$\boxed{\mathrm{Curves}^{(S)}_{F}} \hookrightarrow \boxed{\mathcal{H}ilb_{F}}$

Cartier

$\mathcal{D} \subset F \times S$

$$\begin{array}{c} \mathcal{D} \subset F \times S \\ \text{flat} \searrow \quad \downarrow \\ S \end{array}$$

$$h_{\mathbb{P}^n}(S) = \left\{ \mathcal{O}_S^{n+1} \twoheadrightarrow \mathcal{L} \,\middle|\, \mathcal{L} \text{ a line bundle} \right\}$$

Flattening Stratification

$$\begin{array}{c} \mathcal{F} \sim X \times S \\ \downarrow \\ S \end{array}$$

$\mathcal{F}$ flat/S

iff

$\mathcal{F}_{(x,s)}$ are flat as $\mathcal{O}_{S,s}$ modules

$\mathcal{F} \sim X$

$\mathcal{F}$ flat/$\cancel{X}$

$\mathcal{F}_x$ is flat as $\mathcal{O}_{X,x}$ module

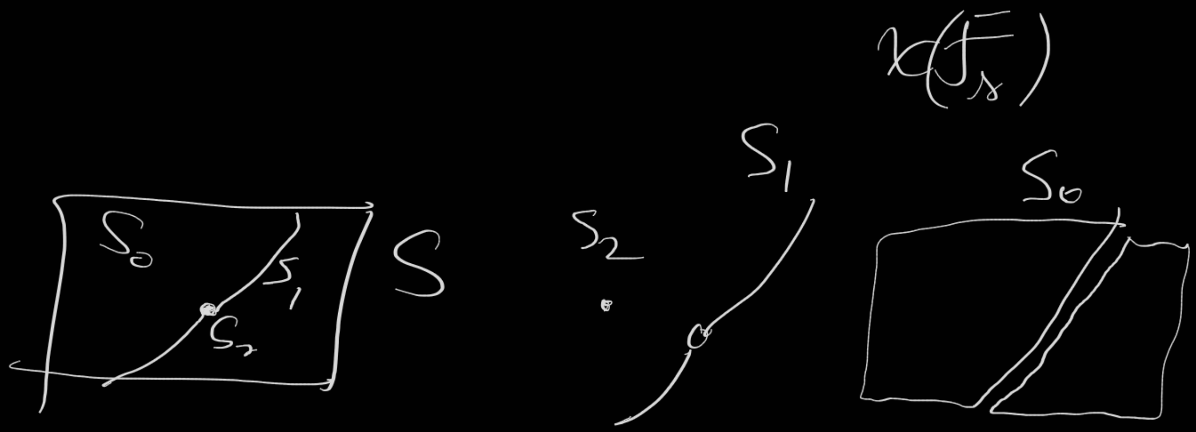
$\Downarrow$

$\mathcal{F}_x$ is free $\mathcal{O}_{X,x}$ module.

$\Downarrow$

$\mathcal{F}$ is vector bundle.

$\chi(\mathcal{F}_s)$ upper semi continuous function.



Def: S/k scheme a <u>stratification</u> is a finite set $S_1, \dots, S_m$ of locally closed subschemes so that $\coprod_{i=1}^{m} S_i \to S$ bijective as sets.

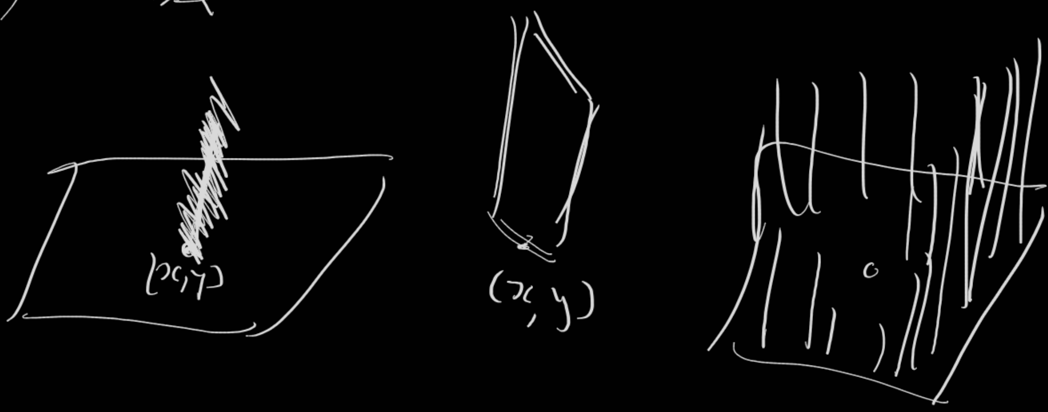
<u>Thm</u>: Let $\mathcal{F}$ on $\mathbb{P}^n \times S$ (S noetherian). Then, there is a stratification of S $S_1, \dots, S_m$ so that for any $g : T \to S$ (noetherian)

$$\begin{array}{ccc} (1,g)^*\mathcal{F} & & \mathcal{F} \\ \mathbb{P}^n \times T & \xrightarrow{(1,g)} & \mathbb{P}^n \times S \\ \downarrow & \ulcorner & \downarrow \\ T & \longrightarrow & S \end{array}$$

$(1,g)^*\mathcal{F}$ flat$/T \iff T \xrightarrow{g} S$ factors through $\coprod_{i=1}^{m} S_i \to S$

Moreover, $\{\mathcal{F}|_s \mid s \in S_i\}$ have constant Hilbert polynomial $\chi(\mathcal{F}|_s(t))$ is constant

$\mathcal{I}_{(x,y)} \subset \mathcal{O}_{\mathbb{A}^2}$



Vanishing thms

Def: $\mathcal{F}$ is m-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ $\forall i > 0$.

Thm: $\mathcal{F}$ m regular, then

$$\left.\begin{array}{l} H^0\mathcal{F}(k) \otimes H^0\mathcal{O}_{\mathbb{P}^n}(1) \twoheadrightarrow H^0\mathcal{F}(k+1) \\ \mathcal{F}(k) \text{ is globally generated} \end{array}\right\} \forall k \geq m$$

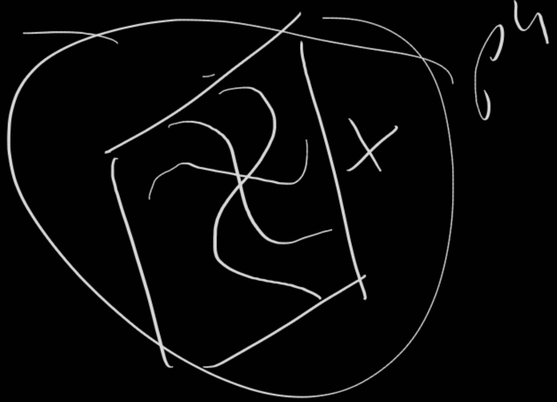
$H^i\mathcal{F}(k) = 0 \quad \forall i > 0, \ k \geq m-i$

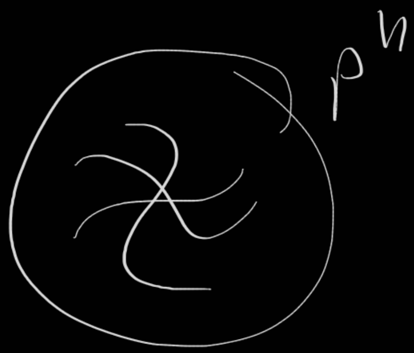
Thm: $\forall n \in \mathbb{N}$, there is a polynomial $F(x_0, \ldots, x_n)$ s.t. for all subschemes in $\mathbb{P}^n$ $\boxed{\text{in } X}$ with fixed Hilbert polynomial

$$\chi(\mathcal{I}(m)) = \sum_{i=0}^{n} a_i \binom{m}{i}, \text{ then,}$$

$\mathcal{I}$ is $F(a_0, \ldots, a_n)$-regular

$I(\underbrace{F(a_0,\dots,a_n)})$





$I_Z \subset \mathcal{O}_X$

$I_Z \subset \mathcal{O}_{\mathbb{P}^n}$

$I \cap$

$I_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X$

$0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$

$\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to \mathcal{O}_Z$

$\exists m$ $\quad I(m), \ \mathcal{O}_Z(m)$

Pf: Induct on n. $n=0$ obvious.

$(n \to n+1)$ $\quad H \subset \mathbb{P}^{n+1}$ be a hyperplane.

$$0 \to I \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_Z \to 0 \quad / \otimes \mathcal{O}_H$$

$x \in \mathbb{P}^n$

$$\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{Z,x}, \mathcal{O}_{H,x}) \to I_{H,x} \to \mathcal{O}_{H,x} \to \mathcal{O}_{Z\cap H,x} \to 0$$

$H = \{f = 0\}$ near x

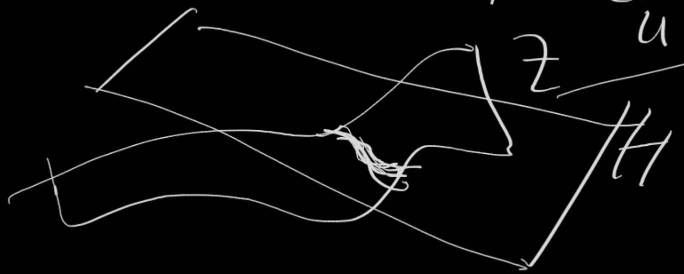
$$\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^n,x}}\left(\mathcal{O}_{Z,x},\ \mathcal{O}_{\mathbb{P}^n,x}/f\cdot\mathcal{O}_{\mathbb{P}^n,x}\right)$$

$x \notin Z$

$x \in Z$

$$\mathrm{Tor}_1^R\left(B,\ R/(f)R\right) = \{u \in B \mid u\cdot f = 0\} = 0$$

(because H doesn't contain any assoc. pt. of Z)



$$\boxed{0 \to I_H \to \mathcal{O}_{H\simeq\mathbb{P}^{n-1}} \to \mathcal{O}_{Z\cap H} \to 0}$$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-H) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0 \quad / \otimes\ \mathcal{I}\mathcal{O}(m)$$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-1) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_H(m) \to 0 \quad / \otimes\ \mathcal{I}$$

$\mathcal{I} \subset \mathcal{O}$

$$\boxed{0 \to \mathcal{I}(m) \to \mathcal{I}(m+1) \to \mathcal{I}_H(m+1) \to 0}$$

$$\chi\,\mathcal{I}_H(m+1) = \chi\,\mathcal{I}(m+1) - \chi\,\mathcal{I}(m)$$

$$= \sum_{i=0}^{n} a_i\binom{m+1}{i} - a_i\binom{m}{i}$$

$$= \sum_{i=0}^{n-1} \underline{a_{i+1}}\binom{m}{i}$$

I_H is $\underline{m \geq G(a_1, \dots, a_{n+1})}$ – regular?

p_1
$\vdots$ after m
p_m
$\vdots$
p_N

$$0 \to H^0 I(m) \to H^0 I(m+1) \xrightarrow{\rho_{m+1}} H^0 I_H(m+1) \to$$

$$H^1(I(G(a_1,\dots,a_n))) \to H^1(I(G(a_1,\dots,a_n)+1))$$

$$\to H^1 I(m) \to H^1 I(m+1) \to 0$$

$$0$$

$$\to H^i I(m) \xrightarrow{\sim} H^i I(m+1) \to 0 \qquad \forall i \geq 2$$

$$\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \underline{\forall m \geq G(a_1, \dots, a_n)}$$

$$\qquad\qquad\qquad\qquad\qquad 0$$

ρ_{m+1} surj

$$\begin{tikzcd}
H^0 I(m+1) \otimes H^0 \mathcal{O}_{\mathbb{P}^n}(1) \arrow[r, two heads, "(\rho_{m+1} \otimes 1)"'] \arrow[d] & H^0 I_H(m+1) \otimes H^0 \mathcal{O}_{\mathbb{P}^n}(1) \arrow[d, two heads, "\text{by induction hypothesis}"] \\
H^0 I(m+2) \arrow[r, two heads, "\rho_{m+2}"'] & H^0 I_H(m+2)
\end{tikzcd}$$

ρ_{m+1} surj $\Rightarrow$ ρ_{m+2} surj.

ρ_{m_0+1} surj $\Rightarrow$ $\underline{H^1 I(m_0+1)} \xrightarrow{\sim} H^1 I(m_0+2)$

ρ_{m_0+2} surj $\Rightarrow$ $H^1 I(m_0+2) \xrightarrow{\sim} H^1 I(m_0+3)$

$m = G(a_1, \dots, a_n)$

$p_m, p_{m+1}, \dots,$

$\dim H^1(\mathcal{I}(G(a_1, \dots, a_n)))$

then $\mathcal{I}$ will become

$$G(a_1, \dots, a_n) + \underline{h^1(\mathcal{I}(\overbrace{G(a_1, \dots, a_n)}^{m_1}))} \text{ - regular}$$

$$\{\mathcal{O}_{\mathbb{P}^r}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k)\}_{k \in \mathbb{N}}.$$

k

$$h^1\mathcal{I}(m_1) = h^0\mathcal{I}(m_1) - \chi(\mathcal{I}(m_1))$$

$$\leq h^0\mathcal{O}(m) - \chi(\mathcal{I}(m_1))$$

$$= \underline{\chi(\mathcal{O}(m))} - \underline{\chi(\mathcal{I}(m_1))}$$

$$\shortparallel$$

$$H(a_0, \dots, a_n, G(a_1, \dots, a_n))$$

$$\shortparallel$$

$$H'(a_0, \dots, a_n)$$

$\mathcal{I}$ is $G(a_0, \dots, a_n) + H'(a_0, \dots, a_n) = F(a_0, \dots, a_n)$.

$0 \to I \to \mathcal{O}_X \quad / \otimes \mathcal{O}(m)$

$0 \to I(m) \to \mathcal{O}_X(m) \to$

$0 \to H^0 I(m) \to H^0 \mathcal{O}_X(m)$ is a point on a Gr

$h^0 I(m) = \chi I(m)$

Curves_F^P as a closed subscheme of a Grassmannian

Linear Systems & Examples

$D \qquad \{E \geq 0 \mid E \underset{\text{lin}}{\sim} D\}$,

$\phi^{-1}([\mathcal{L}]) \qquad \mathcal{O}(E) \simeq \mathcal{O}(D)$

$\mathrm{Curves}_F \xrightarrow{\phi} \mathrm{Pic}_F \ni [\mathcal{L}]$

$\mathcal{D} \longrightarrow \mathcal{O}(\mathcal{D})$

<u>Proposition</u>: Let X projective$/k$.
Assume $\exists$ irred divisor $H \subset X$ so that
$X - H = \mathrm{Spec}(A)$, A UFD

Then, $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Num}(X) = \langle [H] \rangle_{\mathbb{Z}}$

$\mathcal{L}$

$\mathcal{L} \cdot \mathcal{L}_1 = 0 \quad \forall\, \mathcal{L}_1 \in \mathrm{Pic}(X)$

Pf: $\langle [H] \rangle_{\mathbb{Z}} \twoheadrightarrow \mathrm{Pic}(X)$ $\qquad D \in \mathrm{CDiv}(X)$

$\boxed{\text{irreducible}}\ H \longrightarrow \mathcal{O}(H) \qquad D = D^+ - D^-$

$(D \geq 0)$

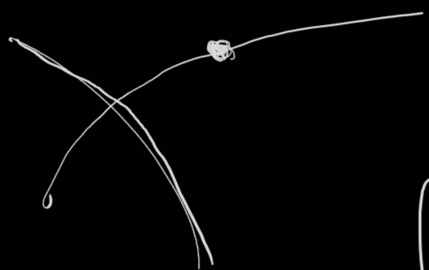
$D \cap \mathrm{Spec}(A) = \mathrm{Spec}(A/\mathcal{J})$

$\mathfrak{p} \in \mathrm{Spec}(A) \qquad \mathcal{J}_{\mathfrak{p}} \subset A_{\mathfrak{p}}$

$\mathcal{J}_{\mathfrak{p}} = (f)_{\mathfrak{p}}$

$\mathcal{J}_{\mathfrak{p}}$ is a PID $\forall \mathfrak{p}$

$\mathrm{Ass}(\mathcal{J}) = \mathrm{Ass}_A(A/\mathcal{J})$.



$\boxed{A \text{ UFD}}$ $\qquad \mathcal{J}$ is a height 1 prime

$\boxed{\mathcal{J} = (f)}$

$D = (f)$ on A

$D = (f) + n \cdot H$

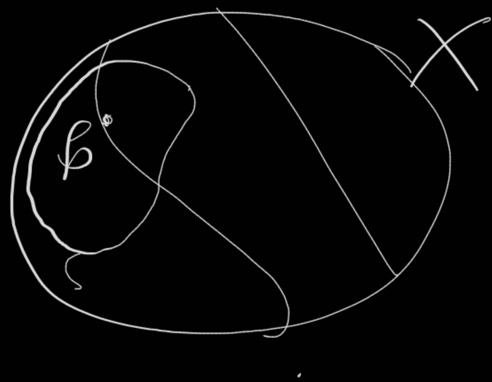
X

$$\tilde{\mathcal J} \subset \mathcal J \subset A \Longrightarrow \tilde{\mathcal J} = 0 \text{ OR } \mathcal J.$$

$$A_{\mathfrak p}/\tilde{\mathcal J}_{\mathfrak p} \twoheadrightarrow A_{\mathfrak p}/\mathcal J_{\mathfrak p} \qquad \mathcal J_{\mathfrak p} = (f)_{\mathfrak p}$$

irreducible

Because $\tilde{\mathcal J}_{\mathfrak p}$ prime



$$\boxed{\tilde{\mathcal J}_{\mathfrak p} = \underline{\mathcal J_{\mathfrak p}} \text{ or } \underline{0_{\mathfrak p}}.}$$

$$\mathcal J_{p} = (f)_{p}$$

$$\tilde{\mathcal J} = \mathcal J \text{ or } \tilde{\mathcal J} = 0.$$

$\Longrightarrow \mathcal J$ height 1.

This shows: $\langle H \rangle_{\mathbb Z} \twoheadrightarrow \mathrm{Pic}(X)$

$$D \geq 0$$

$\mathrm{Supp}(D - (f)) \subset H$ (X integral proj/k)

$$\Longrightarrow D - (f) = nH$$

$$\mathcal O(D) = \mathcal O(nH)$$

$$\mathbb Z \twoheadrightarrow \mathrm{Pic}\, F \twoheadrightarrow \mathrm{Num}(F)$$

$$H \longrightarrow [H]$$

$nH = \mathcal{O}(1)$ $\deg(H)$

$$H^{\dim X - 2} \cdot nH = H^{\dim X - 1} \cdot \mathcal{O}(1) > 0$$

$nH \longrightarrow 0$

$$\underbrace{nH \cdot nH}_{\geq 0} = 0 \cdot nH = 0$$