertices of Q can be ordered to form a sequence adapted to Q.

Proof.

Proceed inductively:

- Since Q is acyclic, it has a sink. Make that x_1.
- Remove x_1 from Q and let x_2 be a sink of the graph that remains.
- Repeat until you run out of vertices.

□

Definition

Let $x_1, \ldots, x_n$ be the ordering just obtained. The **Coxeter element** adapted to Q is the product of reflections

$$c = s_{x_n} \cdots s_{x_2} s_{x_1}.$$

Facts about the Coxeter element

- The linear map c has finite order. (More generally, the reflections of a finite root system generate a finite group: the **Weyl group**.)
- c has no fixed points — intuitively, because it changes each of the coefficients on the simple roots in turn. Thus $c - \mathrm{Id}$ is invertible.

Lemma

For any nonzero v in the space containing the roots, there is some exponent e such that $c^e v$ has a negative coefficient (in the basis of simple roots).

Proof.

Let h be the order of c. Then

$$v + cv + c^2 v + \ldots + c^{h-1} v = \left(\frac{c^h - 1}{c - 1} \right) v = 0$$

So one of the terms on the left must have a negative coefficient. □

Bringing it all together

- Let V be an indecomposable representation.
- If we repeatedly apply c to $\dim(V)$, we'll eventually end up with a negative coefficient.
- Thus, if we repeatedly apply C to V, we'll eventually be applying some s_y to a representation not in Rep_y.
- But that representation is indecomposable, so it must be S_y.
- Then we can undo all the reflections we performed to get there and write V in terms of reflection functors applied to S_y.

Theorem

If Q is a Dynkin quiver, the dimension vector of any indecomposable representation is a positive root, and an indecomposable representation is determined by its dimension vector.

Accounting for every positive root

- It remains to show that, for every orientation, every positive root occurs.
- There are a couple of ways to do this, but we'll see (without proof) an especially clean one, which will be useful later.

Lemma

(1) *There exists a product* $w_0 = s_{x_\ell} \cdots s_{x_2} s_{x_1}$ *of simple root reflections which sends every positive root to a negative root.*

(2) *This can be chosen such that* $x_1, \ldots, x_\ell$ *is adapted to the orientation of* Q*.*

(3) *The sequence of roots of the form* $s_{x_1} \cdots s_{x_{j-1}}(\alpha_{x_j})$, $1 \leq j \leq \ell$, *hits every positive root exactly once.*

(4) *The sequence of representations* $\Phi^{-} x_1 \cdots \Phi^{-}_{x_{j-1}}(S_{x_j})$ *(where* S_{x_j} *is defined on the appropriate reorientation of* Q*) produces an indecomposable representation of* Q *for each positive root.*

Two paths of inquiry

- When root systems show up in a larger context, it's natural to see if we can fit quiver representations into that context. Two directions:
- Coxeter groups and combinatorics!
  - The group generated by the reflections of a root system is an example of a **Coxeter group**.
  - These groups have a lot of interesting combinatorial structure, generalizing that of the symmetric groups.
  - **Do these structures have meaning on the side of quiver representations?**
  - This is the area of my research.
- Lie algebras and *their* representation theory!
  - The original motivation for root systems comes from using them to break down Lie algebras.
  - Indecomposable quiver representations give us the roots. **Can we recover the rest of the algebra in a natural way?**
  - This is what we'll talk about for the rest of the course.

(Re)introduction to Lie algebras

- A **Lie algebra** is a vector space $\mathfrak{g}$ with a bilinear, antisymmetric map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

- A prototypical example: an associative algebra A with $[a, b] = ab - ba$.
- An **ideal** of a Lie algebra is a subspace $\mathfrak{i}$ such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$.
- A Lie algebra is **simple** if it has no proper ideals.
- The prototypical example:

$$\mathfrak{sl}_n := n \times n \text{ matrices of trace } 0$$

with $[a, b] = ab - ba$.

The root space decomposition

- For any element $g \in \mathfrak{g}$, we refer to the linear map $[g, -] : \mathfrak{g} \to \mathfrak{g}$ as $\mathrm{ad}(g)$.
- Given a Lie algebra $\mathfrak{g}$, a **Cartan subalgebra** $\mathfrak{h}$ is a maximal subalgebra consisting of elements h for which $\mathrm{ad}(h)$ is diagonalizable.
- It turns out to be **abelian**: $[\mathfrak{h}, \mathfrak{h}] = 0$.
  - For $\mathfrak{sl}_n$, we use the subalgebra of diagonal matrices.
- The operators $\mathrm{ad}(h)$ for $h \in \mathfrak{h}$ commute, so they are *simultaneously* diagonalizable.
- Let $v \notin \mathfrak{h}$ be an eigenvector; then there exists $\alpha : \mathfrak{h} \to k$ such that $h(v) = \alpha(h)v$ for all $h \in \mathfrak{h}$. We say α is a **root** of $\mathfrak{g}$.

The root space decomposition

- For each root α, consider the **root space**

$$V_\alpha := \{v \in \mathfrak{g} \mid h(v) = \alpha(h)v \ \forall h \in \mathfrak{h}\}$$

- Then we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha V_\alpha$$

- There is a bilinear form on $\mathfrak{g}$ called the **Killing form**, given by

$$\mathrm{tr}(\mathrm{ad}(g)\,\mathrm{ad}(h))$$

  which restricts to an inner product on $\mathfrak{h}$.
- Using the Killing form, we identify $\mathfrak{h}$ with $\mathfrak{h}^*$. In this context:

Theorem

The collection of roots is a root system.

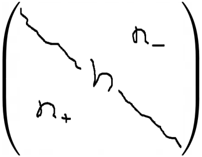
The root space decomposition of $\mathfrak{sl}_n$

- Let e_{ij} be the matrix with a 1 in the (i, j) entry and 0 elsewhere.
- Let $D(\lambda_1, \ldots, \lambda_n)$ be the diagonal matrix with the given entries. Then

$$[D(\lambda_1, \ldots, \lambda_n), e_{ij}] = (\lambda_i - \lambda_j)e_{ij}$$

- The Killing form on diagonal matrices is (up to a scale factor) the standard inner product, so this is precisely the root system A_{n-1} from before.
- Entries below the diagonal correspond to positive roots, and entries above the diagonal to negative roots, giving a coarser decomposition:

$$\mathfrak{sl}_n \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$



- So is there some way in which the indecomposable representations of a type A quiver act like the elements $e_{ij}, i > j$?

Exercise

Exercise

Choose some orientation of a Dynkin diagram. Write down an ordering of its vertices which is adapted to the orientation.

Exercise

Every positive root can be obtained from a simple root by applying reflections of simple roots.
Using this fact, show that every root has a corresponding indecomposable representation for any orientation of the quiver.

Next time. . .

- Reconstructing a Lie algebra from roots!
- All shall be enveloped!
- The Ringel-Hall algebra arrives in town!