Proposition:

$$\underline{\mathrm{LinSys}_L} \simeq h_{\mathbb{P}(\widehat{H^0 L})} \qquad h^0(L)-1$$

Proof: $\underline{\mathrm{LinSys}_L}(S) = \left\{ \begin{tikzcd} \mathcal{D} \subset F \times S \arrow[dr, "\text{flat}"'] \arrow[d] \\ S \end{tikzcd} \text{ rel eff Cartier}/S \;\middle|\; \mathcal{O}(\mathcal{D}) \simeq \pi_F^* L \otimes \pi_S^* K,\ K \in \mathrm{Pic}(S) \right\}$

$$\begin{tikzcd} \mathcal{D} \subset F \times S \arrow[dr] \arrow[d] \\ S \end{tikzcd} \xleftarrow{\hspace{2em}} \xrightarrow{\hspace{2em}} \left(\begin{matrix} s \in H^0(\pi_F^* L \otimes \pi_S^* K) \\ \shortparallel \\ H^0(L) \otimes H^0(K) \end{matrix} \right)$$

$$\mathcal{D} \text{ relative}/S\,? \longleftrightarrow \quad \boxed{s|_s \not\equiv 0 \quad s \in S}$$

$$s_1 \in H^0(\pi_F^* L_1 \otimes \pi_S^* K_1)$$

$$s_2 \in H^0(\pi_F^* L_2 \otimes \pi_S^* K_2)$$

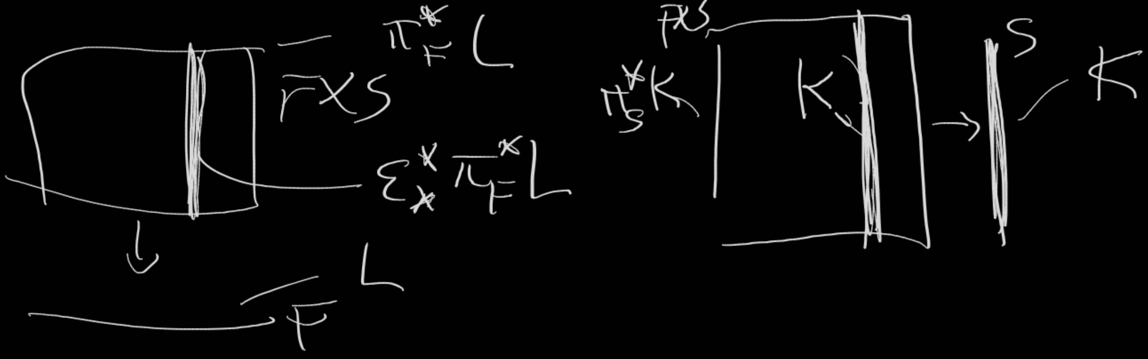
Fix $x \in X$. $\varepsilon_x : S \longrightarrow X \times S$

$$s \longmapsto (x, s)$$

$$K_1 = \varepsilon_x^*(\pi_F^* L_1 \otimes \pi_S^* K_1) \simeq \varepsilon_x^*(\pi_F^* L_2 \otimes \pi_S^* K_2)$$

$$\shortparallel \qquad\qquad\qquad\qquad \simeq K_2$$

$$\underline{\varepsilon_x^*(\pi_F^* L_1)} \otimes \underline{\underline{\varepsilon_x^* \pi_S^* K_1}}$$



$s_1, s_2 \in H^0(L)\otimes H^0(K)$

$$H^0(\mathcal{O}^*_{F\times S}) \overset{\nearrow}{=} \underline{H^0(\mathcal{O}^*_S)}$$

F proj$/_{k=\bar{k}}$

$$\mathrm{LinSys}_L(S) = \left\{ \begin{array}{l} s\in H^0L\otimes H^0K, \\ \text{for some } K\in \mathrm{Pic}(S) \\ \text{that are not zero on} \\ \text{any Fiber} \end{array} \right\} \Big/ H^0(\mathcal{O}^*_S)$$

$\boxed{\text{if } S \text{ proper}/k} \longrightarrow H^0(\mathcal{O}^*_S) \cong k^*$

$H^0L = \langle e_1, \dots, e_n\rangle$

$s = \sum e_i\otimes s_i,\ s_i\in H^0(K)$

$s \neq 0$ on $\pi_S^{-1}(y)$ $\iff$ $\sum s_i(y)\, e_i(x) \neq 0 \quad \forall x\in F\times\{y\} \subset F\times S$

$\iff \sum s_i(y)\, e_i \neq 0 \in H^0(\pi^*L|_{\pi_S^{-1}(y)})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad H^0L$

$\iff s_1(y) = \dots = s_n(y) \neq 0$

$\mathcal{O}^n \longrightarrow K$

$s \not\equiv 0$ on $\pi_S^{-1}(y)$ $\iff$ $\mathcal{O}_y^n \twoheadrightarrow K_y \quad \forall y \in S$

$\forall y \in S$

$\iff \mathcal{O}_S^n \twoheadrightarrow K$

$\cap \qquad \shortparallel$

$h_{\mathbb{P}(H^0(L)^\vee)}(S) \qquad \mathcal{O}_S \otimes H^0(L)$

$$\left\{ \begin{array}{l} s_1, \dots, s_n \in H^0(K) \text{ that} \\ \text{span } K \end{array} \right\} \Big/ \boxed{H^0(\mathcal{O}_S^\times)}$$

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r] \arrow[ddr, bend right] & K & \\ & s \arrow[u] & H^0(\mathcal{O}_S^\times) \arrow[l] \\ & K' & \end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Curves}_F \arrow[r, "\Phi"] & \mathrm{Pic}_F \\
\mathrm{LinSys}_L(k) \arrow[u] \arrow[r] & \mathrm{Spec}(k) \arrow[u]
\end{tikzcd}$$

Proposition*: Assume $H^1\mathcal{O}_F = 0$. S connected.

Let $\mathcal{L} \in \mathrm{Pic}(F\times S)$.

$\Rightarrow \exists\, L \in \mathrm{Pic}(F)$, $K \in \mathrm{Pic}(S)$ so that

$\mathcal{L} \simeq \pi_F^* L \otimes \pi_S^* K$,

Cor: $H^1\mathcal{O}_F = 0 \Rightarrow \mathrm{Pic}_F \simeq h_{\left(\coprod_{\mathrm{Pic}_F(k)} \mathrm{Spec}(k)\right)}$

$$\mathrm{Pic}_F(S) = \mathrm{Pic}(F\times S) / \pi_S^* \mathrm{Pic}(S) = \frac{\mathrm{Pic}(F)\times \mathrm{Pic}(S)}{(0, \mathrm{Pic}(S))}$$

$$= \mathrm{Pic}(F) = \mathrm{Pic}_F(k) \simeq \mathrm{Hom}\left(S, \coprod_{\mathrm{Pic}_F(k)} \mathrm{Spec}(k)\right)$$

$\Rightarrow \mathrm{Pic}_F \simeq h_{\coprod_{\mathrm{Pic}_F(k)} \mathrm{Spec}(k)}$. $\square$

Examples: X birational to $\mathbb{P}^n \Rightarrow H^1(\mathcal{O}_X) = H^1(\mathcal{O}_{\mathbb{P}^n}) = 0$

$\Rightarrow \mathrm{Pic}_X$ is discrete.

$C_0 \subset C$ smooth

$$\{C_0 \longrightarrow \mathrm{Curves}_F\} \simeq \{C \longrightarrow \mathrm{Curves}_F\}$$

$$\begin{tikzcd} \mathcal{D} \arrow[r, phantom, "\subset"] \arrow[dr, bend right, "\text{flat}"'] & F\times C_0 \arrow[d] \\ & C_0 \end{tikzcd}$$

Cartier divisors

$D = \overline{\mathcal{D}} \subset F \times C$.

$\mathcal{D} = \sum' n_i Z_i \,,\ n_i > 0$

$D = \overline{\mathcal{D}} = \sum' n_i \overline{Z_i}$

$\Rightarrow D$ is a Weil divisor

F smooth, C smooth $\Rightarrow F \times C$ smooth $\Rightarrow$ Weil = Cartier.

$\mathrm{Supp}(D) \not\supset$ any fiber. $\mathcal{D}$ rel$/C_0$

$\boxed{\Rightarrow D \text{ rel}/C} \Rightarrow \boxed{?}$

Rk: $$\begin{tikzcd} \mathrm{Curves}^{\xi}_F \arrow[r, bend left, "\simeq"] \arrow[r, "\phi"'] & \mathrm{Pic}^{\xi}_F \end{tikzcd}$$

$$\mathrm{Pic}_F = \coprod_{P \text{ Hilb poly}} \mathrm{Pic}^P_F = \coprod_{\xi \in \mathrm{Num}(F)} \mathrm{Pic}^{\xi}_F$$

for $\mathrm{Pic}^{\xi_0}_F \longrightarrow \overset{L_1}{\overset{\cap}{\mathrm{Pic}^{\xi_1}_F}}$ $(L \longmapsto L \otimes L_1 \otimes L_0^{-1})$

Two results needed for construction:

① $h = [\mathcal{O}(1)] \in \mathrm{Num}\, F$

$w = [K_F]$

$\exists\, c, \varepsilon > 0$ so that if $\lambda \in \mathrm{Num}(F)$ that satisfies $\lambda \cdot h \geq c$ and $\lambda \cdot (\lambda - \omega) \geq (1-\varepsilon)\left(\frac{(\lambda \cdot h)^2}{(h \cdot h)^2}\right)$

$\forall [L] = \lambda \implies L$ is very ample & 0-regular.

② $\exists\, c, \varepsilon > 0$ s.t. $\forall$ line bundles L with
$(F, \mathcal{O}(1))$

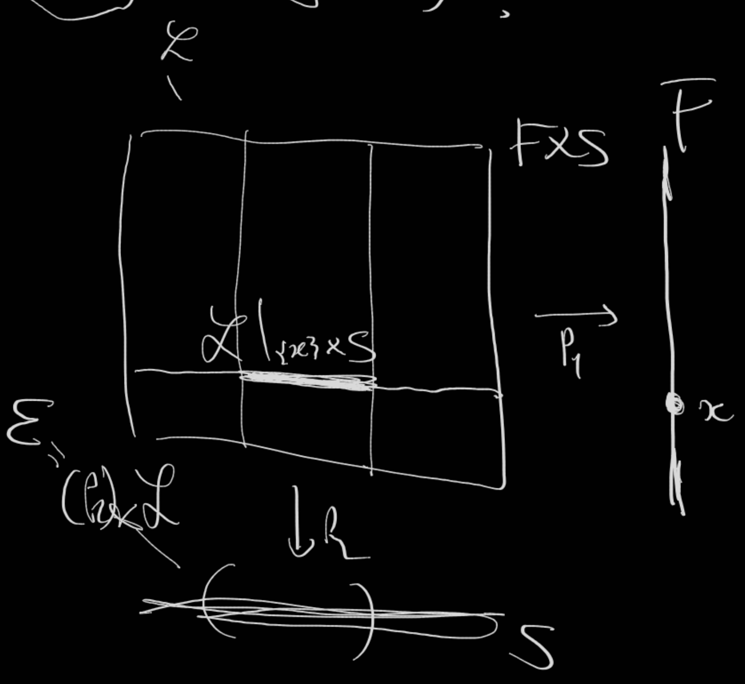
$\deg(L) \geq c$

$$\chi(L) \geq \frac{(1-\varepsilon)}{2(h \cdot h)^2} \deg(L)^2$$

$\implies L$ very ample + 0-regular.

$\boxed{\xi}$ $[\mathcal{L}] = \xi$.

$$\begin{tikzcd} \mathrm{Curves}^{\xi}_F(S) \arrow[r, "\phi"'] & \mathrm{Pic}^{\xi}_F(S) \arrow[l, bend right] \end{tikzcd}$$



$$\begin{tikzcd} \mathcal{D}_{\mathcal{L}} \arrow[r, leftrightarrow] \arrow[d, mapsto] & \mathcal{L} \arrow[d] \\ \mathcal{D}_{g^* \mathcal{L}} \arrow[r, leftrightarrow] & g^* \mathcal{L} \end{tikzcd}$$

$i_x : \{x\} \times S \to F \times S$

$$\boxed{h_x : \mathcal{E} \longrightarrow \mathcal{L}|_{\{x\}\times S} = i_x^* \mathcal{L} =: M_x}$$

$$H^i \mathcal{L}|_{F\times s} = 0 \quad \forall i > 0$$

and $\mathcal{L}|_{F\times s}$ is very ample (0-regular)

$\boxed{\underbrace{\mathcal{E} \text{ is loc free}}_{r = \operatorname{rk}\mathcal{E}}}$. rank is determined by $\chi(\mathcal{L}|_{F\times s})$ by ξ.

$$\tilde{h} = \sum_{i=1}^{r-1} h_{x_i} : \mathcal{E} \longrightarrow \bigoplus_{i=1}^{r-1} M_{x_i} \rightsquigarrow \Lambda^{r-1}\mathcal{E} \longrightarrow \bigotimes_{i=1}^{r-1} M_{x_i}$$

$$\boxed{\Lambda^{r-1}\mathcal{E}^* \simeq (\Lambda^r\mathcal{E})^{-1} \otimes \mathcal{E}}$$

$$\bigotimes_{i=1}^{r-1} M_{x_i}^{-1} \longrightarrow (\Lambda^{r-1}\mathcal{E})^{*} = (\Lambda^r\mathcal{E})^{-1} \otimes \mathcal{E},$$

$$\pi_S^*\left(\mathcal{O}_S \longrightarrow \mathcal{E} \otimes (\Lambda^{r-1}\mathcal{E})^{-1} \bigotimes_{i=1}^{r-1} M_{x_i} \right)$$

$$\mathcal{O}_{F\times S} \longrightarrow \underbrace{L \otimes \pi_S^*\left((\det\mathcal{E})^{-1} \bigotimes_{i=1}^{r-1} M_{x_i} \right)}_{\text{section } \zeta \in H^0(\quad)}$$

$$\zeta|_{\pi_S^{-1}(s)} \neq 0 \quad \forall s \in S,$$

$\mathcal{L}_s$ very ample $\quad r = \operatorname{rk}(\mathcal{E}) = h^0(\mathcal{L}_s)$,

$\Rightarrow \varphi_s : F \longrightarrow \mathbb{P}^{r-1}$. Then,

Lemma: $\delta|_{\pi_S^{-1}(s)} \not\equiv 0 \iff \varphi_s(x_1), \dots, \varphi_s(x_{r-1})$ are in general position.

Pf: $\mathcal{L} = \mathcal{L}_s$, $\mathcal{E} = H^0(\mathcal{L})$, $M_{x_i} = \mathcal{L}|_{x_i}$, $\delta = \delta_s$

$\delta \not\equiv 0 \iff H^0(L) \twoheadrightarrow \bigoplus_{i=1}^{r-1} \mathcal{L}|_{x_i} \iff$

$\{h_{x_i} : H^0 L \to \mathcal{L}|_{x_i}\}$ independent

$\iff \bigcap_{i=1}^{r-1} \ker(h_{x_i})$ 1 dim'l.

$$\begin{tikzcd}
\varphi_s^* : H^0(\mathbb{P}^{r-1}, \mathcal{O}(1)) \ni H \arrow[r, "\sim"] \arrow[rr, bend left, mapsto] & H^0(\mathcal{L}_s) \arrow[r, "h_{x_i}"] & 0
\end{tikzcd}$$

$\iff x_i \in H$

$$h_{\varphi_s(x_i)} : H^0(\mathbb{P}^{r-1}, \mathcal{O}(1)) \longrightarrow \mathcal{O}(1)|_{x_i} = \mathcal{L}|_{x_i}$$

$$H \ni x_1, \dots, x_{r-1}.$$
$$\Updownarrow$$
$$H \in \bigcap \ker h_{x_i}$$

□

For fixed $s \in S$ we could find $x_1, \dots, x_{r-1} \in F$

$\varphi_s(x_1), \dots, \varphi_s(x_{r-1})$ are in general position.

$\Rightarrow \delta|_{\pi_S^{-1}(s)} \not\equiv 0$

$r = \mathrm{rk}(\mathcal{E})$, $N > 0$.

$N \cdot r - 1$ pts on F

$$(N-1)r + r - 1 \atop {\shortparallel \atop N \cdot r - 1}$$

$$(\gamma) \left(\begin{array}{ccc} x_{11} & \cdots & x_{1r} \\ & \vdots & \\ x_{N-1\,1} & \cdots & x_{N-1\,r} \\ (x_{N1} & \cdots & x_{N\,r-1}) \end{array} \right) \begin{array}{l} \mathcal{E} \to \bigotimes_{i=1}^{r} M_{x_{k,i}} \\ \quad \rightsquigarrow \zeta_1, \ldots, \zeta_{N-1} \\ \\ \rightsquigarrow \zeta_N \end{array}$$

$$\zeta^*_\gamma = (\zeta_1)_\gamma \otimes \cdots \otimes (\zeta_{N-1})_\gamma \otimes (\zeta_N)|_\gamma \in H^0(F \times S, \mathcal{K})$$

$$\mathcal{K} = (\Lambda^r \mathcal{E})^{-N} \otimes \Big[\bigotimes_{\text{all } k,i} M_{x_{k,i}} \Big]$$

<u>Thm:</u> For suitable choices of ξ, N, $Nr-1$ pts on F, scalars $\sum' a_\gamma \zeta_\gamma \in H^0(\mathcal{L})$ is never zero

(on $\pi_S^{-1}(s)$ $\forall s \in S$)