§ Construction of Curves_F as a scheme $\leftarrow F$ is smooth

$$\mathrm{Curves}_F(S) = \left\{ \begin{array}{c} \mathcal{D} \subset F \times S \ \boxed{\text{Cartier}} \\ \text{flat} \searrow \downarrow \\ S \end{array} \right\}$$

① I (ideal) with fixed Hilbert polynomials

$\exists m$ independent of I so that I is m-regular.

P Hilbert poly.

Curves_F

$$\mathrm{Curves}^P_F(S) = \left\{ \begin{array}{l} \mathcal{D} \subset F \times S \text{ Cartier} \\ \text{flat} \searrow \downarrow \quad \chi(\mathcal{O}_{\mathcal{D}_s}(t)) \\ \qquad\quad S \qquad \quad \| \\ \qquad\qquad\qquad P(t) \\ \qquad\qquad \forall s \in S \end{array} \right\}$$

$$\coprod_{i=1}^{m} S_i \xrightarrow{P} S$$

$$\mathrm{Curves}^P_F \rightsquigarrow \coprod_{P \text{ Hilb. poly.}} \mathrm{Curves}^P_F(S) \simeq \mathrm{Curves}_F(S)$$

$$S \longrightarrow \coprod_{P \text{ Hilbpoly.}} \mathrm{Curves}^P_F$$

$$\cong \mathrm{Curves}_F(S)$$

$\boxed{\mathrm{Curves}^P_F}$ $\quad D \subset F$ (be Cartier)

$$\chi(\mathcal{O}_F(-D)(t)) = P(t).$$

$\exists m$ s.t. D has Hilb poly P

$\Rightarrow D'$ is m regular.

$$0 \to I_D(m) \to \mathcal{O}_X(m) \to \mathcal{O}_D(m) \to 0$$

$$0 \to H^0 I_D(m) \to H^0 \mathcal{O}_X(m) \to H^0 \mathcal{O}_D(m) \to 0$$

$$\rightsquigarrow \quad G(\underbrace{h^0\mathcal{O}_D(m)}, \ h^0\mathcal{O}_X(m))$$

$$\begin{array}{cc} \| & \| \\ \chi\mathcal{O}_D(m) & \chi\mathcal{O}_X(m) \\ \| & \\ \chi\mathcal{O}_X(m) - P(m) & \end{array}$$

$\mathcal{D} \subset F \times S$ Cartier,

$\mathcal{D} \xrightarrow{\text{flat}} S$, $F \times S \xrightarrow{\pi} S$

$$0 \to I_{\mathcal{D}}(m) \to \mathcal{O}_{F\times S}(m) \to \mathcal{O}_{\mathcal{D}}(m) \to 0 \quad / \pi_*$$

$$0 \to \pi_* I_{\mathcal{D}}(m) \to \pi_* \mathcal{O}_{F\times S}(m) \to \pi_* \mathcal{O}_{\mathcal{D}}(m) \to$$

$$R^1\pi_* I_{\mathcal{D}}(m) = 0$$

$$\begin{tikzcd} F_s \arrow[r] \arrow[d] & F\times S \arrow[d, "\pi"] & - \mathcal{I}_{\mathcal{D}}(m) \\ s \arrow[r] & S & - R^1\pi_* \mathcal{I}_{\mathcal{D}}(m) \end{tikzcd}$$

$$\underline{\underline{R^1\pi_* \mathcal{I}_{\mathcal{D}}(m)|_s}} \longrightarrow H^1\left(\mathcal{I}_{\mathcal{D}}(m)|_{F_s}\right)$$

$$\| $$

$$H^1\left(\mathcal{I}_{\mathcal{D}_s}(m)\right) = 0$$

$$R^i\pi_* \mathcal{I}_{\mathcal{D}}(m) = 0 \quad \forall i > 0.$$

$$R^i\pi_* \mathcal{O}_{\mathcal{D}}(m) = 0 \quad \forall i > 0$$

$$\begin{tikzcd} F_s \arrow[r] \arrow[d] & F\times S \arrow[d] & - \mathcal{O}_{\mathcal{D}}(m) \\ s \arrow[r] & S & - \pi_*(\mathcal{O}_{\mathcal{D}}(m)) \end{tikzcd}$$

$H^0(\mathcal{O}_{\mathcal{D}_s}(m))$ independent of $s \in S$

$R^i\pi_*\mathcal{O}_{\mathcal{D}}(m) = 0$

$\Longrightarrow$ $\pi_* \mathcal{O}_{\mathcal{D}}(m)$ locally free.

$$0 \to \pi_* \mathcal{I}_{\mathcal{D}}(m) \to \pi_* \mathcal{O}_{F\times S}(m) \longrightarrow \pi_* \mathcal{O}_{\mathcal{D}}(m) \to 0$$

$$\wr$$

$$\pi_*(\pi^*\mathcal{O}_S)(m))$$

$$\mathcal{O}_S \otimes H^0\mathcal{O}_F(m) \xrightarrow[\sim]{} \pi_* \mathcal{O}_{F\times S}(m)$$

$$\pi^*\left(\mathcal{O}_S \otimes H^0\mathcal{O}_F(m)\right) \longrightarrow \mathcal{O}_{F\times S}(m)$$

$$\mathcal{O}_{F\times S} \otimes H^0\mathcal{O}_F(m) \longrightarrow \mathcal{O}_{F\times S}(m)$$

$$0 \to \pi_*\mathcal{I}_D(m) \to \mathcal{O}_S \otimes \boxed{H^0\mathcal{O}_F(m)} \to \pi_*\mathcal{O}_D(m) \to 0$$

$$\langle e_1, \dots, e_{h^0\mathcal{O}_F(m)} \rangle \qquad \begin{array}{c} \| \\ \mathcal{O}_S^{\chi\mathcal{O}_F(m)} \end{array}$$

S-val'd pt of $G\left(h^0(\mathcal{O}_{D_s}(m)), \chi(\mathcal{O}_F(m))\right)$

$$\begin{array}{c} \| \\ \chi\mathcal{O}_F(m) - P(m) \end{array}$$

$$\begin{tikzcd} \mathrm{Curves}^P_F \arrow[r] \arrow[dr] & G(r, \chi\mathcal{O}_F(m)) \arrow[d] \\ & \mathrm{Allsubschemes}_F \end{tikzcd}$$

"$\coprod_p \mathrm{Hilb}^p_F S$"

$\|$

$$\mathrm{Allsubschemes}_F(S) = \left\{ \begin{array}{c} I \subset F\times S \\ \text{flat} \searrow \ \downarrow \\ S \end{array} \right\}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\phi"] \arrow[dr, bend right] & h_G \arrow[d] \\ & B \end{tikzcd}$$

$\mathcal{A}$ is representable

$$\begin{tikzcd} h_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{A} \arrow[d] \\ h_X \arrow[r] & B \end{tikzcd}$$

"subscheme subfunctor"

Def: $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a scheme functor if

$\mathcal{F} \times_{\mathcal{G}} h_X \simeq h_Y \quad \forall X \in (Sch/_k)^{op}$

$G(r, \overset{\overset{\eta}{\shortparallel}}{\chi}\mathcal{O}_F(m))(S)$

$0 \to K \to \mathcal{O}_S^n \twoheadrightarrow \mathcal{E} \;/\; \pi_S^*$

$$\begin{tikzcd} 0 \arrow[r] & \pi^*K(-m) \arrow[r] \arrow[dr, "\rho"'] & \mathcal{O}^n_{F\times S}(-m) \arrow[r] \arrow[d, two heads] & \pi^*\mathcal{E}(-m) \arrow[r] & 0 \\ & & \mathcal{O}_{F\times S} \end{tikzcd}$$

$/ \otimes \mathcal{O}(-m)$

$\boxed{H^0\mathcal{O}_F(m)}$

$\boxed{\mathrm{im}(\rho)}$

$0 \to \mathrm{im}\,\rho \to \mathcal{O}_{F\times S}$ (its flat$/S$)

$\mathcal{D} \subset F \times S$ Cartier. m-regular.

flat $\searrow$ $\downarrow$ S

$$0 \to \pi_* I_{\mathcal{D}}(m) \to \mathcal{O}_S \xrightarrow{\chi(\mathcal{O}_F(m))} \pi_* \mathcal{O}_{\mathcal{D}}(m) \to 0 \quad / \otimes \mathcal{O}(-m)$$

$$\mathcal{O}_S \to \mathcal{O}_{F\times S}(m)$$

$$\begin{tikzcd} \pi^* \pi_* \left(I_{\mathcal{D}}(m) \right)^{(-m)} \arrow[r] \arrow[dr, "\rho"'] & \mathcal{O}_{F\times S}^{\chi(\mathcal{O}_F(m))}(-m) \arrow[r] \arrow[d] & \pi^* \pi_* \mathcal{O}_{\mathcal{D}} \arrow[r] & 0 \\ & \mathcal{O}_{F\times S} & & \end{tikzcd}$$

$$\pi_* \mathcal{F} \longrightarrow \pi_* \mathcal{F}$$

$$\pi^* \pi_* \mathcal{F} \longrightarrow \mathcal{F}$$

$$\pi^* \pi_* I_D(m) \longrightarrow I_D(m) \quad / \otimes \mathcal{O}(-m)$$

$$\begin{tikzcd} \pi^* \left(\pi_* I_D(m) \right)(-m) \arrow[r] \arrow[dr] & I_D \arrow[d, hook, "\rho"] \\ & \mathcal{O}_{F\times S} \end{tikzcd}$$

$$\begin{tikzcd} I_D(m) \arrow[d] \\ \mathcal{O}_{F\times S}(m) \end{tikzcd}$$

Abstract claim:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "i"'] & h_G \arrow[d, "\psi"] \\ & B \end{tikzcd}$$

i inclusion of functors

$A \overset{i}{\hookrightarrow} B$ is a subscheme functor

$$\begin{tikzcd} h_Y^{(T)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A(T) \arrow[d, hook, "i"] \\ h_S^{(T)} \arrow[r, "\alpha"] & B(T) \end{tikzcd} \iff \alpha \in B(S) \xrightarrow{B(g)} B(T)$$

$Y \subset S \qquad T \xrightarrow{g} S$

$$\left(\begin{array}{c} g \text{ factors through} \\ Y \end{array} \right) \iff \left(\begin{array}{c} B(g)(\alpha) \text{ land in} \\ i(A(T)) \end{array} \right)$$

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, hook, "i"'] & h_G \arrow[d, "\psi"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} h_Y \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "i"] \\ h_G \arrow[r, "\psi"'] & B \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\phi"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\psi"] & \mathbb{P}^1 \end{tikzcd}$$

$t \longrightarrow (t, \infty)$

$(t, s) \longrightarrow t$

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\text{not iso}"] & \mathbb{P}^1 \arrow[d, "1"] \\ \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\psi"'] & \mathbb{P}^1 \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\exists! \varepsilon"] \arrow[rr, bend left, "1_A"] \arrow[dr, bend right, "\phi"'] & h_Y \arrow[r, "f"] \arrow[d] & A \arrow[d, hook, "i"] \\ & h_G \arrow[r, "\psi"'] & B \end{tikzcd}$$

$f\varepsilon = 1, \quad \boxed{\cancel{\varepsilon f = 1}}$

$$\begin{tikzcd} h_Z \arrow[r, hook, "\eta"] & h_Y \arrow[r, shift left, "\varepsilon f"] \arrow[r, shift right, "1"'] & h_Y \\ & A \arrow[u, "\varepsilon"] \arrow[ul, dashed, "\exists! \rho"] & \end{tikzcd}$$

$\varepsilon f \eta = \eta$

$\eta \rho f \eta = \eta$

$\boxed{\varepsilon f \eta = \eta}$

$\varepsilon(f\varepsilon) = \varepsilon$

$$h_Z \xrightarrow{\eta} h_Y \xrightarrow{f} A \xrightarrow{\rho} h_Z$$

$$A \xrightarrow{\rho} h_Z \xrightarrow{\eta} h_Y \xrightarrow{f} A$$

$\rho f\eta = 1_{h_Z}$? $\underline{f\eta\rho = 1_A}$

$$\begin{tikzcd}
A \arrow[dr, shift left, "\rho"] \arrow[drr, bend left, "\varepsilon"] \arrow[ddr, bend right, "\varepsilon"'] & & \\
& h_Z \arrow[ul, shift left, "f\eta"] \arrow[r, "\eta"] \arrow[d, "\eta"'] & h_Y \arrow[d, "1"] \\
& h_Y \arrow[r, "\varepsilon f"'] & h_Y
\end{tikzcd}$$

$\varepsilon f \varepsilon = \varepsilon$

$\varepsilon f \eta = \eta$

$\boxed{f\eta \text{ iso}}$

$\boxed{\rho f\eta = 1_{h_Z}}$

$$\begin{tikzcd}
h_Z \arrow[r, shift left, "\rho f\eta"] \arrow[r, shift right, "1"'] & h_Z \arrow[r, "\eta"] \arrow[d, "\eta"] & h_Y \\
& h_Y &
\end{tikzcd}$$

$\eta\rho f\eta = \eta$

$\boxed{\iota f\eta\rho = \iota}$

$\iota f\varepsilon = \iota \rightarrow \boxed{f\eta\rho = 1_A}$

A is a subscheme of G

$\boxed{\text{closed?}}$

$\alpha \in B(S)$
$\exists Y \subset S$
$\forall g : T \to S$

$\left(g \text{ factors through } Y \right) \Leftrightarrow \left(B(g)(\alpha) \text{ land in } i_T(A(T)) \right)$

$\alpha \in B(S)$

$T \xrightarrow{g} S$

$I \subset F \times S$

$\xrightarrow{\text{flat}} S$

$A = Curves^P_F$

$I \times_S T \hookrightarrow F \times T$

$\xrightarrow{\text{flat}} T$

$$\begin{tikzcd} & F \\ (1 \times g)^* F & F \times S \arrow[d] \\ F \times T \arrow[ur] \arrow[d] & S & \coprod_{i=1}^{m} S_i \arrow[l] \\ T \arrow[ur, "g"] \end{tikzcd}$$

$\chi(F_s)(t)$ are const $\forall x \in S_i$

$\boxed{S_i \subset S}$

$\Leftrightarrow \quad I_{I \times_S T} = F \times T$

$Curves^P_F \xrightarrow{\phi} Pic^P_F$

$D \longrightarrow O(D)$

$\phi Y = 1$

$$\begin{tikzcd} \mathrm{Pic}_F^P \arrow[r] \arrow[dr, "?"'] & \mathrm{Curves}_F^P \arrow[d] \\ & \mathrm{Pic}_F^{\#} \end{tikzcd}$$

Understand "fibers of a natural transformation"

<u>Def:</u> $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$, $\mathcal{F}, \mathcal{G} : \mathcal{C}^{op} \to \mathrm{Sets}$

Let $S_0 \in \mathcal{C}^{op}$, let $\alpha \in \mathcal{G}(S_0)$

$\phi^\alpha : (\mathcal{C}/S_0)^{op} \to \mathrm{Sets}$ $\quad\quad \mathcal{G}(S) \ni$

$(S \xrightarrow{g} S_0) \longmapsto \{ x \in \mathcal{F}(S) \mid \phi_S(x) = \mathcal{G}(g)(\alpha) \}$

ϕ^α is the fiber functor over α of ϕ.

<u>Rk:</u> $\mathcal{F} = h_X$, $\mathcal{G} = h_Y$, $\phi : X \to Y$, Let $\alpha : \mathrm{Spec}\, k \to Y$

$\phi^\alpha(\alpha) = \{ x \in X(k) \mid \phi(x) = \alpha \}$.

<u>Def:</u> ("LinSys") $\quad \mathrm{Curves}_F \xrightarrow{\phi} \mathrm{Pic}_F$

$L \in \mathrm{Pic}_F(S_0)$.

<u>LinSys_L</u>$(S) = \{ x \in \mathrm{Curves}_F(S) \mid \phi_S(x) \cong \pi_F^* L \otimes \pi_S^* K, \; K \in \mathrm{Pic}(S) \}$

$\mathrm{Pic}_F(S) = \dfrac{\mathrm{Pic}(F \times S)}{\pi_S^*(\mathrm{Pic}(S))}$

$$[L] = [L \otimes \pi^* K]$$

$$\mathrm{LinSys}_L(S) = \left\{ \mathcal{D} \subset F \times S \;\middle|\; \mathcal{O}(\mathcal{D}) \simeq \pi_F^* L \otimes \pi_S^* K \right\}$$

$$\begin{array}{cc} \text{flat} & \downarrow \\ \text{CDiv} & S \end{array} \qquad \text{some } K \in \mathrm{Pic}\, S$$

$$D \quad , \quad |D| = \{ E \geq 0 \mid \mathcal{O}(E) \simeq \mathcal{O}(D) \}.$$