ℓ-adic Cohomology Abstract

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Let X be a smooth projective variety over $\mathbf{F}_q$ (for q a prime power), so X is globally cut out by a finite set of homogeneous polynomials with coefficients in $\mathbf{F}_q$. One important arithmetic question is to determine $N_n = |X(\mathbf{F}_{q^n})|$, i.e. how many simultaneous solutions exist in $\mathbf{F}_{q^n}$ for each $n \geq 1$. One can put this data together into a function $Z_X(t) \in \mathbf{Q}[\![t]\!]$ defined as

$$Z_X(t) = \exp\Big(\sum_{n\geq 1} \frac{N_n}{n} t^n\Big),$$

whose logarithmic derivative is precisely the 'naive' generating function for N_n. Information about the zeros and poles of $Z_X(t)$ can yield information about the growth rate of its logarithmic derivative in a precise way, and vastly generalizing this, Weil conjectured very special structural properties for $Z_X(t)$ informed by certain computations he worked out for curves.

Moreover, Weil noted that if you could define a cohomology theory mimicking certain natural topological properties, then his conjectures (minus the Riemann hypothesis) would more or less follow formally. Such a cohomology theory was realized by Grothendieck and Artin in the 1960s in the form of ℓ-adic cohomology, which is loosely speaking a 'repackaging' of étale cohomology.[1]

In this mini-course we will introduce and develop the theory of étale cohomology and apply it to the study of ℓ-adic cohomology, focusing on certain key properties of étale cohomology most relevant to proving the Weil conjectures.

We will mostly follow the exposition in [FK88] and [LEC], among other references.

[1] Unfortunately, the reason étale cohomology is not by itself sufficient is because étale cohomology only gives sensible cohomology groups using torsion coefficients (whose order is relatively prime to q), which makes it unsuitable as a candidate for a Weil cohomology theory. Precisely, one cannot expect a Lefschetz fixed point formula for étale cohomology with torsion coefficients.

Plan for the course.

- Day 1: What are we trying to do? We're trying to develop a "Weil cohomology theory" in order to prove the Weil conjectures. But it will take some time to define (namely, étale cohomology).
  Preparation on étale cohomology: Étale/unramified/smooth morphisms, sites (in particular, the small étale site), sheaves on sites, stalks, faithfully flat descent.

- Day 2: (cont.) Pushforward and pullback functors, defining étale cohomology.
  Tools to compute étale cohomology: Kummer sequence, Leray spectral sequence. Instead of computing cohomology of curves, we develop more properties of étale cohomology.
  Properties of étale cohomology: Cohomology with support in a closed subscheme, purity, Gysin sequence.

- Day 3: (cont.) Cohomology with compact support, proper base change theorem, cycle map. Poincaré duality for curves, see [LEC, §14, p. 93-99].

- Day 4: Now we can finally define ℓ-adic cohomology.
  ℓ-adic cohomology: Finiteness theorems, constructible sheaves, $\mathbb{Z}_\ell$-sheaves and $\mathbb{Q}_\ell$-sheaves.
  Setting up Weil I: Poincaré duality, Lefschetz fixed point formula. See [LEC, §§24-25].

- Day 5: Traces, introducing some L-functions. Proof of rationality of the zeta function via cohomological interpretation of the L-function, see [D74, §1]. Also see [LEC, §§26-27].

References

[D74] P. Deligne, *La conjecture de Weil : I*, Publications Mathématiques de l'IHÉS, Tome 43 (1974), pp. 273-307. `http://www.numdam.org/item/PMIHES_1974__43__273_0.pdf`

[FK88] E. Freitag, R. Kiehl, *Etale Cohomology and the Weil Conjecture*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 13. Springer-Verlag Berlin Heidelberg, 1988.

[LEC] J. Milne, Lectures on Étale Cohomology. University of Michigan, v. 2.21, 2013. `https://www.jmilne.org/math/CourseNotes/LEC.pdf`