Managing multiple maps' matrices with mirrors

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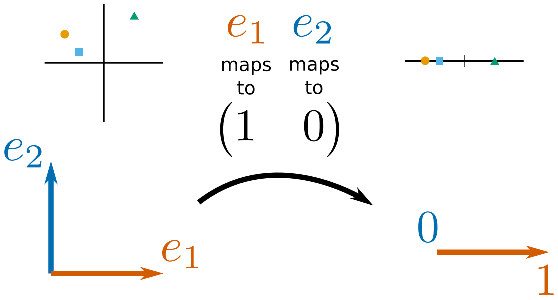
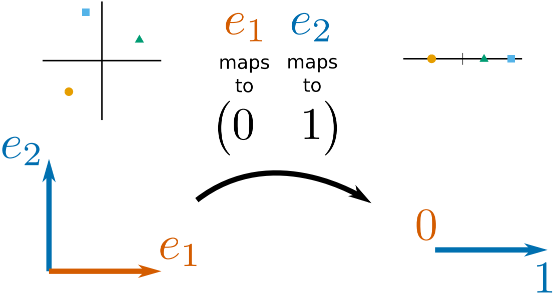
Linear maps

- A **linear map** $f : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $f(av + bw) = af(v) + bf(w)$ for any $a, b \in \mathbb{R}$ and $v, w \in \mathbb{R}^m$.
- It's enough to know where the map sends **coordinate vectors**. We can write this information in the columns of a matrix.

Map 1, $\mathbb{R}^2 \to \mathbb{R}$: forget vertical coordinate



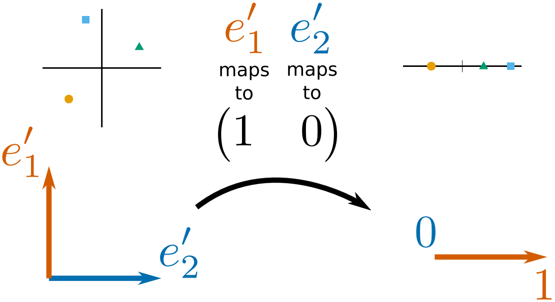
Map 2, $\mathbb{R}^2 \to \mathbb{R}$: forget horizontal coordinate



- Although these maps look different, that's just a matter of **which coordinate is first and which is second**.
- If we label our axes differently, they have the same matrix.

Change of basis

Map 2, $\mathbb{R}^2 \to \mathbb{R}$: forget horizontal coordinate



- Although these maps look different, that's just a matter of **which coordinate is first and which is second**.
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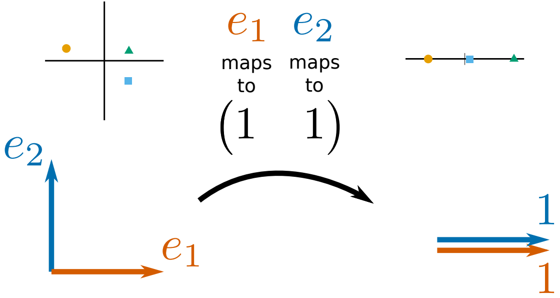
Change of basis

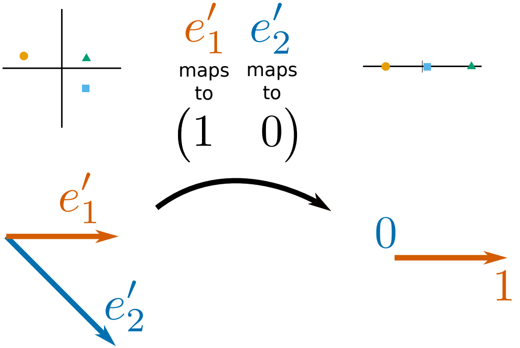
Map 3, $\mathbb{R}^2 \to \mathbb{R}$: add coordinates



- This map seems fundamentally different, but. . .
- if we pick the right coordinates, it's still "keep one piece of information and forget the other."

Change of basis

Map 3, $\mathbb{R}^2 \to \mathbb{R}$: add (original) coordinates



- This map seems fundamentally different, but...
- if we pick the right coordinates, it's still "keep one piece of information and forget the other."

A theorem on change of basis

Theorem

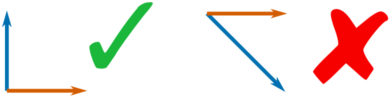
Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Then it is always possible to choose coordinates for $\mathbb{R}^m$ and $\mathbb{R}^n$ such that the matrix of f looks like

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

- Given any map, we can choose tailored coordinates, from whose perspective the map is "keep some information and forget the rest."
- What's important is **rank** — the number of 1's in the matrix.

Directions for generalization

- Direction 1: place restrictions on coordinates (like fixing angles and lengths)



  - Not the subject of this talk, but leads to tools like the **singular value decomposition**.
- Direction 2: consider multiple maps at once

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^p$$

  - Can we choose coordinates of $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^p$ such that f and g both have nice matrices? What does "nice" mean in this context?
  - What are the fundamentally different ways this diagram can behave?

The language of quivers

- A **quiver** is a diagram consisting of **vertices** and **arrows** between them.

$$\bullet \longrightarrow \bullet \qquad\qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad\qquad \begin{tikzcd} & & \bullet \arrow[dl] \\ \bullet \arrow[r] & \bullet & \\ & & \bullet \arrow[ul] \end{tikzcd}$$

- A **representation** of a quiver is an assignment of a vector space to every vertex and a linear map to every arrow.

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \qquad\qquad \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^p \qquad\qquad \begin{tikzcd} & & \mathbb{R}^m \arrow[dl, "f"] \\ \mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^q & \\ & & \mathbb{R}^p \arrow[ul, "h"] \end{tikzcd}$$

The language of quivers

- Two representations of a quiver are **isomorphic** if we can pick coordinates for each one so that they consist of the same matrices.

$$\mathbb{R}^2 \xrightarrow{(1\ 0)} \mathbb{R} \quad \cong \quad \mathbb{R}^2 \xrightarrow{(0\ 1)} \mathbb{R} \quad \cong \quad \mathbb{R}^2 \xrightarrow{(1\ 1)} \mathbb{R}$$

- The **direct sum** of two representations:

$$\begin{array}{c} \mathbb{R}^{m_1} \overset{f_1}{\to} \mathbb{R}^{n_1} \overset{g_1}{\to} \mathbb{R}^{p_1} \\ \oplus \\ \mathbb{R}^{m_2} \overset{f_2}{\to} \mathbb{R}^{n_2} \overset{g_2}{\to} \mathbb{R}^{p_2} \end{array} \quad \cong \quad \begin{pmatrix} \mathbb{R}^{m_1} \\ \mathbb{R}^{m_2} \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} \begin{pmatrix} \mathbb{R}^{n_1} \\ \mathbb{R}^{n_2} \end{pmatrix} \xrightarrow{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}} \begin{pmatrix} \mathbb{R}^{p_1} \\ \mathbb{R}^{p_2} \end{pmatrix}$$

Examples of direct sum

$$(\mathbb{R} \xrightarrow{1} \mathbb{R}) \oplus (\mathbb{R} \xrightarrow{1} \mathbb{R}) \oplus (\mathbb{R} \xrightarrow{0} 0) \;\cong\; \mathbb{R}^3 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} \mathbb{R}^2$$

- In general, any matrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

breaks down as a direct sum of $\mathbb{R} \xrightarrow{1} \mathbb{R}$, $\mathbb{R} \xrightarrow{0} 0$, and $0 \xrightarrow{0} \mathbb{R}$.

Restating the theorem

- A representation is **indecomposable** if it isn't isomorphic to a direct sum of smaller ones.
- With this new language, we can restate the theorem we started with.

Theorem

The only indecomposable representations of the quiver

$$\bullet \to \bullet$$

are

$$\mathbb{R} \xrightarrow{1} \mathbb{R} \qquad \mathbb{R} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{R}$$

so every representation is isomorphic to a direct sum of these.

General quivers

- Our original question becomes: for each quiver, **can we classify the indecomposable representations**?
- We can capture the key features of $\bullet \to \bullet$ with the following property:

Definition

A quiver is **finite type** if it has finitely many indecomposable representations.

- For a representation, the analogue of "rank" is "how many copies of each indecomposable show up?"

Example lists of indecomposables

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$\mathbb{R} \longrightarrow 0 \longrightarrow 0 \qquad 0 \longrightarrow \mathbb{R} \longrightarrow 0 \qquad 0 \longrightarrow 0 \longrightarrow \mathbb{R}$$

$$\mathbb{R} \xrightarrow{1} \mathbb{R} \longrightarrow 0 \qquad \mathbb{R} \xrightarrow{1} \mathbb{R} \xrightarrow{1} \mathbb{R} \qquad 0 \longrightarrow \mathbb{R} \xrightarrow{1} \mathbb{R}$$

Similarly to $\bullet \to \bullet$, we can interpret a direct sum of these as "keep some information, forget the rest" at each step.

$$\begin{array}{c} \mathbb{R} \longrightarrow 0 \longrightarrow 0 \\ \oplus \\ \mathbb{R} \longrightarrow 0 \longrightarrow 0 \\ \oplus \\ \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0 \\ \oplus \\ \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \\ \oplus \\ 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \\ \oplus \\ 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \\ \oplus \\ 0 \longrightarrow \mathbb{R} \longrightarrow 0 \\ \oplus \\ 0 \longrightarrow 0 \longrightarrow \mathbb{R} \end{array}$$

Example lists of indecomposables

$$\begin{tikzcd} & & \bullet \arrow[dl] \\ \bullet \arrow[r] & \bullet & \\ & & \bullet \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \mathbb{R} \arrow[dl] \\ 0 \arrow[r] & 0 \\ & 0 \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \\ \mathbb{R} \arrow[r] & 0 \\ & 0 \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \\ 0 \arrow[r] & 0 \\ & \mathbb{R} \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \\ 0 \arrow[r] & \mathbb{R} \\ & 0 \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \mathbb{R} \arrow[dl, "1"] \\ 0 \arrow[r] & \mathbb{R} \\ & 0 \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \\ 0 \arrow[r] & \mathbb{R} \\ & \mathbb{R} \arrow[u, "1"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & 0 \arrow[dl] \\ \mathbb{R} \arrow[r, "1"] & \mathbb{R} \\ & 0 \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \mathbb{R} \arrow[dl, "1"] \\ \mathbb{R} \arrow[r, "1"] & \mathbb{R} \\ & 0 \arrow[u] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \\ \mathbb{R} \arrow[r, "1"] & \mathbb{R} \\ & \mathbb{R} \arrow[u, "1"] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \mathbb{R} \arrow[dl, "1"] \\ 0 \arrow[r] & \mathbb{R} \\ & \mathbb{R} \arrow[u, "1"] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \mathbb{R} \arrow[dl, "1"] \\ \mathbb{R} \arrow[r, "1"] & \mathbb{R} \\ & \mathbb{R} \arrow[u, "1"] \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \mathbb{R} \arrow[dl, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\ \mathbb{R} \arrow[r, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & \mathbb{R}^2 \\ & \mathbb{R} \arrow[u, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \end{tikzcd}$$

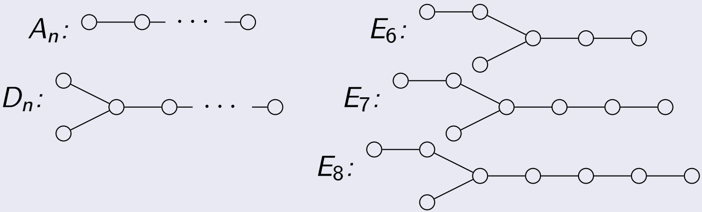
$$\begin{tikzcd} & & \bullet \arrow[d] & & & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

There are 120 indecomposables, with spaces of dimension up to 6.

Gabriel's theorem

Theorem

A quiver is finite type if and only if, ignoring the direction of arrows, it has one of these shapes:



- These diagrams — the **simply laced Dynkin diagrams** — are ubiquitous and important.

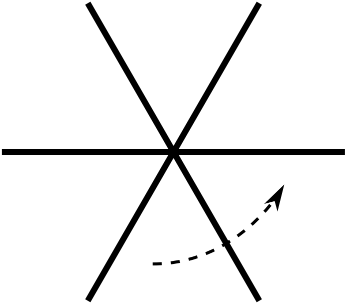
Systems of mirrors

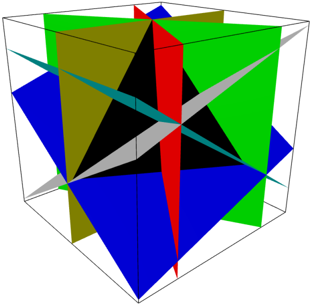
- A **hyperplane** in $\mathbb{R}^n$ is a subspace of dimension $n-1$ through the origin.

Definition

A **closed system of mirrors** is a finite collection $\mathcal{H}$ of hyperplanes such that:

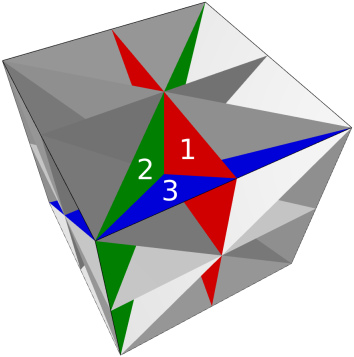
- For $H_1, H_2 \in \mathcal{H}$, the reflection of H_1 through H_2 is also in $\mathcal{H}$.





Notating a system of mirrors

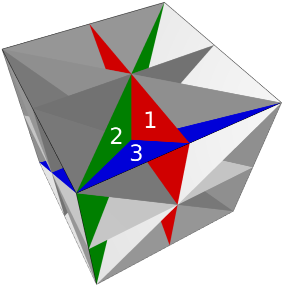
- Knowing the mirrors bordering a single region (**walls**) is enough.
- We record the angle between each pair of walls.



Angle between...

1 and 2	$\pi/4$
2 and 3	$\pi/3$
1 and 3	$\pi/2$

Notating a system of mirrors



Angle between...

1 and 2	$\pi/4$
2 and 3	$\pi/3$
1 and 3	$\pi/2$

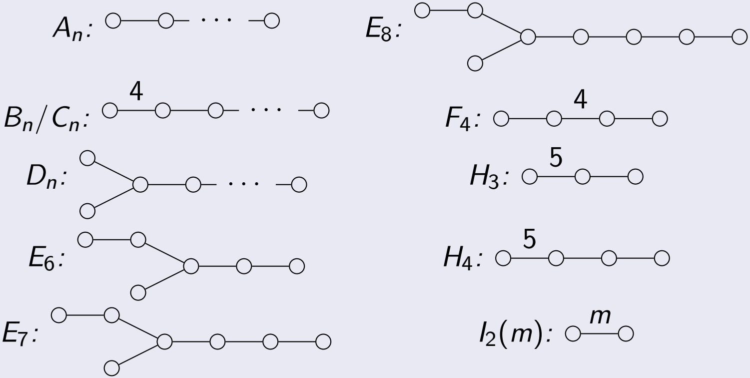
- Now draw a diagram with a vertex for each wall.
- For each pair of walls:
  - If the angle between is $\pi/2$, do nothing.
  - If the angle between is $\pi/3$, draw an edge between them.
  - If the angle between is π/m, $m > 3$, draw an edge between them and label it with m.

$$1 \overset{4}{-\!\!-\!\!-\!\!-} 2 -\!\!-\!\!-\!\!- 3$$

What are the closed systems of mirrors?

Theorem

The closed systems of mirrors correspond to these diagrams.



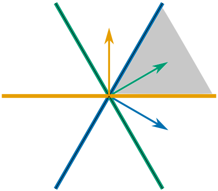
- The unlabeled ones look familiar!

Indecomposables and mirrors

- The **dimension vector** of a quiver representation records the dimension at each vertex.

$$\begin{tikzcd} & 3 \arrow[dl] \\ 1 \arrow[r] & 2 \\ & 4 \arrow[ul] \end{tikzcd} \quad : \quad \begin{tikzcd} & & \mathbb{R} \arrow[dl, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\ \mathbb{R} \arrow[r, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & \mathbb{R}^2 & \\ & & \mathbb{R} \arrow[ul, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \end{tikzcd} \quad \to \quad (1,2,1,1)$$

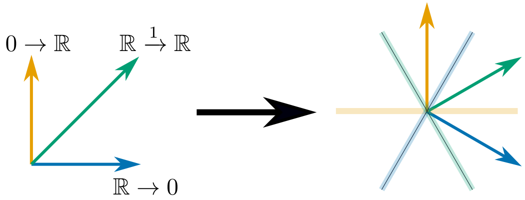
- Choose a region of a system of mirrors, and for each mirror pick the normal vector pointing towards that region. These are **root vectors**.



Indecomposables and mirrors

Theorem

There is a change of basis taking the dimension vectors of indecomposable representations of a finite type quiver to the root vectors of the system of mirrors with the same diagram.



- Or, in other words...

Theorem

Indecomposable representations correspond precisely with mirrors.