$$\begin{tikzcd} \mathrm{Curves}^{\xi}_F \arrow[r, shift left, "s"] \arrow[d, phantom, "\shortparallel"] & \mathrm{Pic}^{\xi}_F \arrow[l, shift left, "\phi"] \arrow[d, phantom, "\shortparallel"] \\ h_G & A = B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[dr, equal, "1"'] & h_G \arrow[d, "\phi"] \\ & B \end{tikzcd}$$

$$\mathrm{Curves}_F = \coprod_{\xi \in \mathrm{Num}\,F} \mathrm{Curves}^{\xi}_F \qquad \xi \in \mathrm{Num}(F)$$

$$\mathrm{Pic}_F = \coprod_{\xi \in \mathrm{Num}\,F} \mathrm{Pic}^{\xi}_F$$

$\xi_1, \quad \xi_2$

$L_1 \qquad L_2$

$$\mathrm{Pic}^{\xi_1} \longrightarrow \mathrm{Pic}^{\xi_2}$$

$$L \longmapsto L \otimes L_2 \otimes L_1^{-1}$$

$\underline{\boxed{\mathcal{O}(1)}}$

Let F be a ~~surj.~~ proj. with fixed $\mathcal{O}(1)$.

$\exists\, c, \varepsilon > 0$ s.t. $\forall$ l. bdl $\mathcal{L}$ with $[\mathcal{L}] = \xi$

$\xi \cdot h = \deg \mathcal{L} \geq c$

$$\tfrac{1}{2}\xi\cdot(\xi - w) - \chi(\mathcal{O}_F) = \chi(\mathcal{L}) \geq \frac{(1-\varepsilon)}{2(\mathcal{O}(1)\cdot\mathcal{O}(1))}(\deg L)^2 = (\xi \cdot h)^2$$

$\Longrightarrow$ L very ample and 0-regular.

Curves^{P}_{F}, $\{C_0\text{-vals}\} \xrightarrow{\sim} \{C\text{-vals}\}$ for all C_0

$\mathcal{D} \subset F \times C_0$ Cartier $\quad \mathcal{D} = \sum_i n_i Z_i$

$\downarrow$ flat

C_0

$D = \overline{\mathcal{D}} = \sum_i n_i \overline{Z_i} \subset F \times C$

smooth smooth

$k = \overline{k}$

Choose "smart" ξ

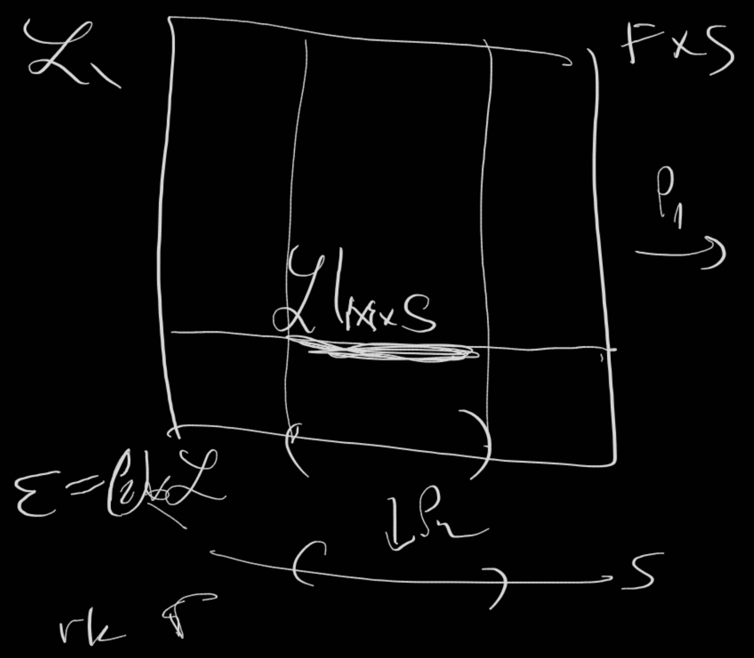
$P_1 \subset^{\xi} (S \underset{\varphi}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Curves}^{\xi}_{1} = (S)$

(P^S_x) $\mathcal{L}(\otimes p^* K) \longrightarrow \mathcal{D}_{\mathcal{L}} = \mathcal{D}_{\mathcal{L} \otimes p^* K} \subset F \times S$

$(K \in P_i^{\vee} \subset (S))$

$T \xrightarrow{g} S$

$g^* \mathcal{L}(\otimes p^* K) \longrightarrow \mathcal{D}_{g^* \mathcal{L}} = \mathcal{D}_{g^* \mathcal{L} \otimes p^* K}$



F

$h_x : \mathcal{E} \to \mathcal{L}|_{x\times S} = M_x$

x

num. criteria

$$\Lambda^{r-1}\mathcal{E}^* \simeq (\Lambda^r\mathcal{E})^{-1} \otimes \mathcal{E}$$

$\forall s \in S$

$\mathcal{L}_s$ 0-regular $\Longleftrightarrow$ $H^i(\mathcal{L}_s) = 0$ $\forall i > 0$, $\forall s \in S$

$\Rightarrow$ $\mathcal{E}$ loc free. rk $H^0(\mathcal{L}_s)$, $s \in S$.

$r = \operatorname{rk}\mathcal{E}$ $\qquad$ $= \chi(\mathcal{L}_s)$

$$\tilde{h} := \sum_{i=1}^{r-1} h_{x_i} : \mathcal{E} \to \bigoplus_{i=1}^{r-1} M_{x_i} \qquad / \Lambda^{r-1}$$

$$\Lambda^{r-1}\mathcal{E} \longrightarrow \bigotimes^{r-1} M_{x_i}$$

$\rightsquigarrow$

$$\mathcal{O}_S \longrightarrow \mathcal{E} \otimes \det(\mathcal{E})^{-1} \bigotimes^{r-1} M_{x_i} \qquad / \pi_S^*$$

$$\mathcal{O}_{F\times S} \longrightarrow \mathcal{L} \otimes \pi_S^*\big(\det(\mathcal{E})^{-1} \bigotimes^{r-1} M_{x_i}\big) \otimes \pi^* K$$

K^{-r} $\quad$ K^{r-1} $\quad$ $\Lambda^r\mathcal{E}$

$\mathcal{E} \to \mathcal{E}\otimes K$ $\quad$ $(\Lambda^r\mathcal{E}\otimes K^r)$

$$L \longrightarrow L \otimes \pi_S^* K$$

$$\Sigma = p\pi_S^* L \longrightarrow (\pi_S)_*(L \otimes (\pi_S)^* K)$$

$$\shortparallel$$

$$(\pi_S)_* L \otimes K = \Sigma \otimes K$$

$$\Lambda^r \Sigma \longrightarrow (\Lambda^r \Sigma) \otimes K^r$$

$$M_{x_i} \longrightarrow M_{x_i} \otimes K$$

$$\delta|_{\pi_S^{-1}(s)} \not\equiv 0 \quad \forall s \in S \;?$$

(Σ smoothness)

Fix ~~s~~ $s \in S$. Then, $\mathcal{L}_s$ very ample

$\rightsquigarrow \varphi_s : F \hookrightarrow \mathbb{P}^{r-1}$

$\delta|_{\pi_S^{-1}(s)} \not\equiv 0 \iff \varphi_s(x_1), \dots, \varphi_s(x_{r+1})$ are in general position.

<u>Calc</u>: $H^1 \mathcal{O}_F = 0 \Rightarrow \mathrm{Pic}_F \cong h_{\left(\coprod_{\mathrm{Pic}_F(k)} \mathrm{Spec}(k)\right)}$

<u>Ex</u>: X birational to $\mathbb{P}^n \Rightarrow H^1 \mathcal{O}_X = H^1 \mathcal{O}_{\mathbb{P}^n} = 0$

$\Rightarrow \mathrm{Pic}_F$ discrete

$$\mathrm{LinSys}_L \qquad \mathrm{Curves}_F \xrightarrow{\phi} \mathrm{Pic}_F \quad L$$

$$\wr\!\shortparallel$$

$$h_{\mathbb{P}(H^0 L^\vee)}$$

$$\begin{array}{ccc} h_X & \xrightarrow{\phi} & h_Y \\ \uparrow & & \uparrow \\ h_{X\times_Y Z} & \xrightarrow{\ \ } & h_Z \end{array}$$

$\mathbb{P}(E)$

$|$

$\mathcal{O}^n \twoheadrightarrow \mathcal{L}$

$N \cdot r - 1$

$$(\gamma) \begin{pmatrix} x_{1,1} & \cdots & x_{1,r} \\ & \vdots & \\ x_{N-1,1} & \cdots & x_{N-1,r} \\ x_{N,1} & \cdots & x_{N,r-1} \end{pmatrix} \begin{array}{l} \rightsquigarrow \sigma_1 \\ \\ \rightsquigarrow \sigma_{N-1} \\ \rightsquigarrow \sigma_N \end{array}$$

$$\sigma_\gamma = \bigotimes_{i=1}^N (\sigma_i)_\gamma \in H^0(F\times S, \mathcal{K})$$

$$\mathcal{K} = (\Lambda^r \mathcal{E})^{-N} \otimes \left[\bigotimes_{\text{all } k_i} M_{x_{k_i}}\right]$$

$$\overset{(\cdots)}{\rightsquigarrow} s_\gamma \in H^0(\mathcal{L})$$

Thm: For suitable choices of ξ, N, $Nr-1$ pts scalars a_γ we get:

$\forall$ llodl. L on F of type ξ, the section

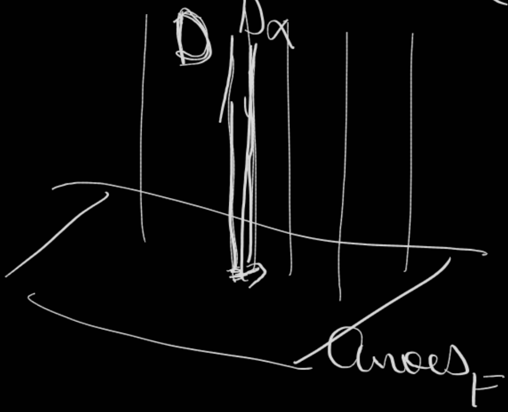
$\sum_\gamma a_\gamma \sigma_\gamma \in H^0(\mathcal{L})$ is not zero on any fibres.

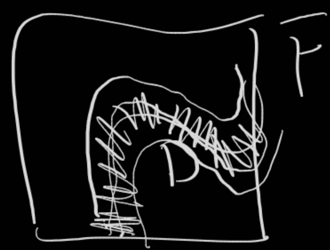
$\boxed{\text{Indepent 0-cycles.}}$

Prop: "$\mathrm{Curves}_F\left(\mathrm{Spec}\,\frac{k[\varepsilon]}{(\varepsilon^2)}\right)$ at $[D]$"

$$\cong$$

$$H^0(N_{D|F})$$





Prop: $0 \to \mathcal{O}_F \to \mathcal{O}^*_{F_\varepsilon} \to \mathcal{O}^*_F \to 0$

$f \mapsto 1+\varepsilon f$

$F_\varepsilon = F \times \mathrm{Spec}\,\frac{k[\varepsilon]}{(\varepsilon^2)}$ $\qquad a+\varepsilon f \to a$

$a \in \mathcal{O}_F^*$

Prop: tg sp at $0 \in \mathrm{Pic}_F$ is $H^1\mathcal{O}_F$

$0 \to H^0\mathcal{O}_F \to H^0\mathcal{O}^*_{F_\varepsilon} \twoheadrightarrow H^0\mathcal{O}^*_F = k^*$

$\to H^1\mathcal{O}_F \hookrightarrow H^1\mathcal{O}^*_{F_\varepsilon} \to H^1\mathcal{O}^*_F$

$H^1\mathcal{O}_F = \ker(H^1\mathcal{O}^*_{F_\varepsilon} \to H^1\mathcal{O}^*_F)$

$$\mathrm{Pic}^1_F\left(\frac{k[\varepsilon]}{(\varepsilon^2)}\right) \longrightarrow \mathrm{Pic}^1_F(k)$$

$$[\mathcal{O}_F]$$

$$h^1\mathcal{O}_F = \dim \mathrm{Pic}^0_F$$

Cor: Let $\mathcal{D} \subset F \times S$ any family of curves.

Fix $s \in S$ $\rightsquigarrow$ $\rho_s : T_s S \longrightarrow H^0(N_{D_s|F})$

Cor: $\mathcal{D} \subset F \times \mathrm{Curves}^{\xi}_F$ univ. family

Then, ρ is iso.

Ex: C elliptic curve, $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C)$

$$\begin{array}{c} \mathcal{O}_C \\ \cong \\ H^1(\mathcal{O}_C) \neq 0 \end{array}$$

$$F = \mathbb{P}(\mathcal{E}) \longrightarrow C$$

Def: $D \subset F$ semi-regular if

$$H^1\mathcal{O}(D) \xrightarrow{0} H^1 N_{D|F}$$

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}_D(D) \to 0$$

Thm (Severi–Kodaira–Spencer) Let $D_0 \subset F$ of type ξ, char$(k) = 0$, D_0 semiregular $\Rightarrow$ $\mathrm{Curves}^{\xi}_{F}$ is nonsingular at s. $\boxed{s = [D_0]}$

(At nonsingular pts of $\mathrm{Curves}^{\xi}_{F}$, D infinitesimal deformations come from actual deformations)

The reason why char $k = 0$ is necessary is because we can define "partial sum of the exponential"

$$(\mathcal{O}_F \otimes A)^* \longrightarrow \left(\mathcal{O}_F \otimes \tfrac{A}{I}\right)^* \quad \text{splits}$$

$$\begin{tikzcd}
(\mathcal{O}_F \otimes A)^* \arrow[r, "\sim"] \arrow[d, shift right] & \mathcal{O}_F^* \cdot (1 + \mathcal{O}_F \otimes M) \arrow[r, "\sim", "\exp"'] & \mathcal{O}_F^* \cdot (\mathcal{O}_F \otimes M)_+ \arrow[d, shift left] \\
(\mathcal{O}_F \otimes A_I)^* \arrow[u, shift right] \arrow[r, "\sim"] & \bar{M} \arrow[r] & \bar{M} \arrow[u, shift left]
\end{tikzcd}$$

R regular local ring

$$\iff \quad \begin{tikzcd}
 & A \arrow[d] \\
R \arrow[r] \arrow[ur, dashed, "\exists"] & A_0
\end{tikzcd} \qquad \begin{tikzcd}
 & k[\varepsilon]/(\varepsilon^2) \arrow[d] \\
R \arrow[r] \arrow[ur, dashed] & k
\end{tikzcd}$$

Let $U \subset \mathrm{Pic}^{\xi}_C$ s.t. $\forall x \in U(k) \Rightarrow H^1 \mathcal{L}_x = 0$

<u>Prop:</u> Over U, ϕ is a projective bundle map

<u>Pf:</u> Let $\mathcal{L}$ univ. l.bdl on $F \times U$.

$\mathcal{E} = \pi_{U*}\mathcal{L}$ loc. free

<u>Wts:</u> $\phi^{-1}(U) \simeq \mathbb{P}(\mathcal{E}) / U$

$$\phi^{-1}(U) \subset \mathrm{Curves}^{\xi}_F$$

$$\left\{ \begin{tikzcd} T \arrow[r] \arrow[dr, "g"'] & \phi^{-1}(U) \arrow[d] \\ & U \end{tikzcd} \right\} \simeq \left\{ \begin{array}{l} D \subset F \times T \text{ s.t. } \exists M \in \mathrm{Pic}(T) \\ \text{s.t. } \mathcal{O}(D) \simeq \mathcal{L} \otimes p^*M \end{array} \right\} = \alpha$$

$$\left\{ \begin{tikzcd} T \arrow[r] \arrow[dr, "g"'] & \mathbb{P}(\mathcal{E}) = \mathrm{Gr}(1,\mathcal{E}) \arrow[d] \\ & U \end{tikzcd} \right\} \simeq \left\{ \begin{array}{c} g^*\mathcal{E} \twoheadrightarrow M \\ \in \\ \mathrm{Pic}(T) \end{array} \right\} = \beta$$

$$\alpha = \left\{ \begin{array}{l} M \in \mathrm{Pic}(T) \\ s \in H^0(h^*\mathcal{L} \otimes q^*M) \\ \text{relative}/T \\ s|_{F \times \{t\}} \end{array} \right\}$$

$$\begin{tikzcd} F \times T \arrow[r, "h"] \arrow[d, "q"] & F \times U \arrow[d, "p"] \\ T \arrow[r, "g"'] & U \end{tikzcd}$$

$\pi_{U*}\mathcal{L} = \mathcal{E}$ lc free

$$H^0(h^*\mathcal{L} \otimes q^*M) \simeq \mathrm{Hom}((g^*p_*\mathcal{L})^*, M)$$

$$\alpha = \left\{ \begin{array}{l} M \in \mathrm{Pic}(T),\ s \in H^0(g^*p_*\mathcal{L}) \otimes M) \\ \quad \text{relative}/T \end{array} \right\}$$

$$g^*\mathcal{E} \simeq \mathrm{Hom}(g^* p_* \mathcal{L}, \mathcal{O}) = (g^* p_* \mathcal{L})^*$$

$$\beta = \left\{ \begin{array}{c} M \in \mathrm{Pic}(T) \\ g^*\mathcal{E} \twoheadrightarrow M \end{array} \right\} = \alpha.$$

$$H^1 \mathcal{L}_x = 0 \Rightarrow R^1 \pi_* \mathcal{L} = 0$$

$$\begin{array}{ccc} D \subset F & & H^1 \mathcal{O}(D) = 0 \\ \ \ \ \downarrow{\scriptstyle \pi_*} & & \\ \ \ \ S & & \end{array}$$