Shard modules of preprojective algebras

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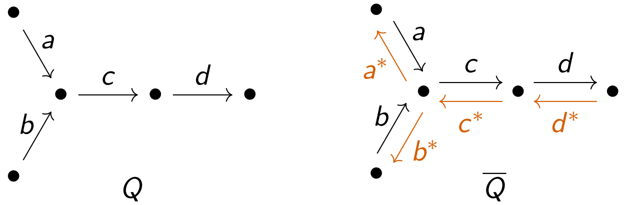
October 27, 2022

The preprojective algebra

- Let k be a field.
- Given a quiver Q with vertices Q_0 and edges Q_1, define a **double quiver** $\overline{Q}$:



Definition

The **preprojective algebra** Λ_Q is the quotient of the path algebra $k\overline{Q}$ by the relation

$$\sum_{a \in Q_1} (a^* a - a a^*) = 0$$

Stability and bricks

Definition

Fix a weight $\theta \in \mathbb{R}^{Q_0}$ and a Λ_Q-module M. Then M is θ-**semistable** if:

- $\langle \theta, \dim M \rangle = 0$;
- $\langle \theta, \dim N \rangle \geq 0$ for any submodule $N \subset M$.

The **stability domain** of M is the collection of all θ for which M is θ-semistable.

Definition

A **brick** is a Λ_Q-module M such that $\mathrm{End}_{\Lambda_Q}(M)$ is a division algebra.

- For any θ, the simple objects in the full subcategory of θ-semistable modules are bricks.

Root systems

- To Q associate a real vector space V with a basis of **simple roots** $\{\alpha_i\}_{i \in Q_0}$, and define a pairing $V \times V \to \mathbb{R}$ by

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & i = j \\ -(\# \text{ of arrows } i \to j \text{ in } \overline{Q}) & \text{otherwise} \end{cases}$$

- We define **reflections** $s_i : V \to V$, $i \in Q_0$:

$$s_i(\beta) := \beta - (\alpha_i, \beta)\alpha_i$$

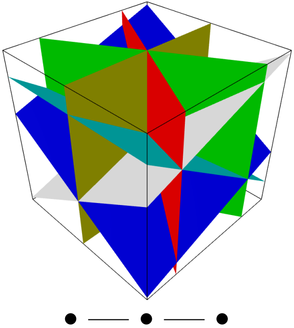
The s_i generate a **Coxeter group** W_Q.

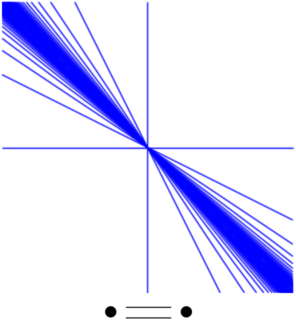
Definition

The **roots** are the elements of the form $w\alpha_i$ for $w \in W_Q, i \in Q_0$.

The Coxeter arrangement

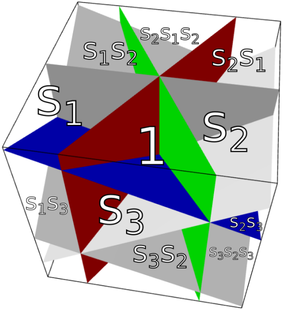
- The **Coxeter arrangement** is an arrangement of hyperplanes in V^*, given by $\beta^\perp$ for all roots β.
- Given the action $W_Q \circlearrowright V$, we get a dual action $W_Q \circlearrowright V^*$; these are the hyperplanes fixed by that action's reflections.

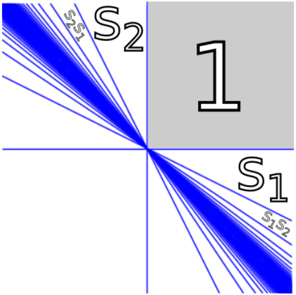




Regions of the arrangement

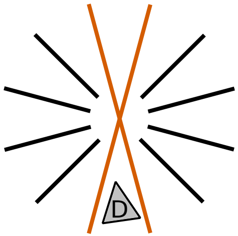
- Let D be the region of the Coxeter arrangement which pairs positively with the simple roots.
- The translates $\{wD \mid w \in W_Q\}$ are disjoint, so they are in bijection with W_Q.





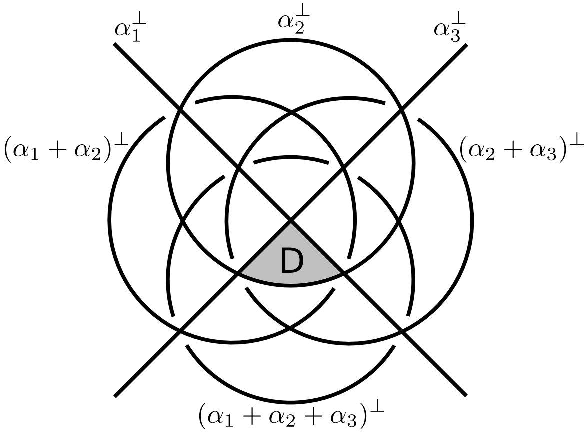
Shards

- Any two hyperplanes intersect in a codimension-2 subspace.
- The collection of hyperplanes containing that subspace is a **rank 2 subarrangement**.
- The two hyperplanes of this subarrangement closest to D are **fundamental**.
- For each rank 2 subarrangement, break all of its non-fundamental hyperplanes at their intersection.
- The result is a collection of convex cones called **shards** (Reading 2004).



Shards

- Here are the shards of the A_3 arrangement, intersected with a sphere and stereographically projected onto the plane.



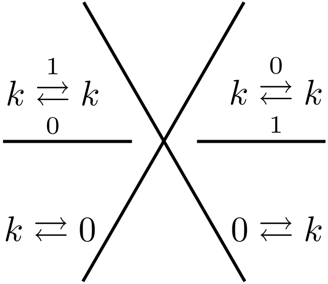
Bricks and shards

Theorem (Iyama-Reading-Reiten-Thomas 2016; Thomas 2017)

Let Q be a finite type quiver. Then there is a bijection

$$\{\text{bricks of } \Lambda_Q\} \leftrightarrow \{\text{shards of the Coxeter arrangement of } Q\}$$

Specifically, the stability domains of bricks are precisely the shards.



Beyond finite type: shard modules

Question

How much of this story is salvageable for non-Dynkin quivers?

Definition

A **shard module** of Λ_Q is a brick M such that:

- $\dim M$ is a root.
  - Equivalently, $\mathrm{Ext}^1(M, M) = 0$.
- The stability domain of M has dimension $\#Q_0 - 1$.
  - i.e., is as big as possible.

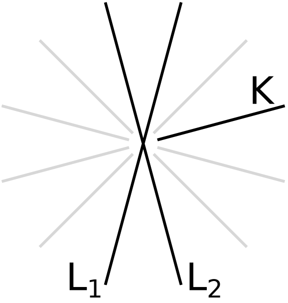
Theorem (D.-Speyer-Thomas)

Taking stability domains gives a bijection

$$\{\textit{shard modules of } \Lambda_Q\} \leftrightarrow \{\textit{shards of the Coxeter arrangement of } Q\}$$

Short exact sequences

- Let K be a shard of the hyperplane $\beta^\perp$.
- Choose a wall of K, and let $\gamma_1^\perp$ and $\gamma_2^\perp$ be the fundamental hyperplanes cutting $\beta^\perp$ at that wall. Assume WLOG that $\langle \gamma_1, - \rangle \geq 0$ on K.
- Suppose $\beta = c_1\gamma_1 + c_2\gamma_2$.
- Let L_1 and L_2 be the shards of $\gamma_1^\perp$ and $\gamma_2^\perp$ which meet K at that wall.
- Let $M(K)$, $M(L_1)$, and $M(L_2)$ be the associated shard modules.



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Theorem (D.)

- $\dim \mathrm{Hom}(M(K), M(L_1)) = \dim \mathrm{Hom}(M(L_2), M(K)) = 0$.
- $\dim \mathrm{Hom}(M(L_1), M(K)) = c_1, \dim \mathrm{Hom}(M(K), M(L_2)) = c_2$.
- *There exists a short exact sequence*

$$0 \to M(L_1)^{\oplus c_1} \to M(K) \to M(L_2)^{\oplus c_2} \to 0$$

with maps defined by bases of the Hom-spaces above.

References

- O. Iyama, N. Reading, I. Reiten, and H. Thomas. Lattice structure of Weyl groups via representation theory of preprojective algebras. *Compos. Math.* **154** (2018), no. 6, 1269–1305.
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