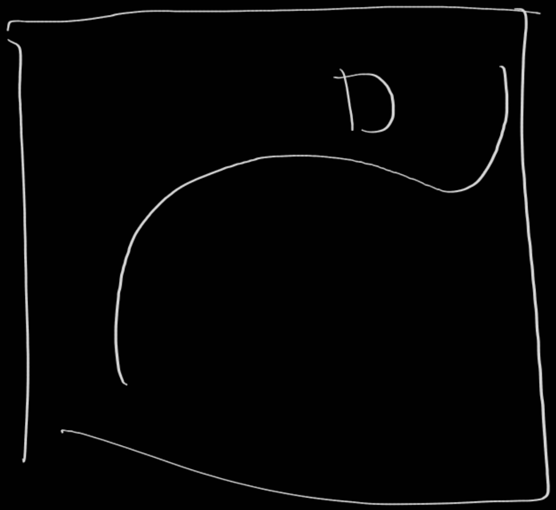
Curves on Surfaces



X variety $/k$

How can we "deform" D in X?

lin equivalence

$D \sim E$

$$\begin{tikzcd} V \arrow[dr, "f"'] \arrow[r, phantom, "\subset"] & X \times \mathbb{P}^1 \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

$$V \cap X \times \{0\} - V \cap X \times \{\infty\} = D - E.$$

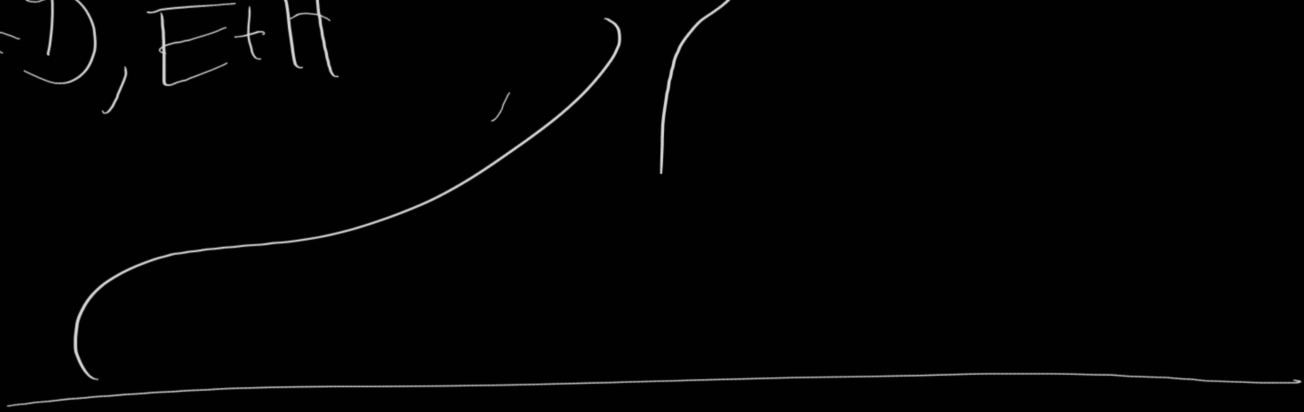
#D, E#

algebraic equiv.

$D \sim E$

$$V \subset X \times C \to C \quad \text{flat}$$

$$V \cap X \times \{a\} - V \cap X \times \{b\} = D - E$$



$F = \mathbb{P}^2$, $\quad F(x_0, x_1, x_2) \quad d$

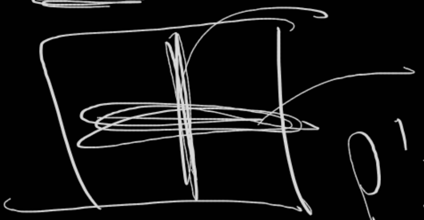
$\mathbb{P}^N$, $\quad N = \binom{d+2}{2} - 1$

$\mathcal{O}(d) \qquad D \underset{\text{lin}}{\sim} E$

$\mathcal{O}(D) \simeq \mathcal{O}(E)$

$\boxed{\text{alg} = \text{lin}}$

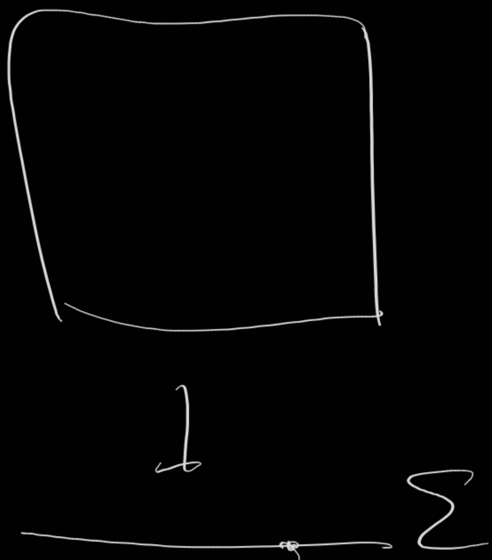
$F = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathcal{O}(d_1, d_2)$



$\mathbb{P}^1 \times \mathbb{P}^1$

alg = lin

$\boxed{F = \mathbb{P}^1 \times \mathcal{E}}$ $\quad \mathcal{E}$ elliptic curve



$p_1 : F \to \mathbb{P}^1$

$\boxed{\text{alg} \neq \text{lin}}$

$p_2 : F \to \Sigma$

if the degree $D \subset F$
$\downarrow$
$\mathbb{P}^1$

$\rightsquigarrow \boxed{\text{alg} \neq \text{lin}}$

How can we "deform" a line bundle

$$\begin{array}{c} \mathcal{L} \to X \times S \\ \downarrow \\ S \end{array}$$

Familiès line bundles $/S$

$$\mathcal{L} \to X \times S, \quad \downarrow \; S \qquad \Big| \qquad \boxed{\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes p_S^* M, \quad M \in \operatorname{Pic} S}$$

<u>Def:</u> $F : (Sch_{/S})^{op} \to Sets$

F is a Zariski sheaf if for all X/S and for all open covers

$$\{U_\alpha\}_\alpha \text{ of } X, \quad U_\alpha \to S, \quad X \to S$$

$$0 \to F(X) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_{\alpha\beta})$$

<u>Rk:</u> $F \simeq h_T \Rightarrow F$ is a Zariski sheaf

(Absolute Picard functor)

Fix X/S. $F : (Sch/S)^{op} \longrightarrow Sets$

$T/S \longrightarrow \underline{Pic(X \times_S T)}$

$Pic(X) = \{ \mathcal{L} \sim X \}/\sim$

$Pic(\cancel{X \times_S T}) \neq 0 \qquad T = \mathbb{P}^1_{\mathbb{R}} \supset \mathbb{A}^1_{\mathbb{R}} \cup \mathbb{A}^1_{\mathbb{R}}$

$\ni \mathcal{L} \qquad$ Take $\{\underline{U_\alpha}\}$ for $\mathcal{L}|_{U_\alpha} \simeq \mathcal{O}|_{U_\alpha}$

$0 \to \underline{F(T)} \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_{\alpha,\beta})$

$\mathcal{O}_{U_\alpha} \simeq \mathcal{L}|_{U_\alpha} \in F(U_\alpha) \longleftarrow F(U)$

$\mathcal{L} \longmapsto (0, 0, \ldots 0) = 0$

$S = Spec(k)$, $T/\mathbb{R}$ $\quad$ ~~Pic~~ $F(T) = Pic(X \times T)$

$\boxed{M \in Pic(T)} \quad U_\alpha$

$\underline{Pic(X \times T)} \longrightarrow \prod \underline{Pic(X \times U_\alpha)}$

$p_T^* Pic\, T \qquad p_{U_\alpha}^* Pic(U_\alpha)$

$p^* M|_{X \times U_\alpha} = p_{U_\alpha}^*(M') \qquad \boxed{M' = M|_{U_\alpha}}$

$$U_\alpha \hookrightarrow X$$
open embeddings $\Rightarrow$ flat

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rel Picard functor

$$F : (Sch/S)^{op} \to Sets$$
$$T/S \longrightarrow \operatorname{Pic}(X \times_S T) / p_T^* \operatorname{Pic}(T)$$

Rk: X/S is an algebraic fibes space and has $(f_* \mathcal{O}_X = \mathcal{O}_S)$

an S-valued point $\Rightarrow$ rel. Picard functor is a Zariski sheaf.

Thm: $f: X \to S$ be a flat projective with geometrically integral fibers so that f is a "universal" algebraic fiber space with S-value

$\Rightarrow \operatorname{Pic}_{X/S}$ representable by a loc finite type seperated S-scheme with qproj components

Special (improved) case:

$S = \operatorname{Spec}(k)$ X smooth projective integral$/k$

$\Rightarrow \operatorname{Pic}_{X/k}$ exists and has projective components$/k$

What if X is a curve? $\quad \mathrm{Pic}(X) \ni [\mathcal{O}_X]$

$$\mathrm{Pic}^0(X) = \frac{\{\sum a_i p_i \mid \sum a_i = 0\}}{\{\sum a_i p_i \text{ lin. eq to } 0\}} = \mathrm{Jac}(X)$$

$\mathcal{O}_X(\sum a_i p_i) \simeq \mathcal{O}_X$

$$\boxed{\begin{array}{ccc} \mathrm{Pic}^d(X) & \overset{\rho}{\underset{\pi}{\rightleftarrows}} & S^d X \\ \mathcal{O}(\sum p_i) & \longleftarrow & (p_1, \dots, p_d) \\ & \pi\rho = 1 & \end{array}}$$

Over $\mathbb{C}$

$$\mathrm{Jac}(C) = \frac{H^0(\Omega^1)^*}{H_1(C, \mathbb{Z})}$$

$$\mathrm{Pic}^0(X) = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})} = \frac{H^1(X, \mathbb{C})}{F^1 H^1(X, \mathbb{C}) + H^1(X, \mathbb{Z})}$$

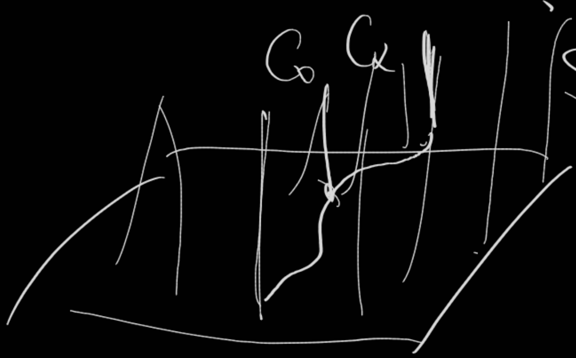
$$H^k(X, \mathbb{C}) = \bigoplus_{j+i=k} H^i(\Omega^j_X) \qquad dz_\alpha, d\bar{z}_\alpha$$

$$F^{j_0} H^k(X, \mathbb{C}) = \bigoplus_{\substack{j+i=k \\ j \geq j_0}} H^i(\Omega^j_X)$$

CCF
curve surface

$S = \{ C_\alpha \mid \alpha \}$ algebraically equivalent curves to $C_0 = C$. Let $S_0 \subset S$

linear subfamily corresponding to C_0



$$\mathcal{O}(C_\alpha) \simeq \mathcal{O}(C_0)$$

B, B_0

$$B/B_0 \xrightarrow{\sim} \boxed{\frac{\text{curves alg eq } 0 \;\; \leftarrow \mathrm{ZaF}}{\text{-- '' -- lin eq } 0 = \mathrm{ZeF}}}$$

$$\alpha \mapsto C_\alpha - C_0$$

For "good" C this is bijective.

$$0 \to \mathcal{O}^\times \to K^\times \to K^\times/\mathcal{O}^\times \to 0$$

$$0 \to \underset{k^\times =}{H^0\mathcal{O}^\times} \to H^0K^\times \to \underbrace{H^0(K^\times/\mathcal{O}^\times)}_{\text{Cartier divisors}}$$

$$\to H^1\mathcal{O}^\times \to 0$$

$$0 \to \boxed{H^0K^\times / k^\times} \hookrightarrow H^0(K^\times/\mathcal{O}^\times) \to \underset{\mathrm{Pic}(X)\,=}{H^1\mathcal{O}^\times} \to 0$$

$$\boxed{D \text{ s.t. } \mathcal{O}(D) \simeq \mathcal{O}_X}$$

ZeF

$$\frac{Z_a F}{Z_\ell F} \hookrightarrow \frac{Z(F)}{Z_\ell F} \simeq H^1\mathcal{O}_X^*$$

$$\frac{Z_a F}{Z_\ell F} = \mathrm{Pic}^0(X)$$

Conj of Severi $\dim \frac{Z_a F}{Z_\ell F} = h^1\mathcal{O} = \dim_k H^1(\mathcal{O}_X)$

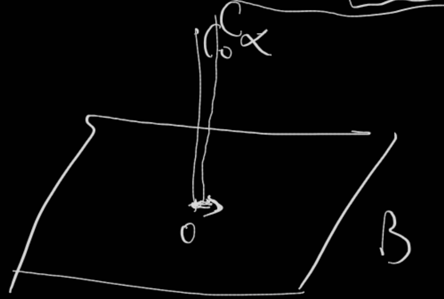
1909 was proved by Poincaré for char $k = 0$

disproved by Igusa for char $k \neq 0$.

Over $\mathbb{C}$, $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$

$$H^1\mathcal{O} \to H^1\mathcal{O}^*$$

Rk: $\dim \frac{Z_a F}{Z_\ell F} = \boxed{h^1\mathcal{O}_F} \iff \rho : T_0 B \xrightarrow[\text{bijective}]{} H^0(N_{C/F})$ "characteristic map"



if we have an "infinitesimal" deformation of C

can we actually upgrade that to an "actual deformation" of C

$$\left\{ 0 \to \mathcal{O}(C) \to \underset{= \mathcal{O}_C(C)}{N_{C/F}} \to 0 ,\ B_0 \subset B,\ B/B_0 \simeq \frac{Z_a F}{Z_\ell F} \right\}$$

~~Def:~~ $H^1\mathcal{O}$ " $\stackrel{\dim}{=}$ tgsp at $[\mathcal{O}_X] \in \operatorname{Pic}^0 X$ "

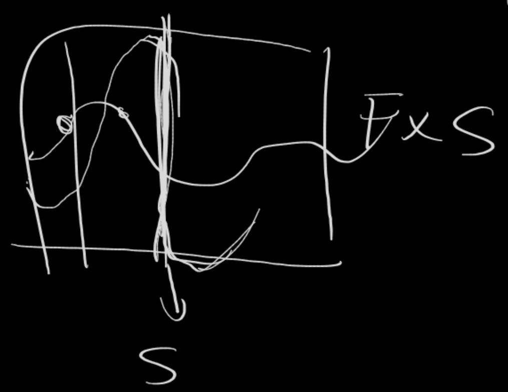
$$\geq \text{" } \dim \operatorname{Pic}^0 X \text{ "}$$

Def: S/k separated $\left(\mathrm{Hilb}_{X/k} = \text{all relative ideal sheaves} \right)$

$$\underset{(F/k)}{\mathrm{Curves}(S)} = \text{" families of curves on } F/S \text{ "}$$

$$= \left\{ \begin{array}{l} \mathcal{D} \subset F \times S \ \text{ Cartier divisor} \\ \text{flat} \searrow \ \downarrow p \\ \qquad\quad S \end{array} \right\} =$$

$$= \left\{ \begin{array}{l} \exists \text{ an open cover } \{U_\alpha\} \text{ of } S \text{ so that} \\ \mathcal{D} \times_S U_\alpha \text{ is given by } 1 \text{ equation } f = 0 \\ \qquad \text{where } f \text{ is not a zerodivisor} \\ \qquad\qquad \text{in } \mathcal{O}_{p^{-1}(s)} \ \forall s \in F(\bar{k}) \end{array} \right\}$$



$$\begin{array}{ccc} \mathrm{Curves}_F & \xrightarrow{\ \phi\ } & \mathrm{Pic}_F \\ D \subset F\times S & & (\mathcal{O}(D)) \sim F\times S \\ \text{flat} \searrow \downarrow & & \downarrow \\ S & & S \end{array}$$

$$\mathcal{O}_{F\times S} \qquad \mathcal{O}_S$$

<u>Def:</u> C_1, C_2 curves in F

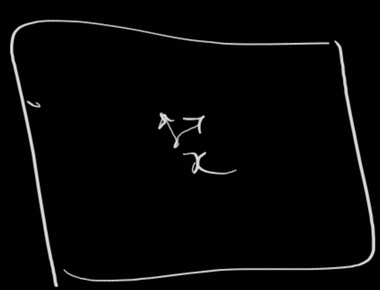
$$C_1 \underset{num}{\sim} C_2 \quad \text{iff} \quad C_1 \cdot C = C_2 \cdot C \quad \forall \text{ curves } C \subset F.$$

<u>Facts:</u>

•) $x : \operatorname{Spec} k \to X$ closed pt.

$$T_x X = \left(\frac{m_x}{m_x^2}\right)^*, \quad \mathrm{Hom}\left(\operatorname{Spec} \frac{k[\varepsilon]}{(\varepsilon^2)}, X\right) \times_{X(k)} x$$

X



$$\begin{tikzcd} \operatorname{Spec}(k) \arrow[r] \arrow[d] & X \\ \operatorname{Spec} \frac{k[\varepsilon]}{(\varepsilon^2)} \arrow[ur, dotted] & \end{tikzcd}$$

$$h_{G_k(r,n)}(S) = \left\{ \mathcal{O}_S^n \twoheadrightarrow \mathcal{E} \;\middle|\; \begin{array}{l} \mathcal{E} \text{ loc free} \\ \mathrm{rk}\,\mathcal{E} = n-r \end{array} \right\} \Big/ \text{iso}$$

$$\begin{tikzcd} \mathcal{O}^n \arrow[r, two heads] \arrow[dr, two heads] & \mathcal{E} \arrow[d, "\wr"] \\ & \mathcal{E}' \end{tikzcd}$$

$$h_{G(n,n+1)} \simeq h_{\mathbb{P}^n}$$

$$\Updownarrow$$

$$G(n,n+1) \simeq \mathbb{P}^n$$

$$h_{\mathbb{P}^n}(S) = \left\{ \mathcal{O}_S^{n+1} \twoheadrightarrow \mathcal{L} \;\middle|\; \mathcal{L} \text{ line bdl.} \right\}.$$

·) D_1, D_2 divisors on a surface F

$$D_1 \cdot D_2 = \chi\mathcal{O}_F - \chi\mathcal{O}_F(-D_1) - \chi\mathcal{O}_F(-D_2) + \chi\mathcal{O}_F(-D_1-D_2).$$

$$0 \to \mathcal{O}(-D_i) \to \mathcal{O}_F \to \mathcal{O}_{D_i} \to 0, \quad i = 1,2.$$

·) (Serre Duality) K

$$H^i(X, \mathcal{E})^* = H^{n-i}(X, \mathcal{E}^* \otimes K)$$

$$H^i(\mathbb{P}^n, \mathcal{F})^\vee \simeq \operatorname{Ext}^{n-i}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \underbrace{\mathcal{O}_{\mathbb{P}^n}(-n-1)}_{= K})$$

- Riemann Roch (for surfaces)

$$\chi\mathcal{L} = \frac{1}{2} L\cdot(L-K) + \chi(\mathcal{O}_F)$$

Flattening Stratification

& Vanishing theorems