

Construction of  $\text{Curves}_F$  as a scheme  $X = F$  is smooth

$$\text{Curves}_F(S) = \left\{ \begin{array}{l} D \subset F \times S \text{ Cartier} \\ \text{flat} \downarrow \\ S \end{array} \right\}$$

①  $\mathcal{I}$  with fixed Hilbert polynomials  
(ideal)

$F$  is independent of  $\mathcal{I}$  so that  $\mathcal{I}$  is  $m$ -regular.  
 $P$  Hilbert poly.

$$\text{Curves}_F^P(S) = \left\{ \begin{array}{l} D \subset F \times S \text{ Cartier} \\ \text{flat} \downarrow \\ S \end{array} \right\} \left\{ \begin{array}{l} \chi(\mathcal{O}_D(t)) \\ P(t) \\ t \in S \end{array} \right\}$$

$$\coprod_{i=1}^m S_i \xrightarrow{P} S$$

$$\text{Curves}_F^P \rightsquigarrow \coprod_{P \text{ Hilb. poly.}} \text{Curves}_F^P \simeq \text{Curves}_F(S)$$

$$S \longrightarrow \coprod_{P \text{ Hilb. poly.}} \text{Curves}_F^P$$

$$\simeq \text{Curves}_F(S)$$

$$\boxed{\text{Curves}_F^P} \quad D \subset F \text{ (be Cartier)}$$

$$\chi(\mathcal{O}_F(-D)(t)) = P(t)$$

$\exists m \text{ s.t. } D \text{ has Hilb poly } P$   
 $\implies D' \text{ is } m \text{ regular.}$

$$0 \rightarrow I_D^{(m)} \rightarrow \mathcal{O}_X^{(m)} \rightarrow \mathcal{O}_D^{(m)} \rightarrow 0$$

$$\implies H^0 I_D^{(m)} \rightarrow H^0 \mathcal{O}_X^{(m)} \rightarrow H^0 \mathcal{O}_D^{(m)} \rightarrow 0$$

$$\rightsquigarrow G \left( \underbrace{H^0 \mathcal{O}_D^{(m)}}_{\cong \mathcal{X}(\mathcal{O}_D^{(m)})}, \underbrace{H^0 \mathcal{O}_X^{(m)}}_{\cong \mathcal{X}(\mathcal{O}_X^{(m)})} \right)$$

$$\mathcal{X}(\mathcal{O}_X^{(m)} - P(m))$$

$\mathcal{D} \subset \text{FXS}$  Cartier,  
 $\text{flat} \searrow \int \pi$

$$0 \rightarrow I_{\mathcal{D}}^{(m)} \rightarrow \mathcal{O}_{\text{FXS}}^{(m)} \rightarrow \mathcal{O}_{\mathcal{D}}^{(m)} \rightarrow 0 \quad / \pi_X$$

$$0 \rightarrow \pi_X I_{\mathcal{D}}^{(m)} \rightarrow \pi_X \mathcal{O}_{\text{FXS}}^{(m)} \rightarrow \pi_X \mathcal{O}_{\mathcal{D}}^{(m)} \rightarrow 0$$

$$\hookrightarrow R^1 \pi_X I_{\mathcal{D}}^{(m)} = 0$$

$$\begin{array}{ccc}
 F_S & \longrightarrow & F \times_S - \mathcal{I}_D^{(m)} \\
 \downarrow & & \downarrow \pi \\
 S & \longrightarrow & S - R^1 \pi_* \mathcal{I}_D^{(m)}
 \end{array}$$

$$\underline{\underline{R^1 \pi_* \mathcal{I}_D^{(m)}|_S}} \longrightarrow H^1(\mathcal{I}_D^{(m)}|_{F_S})$$

$$H^1(\mathcal{I}_D^{(m)}|_{F_S}) = 0$$

$$R^i \pi_* \mathcal{I}_D^{(m)} = 0 \quad \forall i > 0$$

$$R^i \pi_* \mathcal{O}_D^{(m)} = 0 \quad \forall i > 0$$

$$\begin{array}{ccc}
 F_S & \longrightarrow & F \times_S - \mathcal{O}_D^{(m)} \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & S - \pi_* (\mathcal{O}_D^{(m)})
 \end{array}$$

$H^0(\mathcal{O}_D^{(m)})$  independent of  $s \in S$   
 $R^i \pi_* \mathcal{O}_D^{(m)} = 0 \implies \pi_* \mathcal{O}_D^{(m)}$  locally free

$$0 \rightarrow \pi_* \mathcal{I}_D^{(m)} \rightarrow \pi_* \mathcal{O}_{F \times_S}^{(m)} \rightarrow \pi_* \mathcal{O}_D^{(m)} \rightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow \\
 \quad \quad \quad K$$

$$\pi_* (\pi^* \mathcal{O}_S)(m)$$

$$\mathcal{O}_S \otimes H^0(\mathcal{O}_F(m)) \xrightarrow{\sim} \pi_* \mathcal{O}_{F \times S}(m)$$

$$\pi^*(\mathcal{O}_S \otimes H^0(\mathcal{O}_F(m))) \longrightarrow \mathcal{O}_{F \times S}(m)$$

$$\mathcal{O}_{F \times S} \otimes H^0(\mathcal{O}_F(m)) \longrightarrow \mathcal{O}_{F \times S}(m)$$

$$0 \rightarrow \pi_* \mathcal{O}_D(m) \rightarrow \mathcal{O}_S \otimes \boxed{H^0(\mathcal{O}_F(m))} \rightarrow \pi_* \mathcal{O}_D(m) \rightarrow 0$$

$$\langle e_1, \dots, e_n \rangle \in H^0(\mathcal{O}_F(m)) \xrightarrow{\sim} \mathcal{O}_S^{\oplus n} \otimes \mathcal{O}_F(m)$$

$S$ -val'd pt of  $G(r, H^0(\mathcal{O}_F(m)), \chi(\mathcal{O}_F(m)))$

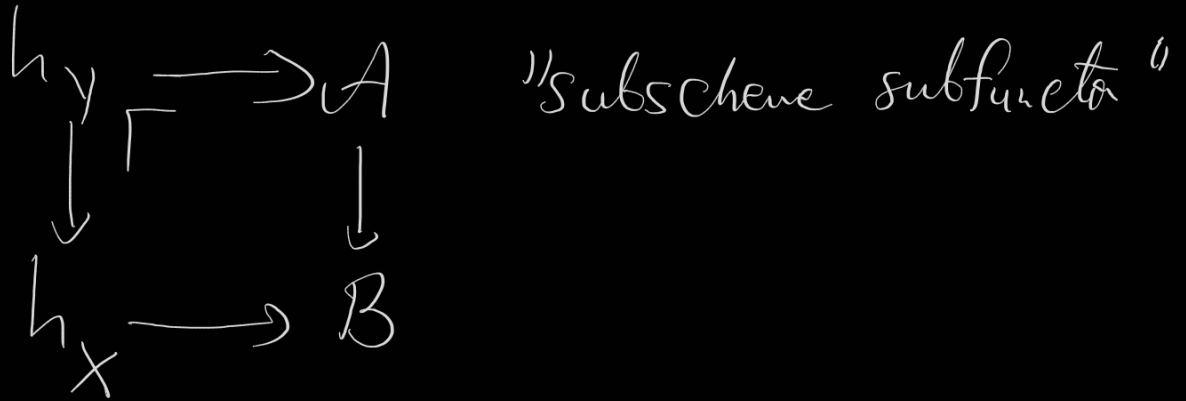
$$\mathcal{Z}(\mathcal{O}_F(m) - \mathcal{P}(m))$$

$$\text{Curves}_F^p \longrightarrow G(r, \chi(\mathcal{O}_F(m)))$$

"Hilb<sub>F</sub><sup>p</sup>(S)"

All subschemes<sub>F</sub>

$$\text{All subschemes}_F(S) = \left\{ \begin{array}{l} I \subset F \times S \\ \text{flat} \downarrow \\ S \end{array} \right\}$$

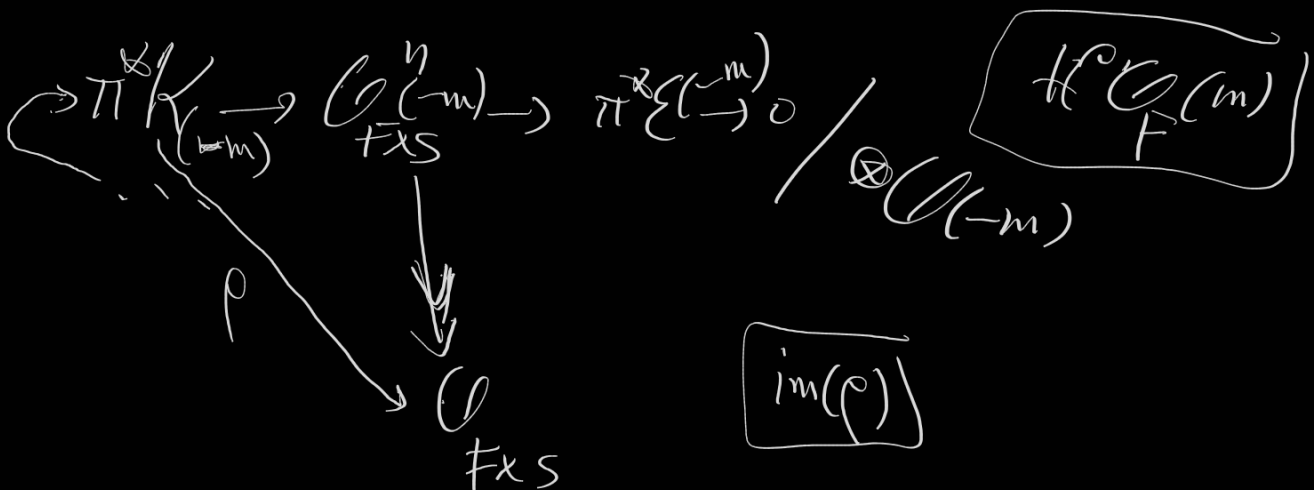


Def:  $F \xrightarrow{\phi} G$  is a scheme functor if

$$F \circ g_{h_X} \simeq h_Y \quad \forall X \in (\text{Sch}/k)^{\text{op}}$$

$$G(r, \mathcal{O}_{F(m)}^{\oplus n})(S) \rightarrow \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{E} / \pi_S^* \mathcal{E}$$

$0 \rightarrow K \rightarrow \mathcal{O}_S^{\oplus n}$



$$0 \rightarrow \text{im}(\rho) \rightarrow \mathcal{O}_{F(m)} \quad (\text{etc flat}/S)$$

OCFS Cartier,  $m$ -regular.

Flat  $S$

$$0 \rightarrow \pi_* I_D(m) \rightarrow \mathcal{O}_S \xrightarrow{\otimes \mathcal{O}_F(m)} \pi_* \mathcal{O}_D(m) \rightarrow 0$$

$$\downarrow$$

$$\mathcal{O}_{FXS}(m)$$

$$\pi^* \pi_* (I_D(m))(-m) \rightarrow \mathcal{O}_{FXS}(-m) \xrightarrow{\otimes \mathcal{O}_F(m)} \pi^* \pi_* \mathcal{O}_D \rightarrow 0$$

$$\rho \searrow$$

$$\downarrow$$

$$\mathcal{O}_{FXS}$$

$$\pi_* F \rightarrow \pi_* F$$

$$\pi^* \pi_* F \rightarrow F$$

$$\pi^* \pi_* I_D(m) \rightarrow I_D(m) / \otimes \mathcal{O}(-m)$$

$$\pi^* (\pi_* I_D(m))(-m) \rightarrow I_D$$

$$I_D(m)$$

$$\downarrow$$

$$\mathcal{O}_{FXS}(m)$$

$$\downarrow$$

$$\mathcal{O}_{FXS}$$

Abstract claim:

$$A \xrightarrow{\phi} h_G$$

$$\downarrow \psi$$

$$i \rightarrow B$$

$i$  inclusion of functors

$A \hookrightarrow B$  is a subscheme functor

$$\begin{array}{ccc}
 h_y^{(T)} & \xrightarrow{\quad} & A(T) \\
 \downarrow & \nearrow & \downarrow i \\
 h_S^{(T)} & \xrightarrow{\alpha} & B(T) \Leftrightarrow \alpha \in B(S) \xrightarrow{B(g)} B(T)
 \end{array}$$

YES  $T \xrightarrow{g} S$

(f factors through  $\gamma$ )  $\Leftrightarrow$  ( $B(g)(\alpha)$  land in  $i(A(T))$ )

$$\begin{array}{ccc}
 A & \xrightarrow{h_g} & B \\
 \downarrow i & \searrow & \downarrow \psi \\
 & & B
 \end{array}$$

$$\begin{array}{ccc}
 h_y & \xrightarrow{f} & A \\
 \downarrow & \nearrow & \downarrow i \\
 h_g & \xrightarrow{\psi} & B
 \end{array}$$

$$\begin{array}{ccccc}
 \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \\
 t & \xrightarrow{\quad} & (t, \infty) & & \\
 & & (t, s) & \xrightarrow{\quad} & t
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \\
 \downarrow \text{not iso} & & \downarrow 1 \\
 \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 \downarrow h_y & \searrow f & \downarrow i \\
 A & \xrightarrow{f} & A \\
 \downarrow h_g & \searrow \psi & \downarrow i \\
 B & \xrightarrow{\psi} & B
 \end{array}$$

$$\varepsilon \neq \varepsilon = \varepsilon$$

$$f\varepsilon = 1, \quad \boxed{\varepsilon \neq 1}$$

$$\begin{array}{ccc}
 h_c & \xrightarrow{\eta} & h_y & \xrightarrow{\varepsilon f} & h_y \\
 \downarrow & \nearrow & \downarrow & & \downarrow \\
 h_g & \xrightarrow{\psi} & B & & B
 \end{array}$$

$$\eta \circ f \varepsilon = \eta$$

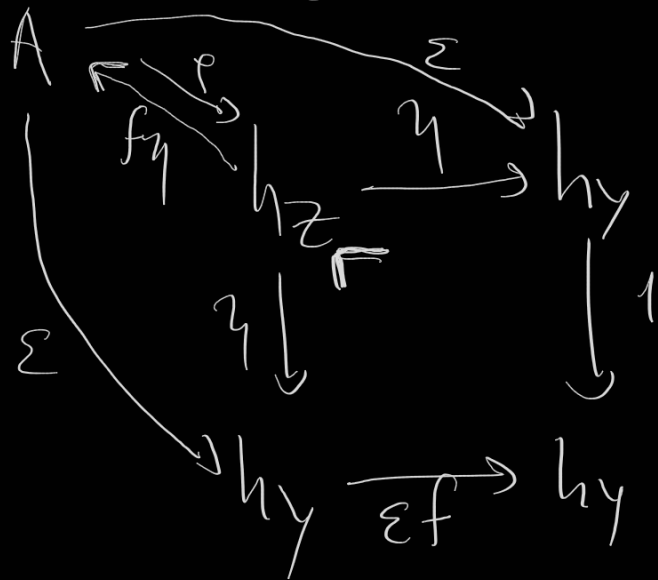
$$\eta \circ f \eta = \eta$$

$$\boxed{L = L \circ f \varepsilon}$$

$$h_z \xrightarrow{\eta} h_y \xrightarrow{f} A \xrightarrow{p} h_z$$

$$A \xrightarrow{p} h_z \xrightarrow{\eta} h_y \xrightarrow{f} A$$

$$\boxed{pf\eta = 1_{h_z}} \quad \underline{f\eta p = 1_A}$$

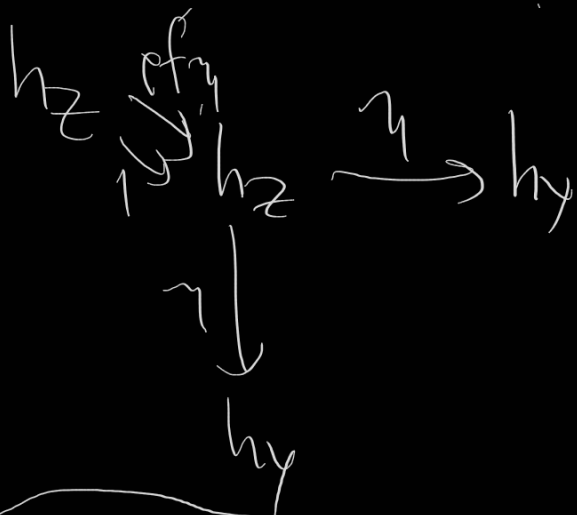


$$\epsilon f \epsilon = \epsilon$$

$$\boxed{\epsilon f \eta = \eta}$$

$$\boxed{f\eta \text{ iso}}$$

$$\boxed{pf\eta = 1_{h_z}}$$



$$\eta pf\eta = \eta$$

$$\boxed{\text{if } \eta p = i}$$

$$\text{if } \epsilon = i \Rightarrow$$

$$\boxed{f\eta p = 1_A}$$



$A$  is a subscheme of  $G$

$x \in B(S)$   
 $\exists y \in S$

closed?

$\forall g: T \rightarrow S$

$(g \text{ factors through } Y) \Leftrightarrow (B(g)(x) \text{ land in } \mathbb{A}^1 / \text{ACTV})$

$x \in B(S)$

$I \subset F \times S$

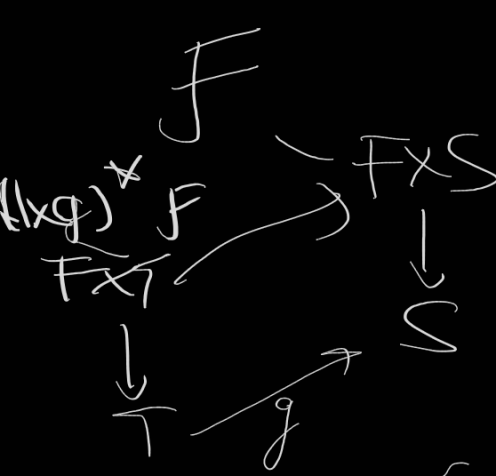
$T \xrightarrow{g} S$

flat  $\downarrow$   
 $S$

$A = \text{Curves}_F^P$

$I \times_S T \hookrightarrow F \times T$

flat  $\downarrow$   
 $T$



$\coprod_{i=1}^m \mathbb{A}^1_{S_i}$   
 $\rightarrow \mathbb{A}^1$

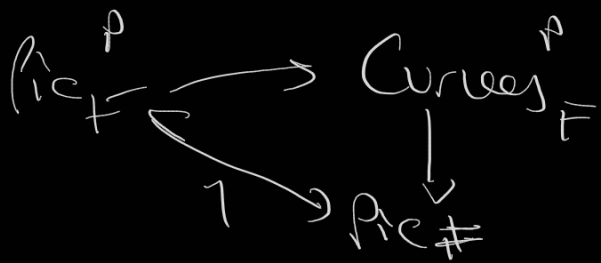
$\mathcal{L}(F/S)$  are closed  
 $\forall x \in S_i$

$\mathbb{A}^1_{S_i} \subset \mathbb{A}^1$

$\Leftrightarrow I_{I \times_S T} = F \times T$

$\text{Curves}_F^P \xrightarrow{\phi} \text{Pic}_F^P$   
 $\mathcal{D} \longrightarrow \mathcal{O}(\mathcal{D})$

$\phi \psi = 1$



Understand "fibers of a natural transformation"

Def:  $F \xrightarrow{\phi} G, \quad F, G: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$

Let  $S_0 \in \mathcal{C}^{\text{op}}, \text{ let } \alpha \in G(S_0)$

$\phi^\alpha: (\mathcal{C}/S_0)^{\text{op}} \rightarrow \text{Sets}$

$G(S)$   
 $\cup$

$(S \xrightarrow{f} S_0) \mapsto \{x \in F(S) \mid \phi_S(x) \stackrel{\cong}{=} G(f)(\alpha)\}$

$\phi^\alpha$  is the fiber functor over  $\alpha$  of  $\phi$ .

Rk:  $F = h_X, G = h_Y, \phi: X \rightarrow Y, \text{ let } \alpha: \text{Spec } k \rightarrow Y$

$\phi^\alpha(x) = \{x \in X(k) \mid \phi(x) = \alpha\}$

Def: ("LinSys")  $\text{Curves}_F \xrightarrow{\phi} \text{Pic}_F$

$L \in \text{Pic}_F(S_0)$

$\text{LinSys}_L(S) = \left\{ x \in \text{Curves}_F(S) \mid \phi_S(x) \cong \pi_F^* L \otimes \pi_S^* K \right\}$   
 $K \in \text{Pic}(S)$

$\text{Pic}_F(S) = \frac{\text{Pic}(F \times S)}{\pi_S^* (\text{Pic}(S))}$

$$[L] = [L \otimes \pi^* K]$$

$$Z_{\text{mSys}}(s) = \left\{ \begin{array}{c} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(s) \\ \downarrow \text{flat} \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(s) \end{array} \middle/ \left( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(s) \right) \otimes_{\mathbb{F}} \left( \pi^* L \otimes \pi^* K \right) \right\}$$

$\downarrow$   
 $s$

$\downarrow$   
 $\text{some } K \in \text{Pic } S$

$$D) \quad |D| = \{ E \geq 0 \mid \mathcal{O}(E) \cong \mathcal{O}(D) \}$$


---