A QUICK TOUR TOWARDS THE LOCAL LANGLANDS CORRESPONDENCE

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Abstract. These are the notes I used for the graduate minicourse on local Langlands correspondence during summer 2022. Local Langlands correspondence is a conjectured parametrization of irreducible admissible representation of a p-adic group by the so called Langlands parameters, which are generalization of the Galois representations. In this course we define these objects, explain some conjectured properties of the correspondence and work explicitly with the case of GL_2. We also introduce the theory of endoscopy, in particular the endoscopic character identity conjecture in the local Langlands correspondence.

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Introduction

When we talk about a correspondence, there will be two sides of the story. In the case of Langlands correspondence, one is called the *automorphic side*, and the other is called the *Galois* *side*. The local Langlands correspondence aims to establish a map

$$\text{Automorphic side} \longrightarrow \text{Galois side.}$$

The automorphic side consists of objects called *irreducible admissible representations* while the Galois side of objects called *Langlands parameters*. This map is expected to be surjective and finite-to-one, so one can also think of the local Langlands correspondence as a parameterization of irreducible admissible representations by Langlands parameters, which are in some sense generalization of Galois representations.

This map is of course not a random assigment, and it is expected to satisfy some nice properties, like preserving certain operations of representations or invariants.

These phenomena have been observed in many known examples, like the abelian case from the local class field theory, and the case of GL_n by the work of Harris-Taylor, Henniart and Scholze-Fargues. We will later see how LLC works in the case of GL_2, what these recipes are, and verify some of the expectations explicitly.

Finally we introduce the theory of endoscopy for quasi-split reductive groups. This is a theory that relates the L-packets of a reductive group with the L-packets of its endoscopic groups. It is an instance of the so-called Langlands functoriality.

1. Explanation on the automorphic side

1.1. **Reivew of reductive groups (absolute theory).**

1.1.1. *Affine algebraic groups.* Let F be an algebraically closed field (can think of $F = \mathbb{C}$).

Definition. An *affine algebraic group* G is an affine variety equipped with polynomial maps

$$m : G \times G \to G, \quad i : G \to G$$

that make G into a group.

Example 1.1.
- $\mathbb{G}_a = \mathbb{F}$ with $m(x, y) = x + y$, $i(x) - x$.
- $\mathbb{G}_m = \{(x, x') \in F^2 \mid x \cdot x' = 1\}$ with $m(x, y) = xy$ and $i(x) = x^{-1}$.

Remark. Over R we have $\exp : \mathbb{R} \to \mathbb{R}^\times$ which is a group homomorphism, almost are isomorphism. This is not algebraic. There are non-trivial group homomorphism $\mathbb{G}_a \to \mathbb{G}_m$ or $\mathbb{G}_m \to \mathbb{G}_a$.

Theorem 1.1. $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$ *given by taking powers.*

Theorem 1.2. *Every affine algebraic group G is isomorphic to a closed subgroup of some* GL_n *(vanishing locus of polynomials).*

From now on, consider $G \subseteq \mathrm{GL}_n$.

1.1.2. *Jordan decomposition.* In GL_n we have the Jordan decomposition: every $g \in \mathrm{GL}_n$ there are $s, u \in \mathrm{GL}_n$ commuting such that $g = s \cdot u$ with s semisimple (diagonalizable) and u unipotent (if $g = s + n$ is Jordan normal form, then $u = I + s^{-1} \cdot n$).

Theorem 1.3. *If $g \in G$, then $s, u \in G$.*

Definition. G is called *unipotent* if all $g \in G$ are unipotent.

Example 1.2. $\mathbb{G}_a$ is unipotent. Indeed, $\mathbb{G}_a \cong \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$.

Theorem 1.4. *If G is abelian, then*

$$G_{ss} = \{g \in G \mid g \text{ is semisimple}\}, \quad G_{uni} = \{g \in G \mid g \text{ is unipotent}\}$$

are closed subgroups and $G = G_{ss} \times G_{uni}$. In general, G_{ss} and G_{uni} are not subgroups.

Definition. An affine algebraic group is called
- a *torus*, if $G \cong (\mathbb{G}_m)^n$

- *reductive*, if G has no closed connected unipotent normal subgroups.

Example 1.3.
- Any torus is reductive (it has not unipotent elements)
- The group $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subseteq \mathrm{GL}_2$ is not reductive because $\left(\begin{smallmatrix} 1 & * \\ & * \end{smallmatrix}\right)$ is normal.

Theorem 1.5. *Any affine algebraic group G sits in an exact sequence*

$$1 \to R_u(G) \to G \to G^{red} \to 1.$$

Here $R_u(G)$ is the unipotent radical and G^{red} is the reductive quotient. A unipotent G is a repeated extension of copies of $\mathbb{G}_a$.

1.1.3. *Tori.* Let T be a torus.

Definition. $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ is called the *character module*, and $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ is the *cocharacter module.*

These are finitely generated free abelian groups. The pairing

$$\langle -, - \rangle : X_*(T) \otimes X^*(T) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}, \quad (\chi, \lambda) \mapsto \chi \circ \lambda$$

is perfect.

1.1.4. *Lie algebra.* Let $T \to \mathrm{GL}(V)$ for some F-vector space. Then $V = \bigoplus_{\alpha \in X^*(T)} V_\alpha$ where $V_\alpha = \{v \in V \mid t \cdot v = \alpha(t) \cdot v, t \in T\}$.

For any affine algebraic group G we have $\mathrm{Lie}(G) =$ tangent space of G at $e \in G$. The action of G on itself by conjugate differentiates to $G \to \mathrm{GL}(\mathrm{Lie}(G))$.

1.1.5. *Classification of reductive groups.* Let G be a connected reductive group. Let G be a maximal torus. Then T acts in $\mathrm{Lie}(G)$ and we have the eigenspace decomposition:

$$\mathrm{Lie}(G) = \bigoplus_{\alpha \in X^*(T)} \mathrm{Lie}(G)_\alpha.$$

We have $\mathrm{Lie}(G)_0 = \mathrm{Lie}(T)$.

Let $R(T, G) = \{0 \neq \alpha \in X^*(T) \mid \mathrm{Lie}(G)_\alpha \neq \{0\}\}$. Then

- $R(T, G)$ is a reduced root system.
- The action of $N_G(T)$ on $R(T, G)$ identifies $N_G(T)/T$ with the Weyl group of $R(T, G)$.
- $\dim(\mathrm{Lie}(G)_\alpha) = 1$.
- For $\alpha \in R(T, G)$ there is a unique unipotent subgroup U_α normalized by T and $\mathrm{Lie}_\alpha = \mathrm{Lie}(G)_\alpha$. It is isomorphic to $\mathbb{G}_a$ but not canonically.
- For any choice of positive roots $R^+ \subseteq R(T, G)$, $U = \prod_{\alpha \in R^+} U_\alpha$ is a maximal connected closed unipotent subgroup, and $B = U \cdot T$ is a maximal connected closed solvable subgroup, called *Borel subgroup.* $R^+ \mapsto B$ is a 1-1 correspondence between choices of positive roots and Borel subgroup containing T.

Definition. The tuple $(X^*(T), R(T, G), X_*(T), R^\vee(T, G))$ is called the *root datum* for G and T.

Theorem 1.6. *All maximal tori in G are conjugate.*

Corollary 1.1. *The root datum of G, T is up to isomorphism independent of T.*

Theorem 1.7. *The isomorphism classes of reductive groups are in 1-1 correspondence with the isomorphism classes of root data.*

Remark. The root datum contains more information than the root system. For example, all tori are reductive groups with trivial root system.

1.1.6. *Semisimplicity.*

Definition. Let G be reductive. Its *derived subgroup* is $G_{\mathrm{der}} := [G, G]$. G is called *semisimple* if $G = G_{\mathrm{der}}$.

Theorem 1.8. *The following are equivalent.*

(1) G is semisimple.
(2) $Z(G)$ is finite.
(3) The $\mathbb{Z}$-span of $R(T, G)$ in $X^*(T)$ is of finite index.

Proposition 1.1. *Let G be a reductive group.*

(1) G is an alomost direct product of G_{der} and $Z(G)^\circ$ which is a torus.
(2) If G is semisimple, then G is an almost direct product of semisimple subgroups with irreducible root systems.

Remark. (1) $\mathbb{Z}R(T, G) = X^*(T)$ iff $Z(G) = \{1\}$.
(2) $\mathbb{Z}R^\vee(T, G) = X_*(T)$ iff G has no central isogeny.

1.2. **Review of reductive groups (relative theory).** Let F be a field, not necessarily algebraically closed. Consider an affine algebraic group G defined over F. Again we can think of $G \subseteq \mathrm{GL}_n$ descriped by polynomial equations with coefficients in F.

Remark. When F was algebraically closed, we identify G with the abstract group $G(F) \subseteq \mathrm{GL}_n(F)$. Now we dont want to do this. There are two ways to think of G.

(1) For any field extension F'/F, we have $G(F') \subseteq \mathrm{GL}_n(F')$. More generally, for any F-algebra R, we have $G(R) \subseteq \mathrm{GL}_n(R)$. In addition, for any F-algebra homomorphism $R_1 \to R_2$, we have $G(R_1) \to G(R_2)$. Therefore, we have a functor $\{F\text{-alg}\} \to \{\text{groups}\}$.
(2) Consider the coordinate ring $\overline{F}[G]$, the ring of polynomial functions on the variety $G(\overline{F})$. The fact that G is defined over F means that there is an F-algebra $F[G] \subseteq \overline{F}[G]$ such that $F[G] \otimes_F \overline{F} = \overline{F}[G]$.

Remark. For any field extension F'/F, we have the base change $G \times_F F'$ an affine algebraic group over F'. In terms of (1) above, we just "forget" that we can plug in F-algebra, and only plug in F'-algebra. In terms of (2), we replace $F[G]$ by $F[G] \otimes_F F'$.

1.2.1. *Torus.*

Definition. G is a *torus* if $G \times_F \overline{F}$ is a torus. G is a *split torus* if $G \cong \mathbb{G}_{m,F}^r$. G is *reductive* if $G \times_F \overline{F}$ is reductive.

Theorem 1.9. *Let T be a torus. Then*

(1) $T \times_F F^s \cong (\mathbb{G}_{m,F^s})^r$
(2) $X^*(T), X_*(T)$ are lattices with Γ-action, where $\Gamma = \mathrm{Gal}(F^s/F)$, and their pairing is F-invariant.
(3) T is split iff the Γ-action of $X^*(T)$ is trivial.
(4) The functor $T \mapsto X^*(T)$ is an equivalence of categories between F-torus and finitely generated free abelian groups with Γ-action.

Example 1.4. Let $F = \mathbb{R}$. We have two non-isomorphic 1-dimensional tori:

- $T = \mathbb{G}_m$, $X^*(T) = \mathbb{Z}$ with trivial Γ-action, $T(\mathbb{R}) = \mathbb{R}^\times$
- $T = R^1_{\mathbb{C}/\mathbb{R}}\mathbb{G}_a$, $X^*(T) = \mathbb{Z}$ with complex conjugate action, $T(\mathbb{R}) = \mathbb{C}$

Remark. Over non-algebraically closed F, not all maximal tori in a reductive group are conjugate under $G(F)$. For example, $G = \mathrm{SL}_2$ over $\mathbb{R}$, we have $\mathbb{G}_m \subseteq G$ given by $\begin{pmatrix} * & \\ & * \end{pmatrix}$ and $R^1_{\mathbb{C}/\mathbb{R}} \subseteq G$ given by $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \mathrm{SO}(2)$.

Theorem 1.10 (Grothendieck)**.** *Any affine algebraic F-group has a maximal torus defined over F.*

Remark. For any maximal F-torus $T \subseteq G$, $T_{\overline{F}} \subseteq G_{\overline{F}}$ is maximal.

Definition. Let G be a connected reductive group. G is called

- *split*, if G has a maximal torus that is split.
- *quasi-split*, if G has a Borel subgroup defined over F.

Remark. Let $T \subseteq G$ be a maximal torus, we can consider $R(T, G) \subseteq X^*(T)$ carrying an action of Γ. In this notation, G is

- quasi-split, if we can find T such that Γ presereves a set of positive roots in $R(T, G)$.
- split, if we can find T such that Γ acts trivially on $X^*(T)$.

1.2.2. *Classification of quasi-split reductive groups.*

Definition. Let G be quasi-split connected reductive group. A *Borel pair* is a tuple (T, B) consisting of a Borel subgroup $B \subseteq G$ defined over F, and a maximal torus $T \subseteq B$ defined over F.

Theorem 1.11. *All Borel pairs in G are conjugate under $G(F)$.*

Definition. Consider a Borel pair (T, G), we get the *based root datum* $(X^*(T), R(T, B), X_*(T), R^\vee(T, B))$ with an action of Γ, where $R(T, B)$ (resp. $R^\vee(T, B)$) is the set of positive roots (resp. coroots).

Theorem 1.12. *The isomorphism classes of quasi-split connected reductive groups are in 1-1 correspondence with isomorphism classes of based root data.*

1.3. **Representations of p-adic groups.** For now let F be a p-adic field, i.e., a finite extension of $\mathbb{Q}_p$. Let G be a reductive group defined over F, so that we can think of $G(F) \subseteq GL_n(F)$. $G(F)$ has a structure of locally profinite topological group, meaning that it has a neighbourhood basis of compact open subgroups.

Definition. Let (V, π) be a complex representation of $G(F)$.

(1) (V, π) is *smooth* if for every $v \in V$, its stabilizer $\mathrm{Stab}_{G(F)}(v)$ is open.
(2) (V, π) is *admissible* if for each compact open $U \subseteq G(F)$, $\dim V^U$ is finite.

Denote by $\mathrm{Rep}(G(F))$ the category of smooth representations of $G(F)$.

Definition. Let V be a smooth representation of $G(F)$. We define its *contragradient* $V^\vee$ to be the space of smooth vectors in $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ equipped with the left regular action

$$(g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v).$$

Definition. For $\lambda \in V^\vee$, $v \in V$, we call the function

$$m_{\lambda,v} : G(F) \to \mathbb{C}, g \mapsto \lambda(gv)$$

the *matrix coefficient* of G corresponding to v and λ.

Definition. A smooth representation (π, V) of $G(F)$ is called *supercuspidal* provided that its matrix coefficients are compactly supported modulo the center Z.

Definition. A smooth representation (π, V) of $G(F)$ is a *(essentially) discrete series* representation provided that (up to a twist by a smooth character)

(1) the center acts on V by a unitary character χ, and
(2) for all $v \in V$, $\lambda \in V^\vee$, the matrix coefficient $m_{\lambda,v} \in L^2(G(F)/Z)_\chi$.

Definition. A smooth representation (π, V) of $G(F)$ is *a (essentially) tempered* provided that (up to a twist by a smooth character)

(1) the center acts on V by a unitary character χ, and
(2) for all $v \in V$, $\lambda \in V^\vee$, the matrix coefficient $m_{\lambda,v} \in L^{2+\epsilon}(G(F)/Z)_\chi$ for any $\epsilon > 0$.

Remark. (1) Apparently we have supercuspidal $\Rightarrow$ essentially discrete series $\Rightarrow$ essentially tempered.
(2) Later we will see that how these representations behave under the local Langlands correspondence.

Definition. Let $P = MN$ be a F-rational parabolic subgroup of G with unipotent radical N and Levi subgroup M. Let (τ, W) be a smooth representation of $M(F)$. We define the *normalized induced representation* $\mathrm{Ind}_P^G \tau$ to be space of functions $f : G(F) \to W$ such that

(1) $f(mng) = \tau(m)\delta_p(mn)^{\frac{1}{2}} f(g)$, $m \in M(F), n \in N(F), g \in G(F)$, and
(2) there exists a compact open subgroup $K \subseteq G(F)$ such that $f(gk) = f(g)$ for all $g \in G(F)$, $k \in K$.

Here $\delta_P : P(F) \to \mathbb{C}^\times$ is the *modulus character* given by $\delta_P(mn) = \delta_P(m) = |\det(\mathrm{Ad}(m))|_{\mathrm{Lie}(N)}|$. G acts on $\mathrm{Ind}_P^G \tau$ be right translation: $(g \cdot f)(h) = f(hg)$.
This defines a functor $\mathrm{Ind}_P^G : \mathrm{Rep}(M(F)) \to \mathrm{Rep}(G(F))$.

Remark. We need the normalization by $\delta_P^{\frac{1}{2}}$ so that

(1) the functor Ind_P^G takes unitary representations to unitary representations, and
(2) it commutes with taking contragradient, i.e., $\mathrm{Ind}_P^G(\pi)^\vee \cong \mathrm{Ind}_P^G(\pi^\vee)$.

Remark. Even though the representation $\mathrm{Ind}_P^G \tau$ depends on the choice of parabolic P containing M, its Jordan-Holder factor, on the other hand, doesn't depend on the choice of P. So if we only care about constituents of $\mathrm{Ind}_P^G \tau$, it makes sense to use the notation $\mathrm{Ind}_M^G \tau$ instead.

It turns out that parabolic induction is closely related to classification of irreducible representations of $G(F)$. For example, supercuspidal representations are exactly those representations that cannot be obtained by parabolic induction. To make this statement precise, we need to introduce the Jacquet functor (which turns out to be the left adjoint of Ind_P^G).

Definition. Let (π, V) be a smooth representation of $G(F)$. We define the *normalized Jacquet module* $r_M^G(\pi) = \delta_p^{-\frac{1}{2}} V_N$ where $V_N = V/\mathrm{span}\{v - n \cdot v \mid v \in V, n \in N(F)\}$. This defines a functor $r_M^G : \mathrm{Rep}(G(F)) \to \mathrm{Rep}(M(F))$.

Proposition 1.2. *Let (π, V) be a smooth representation of $G(F)$. The following two conditions are equivalent:*

(1) (π, V) is supercuspidal;
(2) $r_M^G \pi = 0$ for all ***proper*** *parabolic subgroup $P = MN$.*

2. Explanation on the Galois side

Throughout this section, let F be a p-adic field with ring of integer $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$, residue field $k \cong \mathbb{F}_q$ and uniformizer ϖ. Let F^{s} be a separable algebraic closure with ring of integer $\mathcal{O}$, maximal ideal $\mathfrak{p}$, residue field $\bar{k}$ and uniformizer ϖ. Let $\omega : F \to \mathbb{Z}$ be the canonical non-archimedean valuation on F.

2.1. Weil-Deligne representations.

2.1.1. *Weil group.* F admits a unique unramified extension F_m of degree m, for each $m \geq 1$. Let F_∞ denote the composite of all these fields, called the *maximal unramified extension.*

The extenison F_m/F is Galois with $\mathrm{Gal}(F_m/F)$ cyclic. An element $\sigma \in \mathrm{Gal}(F_m/F)$ is determined by its action on $k_{\mathbb{F}_m} = \mathbb{F}_{q^m}$. In particular, there is an element Φ_m that acts on $\mathbb{F}_{q^m}$ by

$x^q \mapsto x$ such that $\mathrm{Gal}(F_m/F) = \langle \Phi_m \rangle \cong \mathbb{Z}/m\mathbb{Z}$. Taking the limit over m, we get a canonical isomorphism of topological groups

$$\mathrm{Gal}(F_\infty/F) \cong \varprojlim_{m\geq 1} \mathbb{Z}/m\mathbb{Z} = \hat{\mathbb{Z}} \cong \prod_{\text{prime } l} \mathbb{Z}_l. \tag{2.1}$$

There is a unique element $\Phi_F \subseteq \mathrm{Gal}(F_\infty/F)$ which acts on F_m as Φ_m for all m, called the *geometric Frobenius.*

We set $\mathcal{I}_F = \mathrm{Gal}(F^s/F_\infty) \subseteq \mathrm{Gal}(F^s/F)$: this is called the *inertia group* of F. We have an exact sequence of topological groups

$$1 \to \mathcal{I}_F \to \mathrm{Gal}(F^s/F) \xrightarrow{\text{res}} \underbrace{\mathrm{Gal}(F^\infty/F)}_{\hat{\mathbb{Z}}} \to 1. \tag{2.2}$$

The *Weil group* $\mathcal{W}_F$ is the inverse image in $\mathrm{Gal}(F^s/F)$ of the cyclic subgroup $\langle \Phi_F \rangle$, i.e., $\mathrm{res}^{-1}(\mathbb{Z})$ using the notation in (2.2). Therefore, $\mathcal{W}_F = \mathbb{Z} \ltimes \mathcal{I}_F$ fits into the exact sequence of abstract groups

$$1 \to \mathcal{I}_F \to \mathcal{W}_F \xrightarrow{v_F} \mathbb{Z} \to 1. \tag{2.3}$$

The natural valuation v_F induces a norm map

$$|\cdot| : \mathcal{W}_F \to q^{\mathbb{Z}}$$

such that $\mathcal{I}_F = \ker(|\cdot|)$ and $|\Psi_F| = q^{-1}$.

We can equip $\mathcal{W}_F$ with a topology such that

(1) $\mathcal{I}_F$ is an open subgroup of $\mathcal{W}_F$, and
(2) the topology of $\mathcal{I}_F$ in $\mathcal{W}_F$ coincides with its subgroup topology as $\mathrm{Gal}(F^s/F_\infty) \subseteq \mathrm{Gal}(F^s/F)$.

Remark. This topology is different from the subspace (profinite) topology of $\mathcal{W}_F \subseteq \mathrm{Gal}(F^s/F)$. In fact, it's finer (with more open subsets) than the subspace topology, so that there are more continuous map coming out from $\mathcal{W}_F$.

Definition. The *Weil-Deligne group* of F is $WD_F := \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C})$.

Definition. Let $\mathcal{G}$ be a complex Lie group whose identity component $\mathcal{G}^\circ$ is reductive. A *Weil-Deligne representation* of $\mathcal{W}_F$ is a $\mathcal{G}^\circ$-conjugacy class of homomorphism $\varphi : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{G}$ which is continuous on $\mathcal{I}_F \subseteq \mathcal{W}_F$, algebraic on $\mathrm{SL}_2(\mathbb{C})$ with $\varphi(\Phi_F)$ semisimple.

2.2. **Langlands parameters.** Let G be a connected, quasi-split reductive algebraic group over F with maximal torus T in a Borel B defined over F. Let F_0 be the splitting field of T, and $S \subseteq T$ the maximal F-split torus. Associated to these notations, we have the root datum $(X^*(T), R(T,B), X_*(T), R^\vee(T,B))$ together with the action of $\Gamma = \mathrm{Gal}(F^s/F)$.

2.2.1. *The L-group.* By Theorem 1.7 (classification of reductive group), there is a connected, reductive group $\hat{G}$ over $\mathbb{C}$ with the dual root datum $(X_*(T), R^\vee(T,G), X^*(T), R(T,G))$ with a maximal torus in a Borel subgroup $\hat{T} \subseteq \hat{B}$, called the *Langlands dual group.*

G	GL_n	SL_n	PGL_n	Sp_{2n}	SO_{2n}	U_n	G_2
$\hat{G}$	GL_n	PGL_n	SL_n	SO_{2n+1}	SO_{2n}	GL_n	G_2

TABLE 1. The dual group of some semisimple groups

Remark. For semisimple groups, taking $\hat{G}$ has the effect of exchanging the long and short roots, the simply connected and adjoint forms, etc.

Fix a set $\{x_{\hat{a}} \mid \hat{a} \in \hat{\Delta}(T,B)\}$ of nonzero vectors in each simple root space in $\hat{\mathfrak{b}} = \mathrm{Lie}(\hat{B})$, thereby giving a pinning $\mathcal{E} = (\hat{T}, \hat{B}, \{x_{\hat{a}}\})$ in $\hat{G}$.

Remark. Roughly speaking one can think of a pinning as a set of coordinates on $\hat{G}$.

The action of Γ on these objects an automorphism of $\hat{G}$ that preserves the pinning, i.e., an embedding of Γ in $\mathrm{Aut}(\hat{G}, \mathcal{E})$, the group of *pinned automorphisms*.

Definition. The *L-group* of G is the semidirect product ${}^L G = \mathcal{W}_F \ltimes \hat{G}$ with $\mathcal{W}_F$ acting a subgroup of Γ.

Definition. A *L-parameter* for G is a homomorphism

$$\varphi : \mathcal{W}_F \times \mathrm{SL}_2 \to {}^L G$$

such that

(1) $\varphi|_{\mathcal{W}_F} \circ \mathrm{pr} : \mathcal{W}_F \to {}^L G \to \mathcal{W}_F$ is the identity map;
(2) φ is continuous on $\mathcal{I}$ and $\varphi(\Phi_F)$ is semisimple
(3) $\varphi|_{\mathrm{SL}_2}$ is a homomorphism of algebraic groups over $\mathbb{C}$.

Two parameters are considered equivalent if they are conjugate by $\hat{G}$. Denote by $\mathrm{LP}(G/F)$ the set of equivalent classes of L-parameters of G over F.

3. Local Langlands conjecture

It is conjectured (and proved in many cases) that there is a canonical parametrization

$$\mathcal{L} = \mathcal{L}_{G/F} : \Pi(G/F) \to \mathrm{LP}(G/F)$$

where the former is the set of irreducible admissible representation that satisfies many nice properties. In this secion we will focus on the expected properties of the correspondence. In this section unless specified, all "Propositions" are only conjectured properites of the local Langlands correspondence.

3.1. L-packets.

Definition. For each $\varphi \in \mathrm{LP}(G/F)$, the fiber over φ is called a *L-packet*, denoted by Π_φ.

Proposition 3.1. *Each L-packets is finite and non-empty, i.e., $\mathcal{L}$ is finite-to-one surjective.*

Definition. A L-parameter $\varphi : \mathcal{W}_F \times \mathrm{SL}_2 \to {}^L G$ is *tempered* if its restriction to $\mathcal{W}_F \times \mathrm{SU}(2)$ has bounded image after projection on $\hat{G}$.

3.2. **Basic properties.** The first property implies that the local Langlands conjecture can be considered as a nonabelian version of local class field theory.

Proposition 3.2. *For $G = \mathrm{GL}_1$, $\mathcal{L}$ is induced by Artin map $\mathrm{GL}_1(F) = F^\times \cong \mathcal{W}_F^{ab}$ in the local class field theory.*

The parametrization $\mathcal{L}$ is also expected to respect some operations.

Proposition 3.3. *$\mathcal{L}$ is compatible with characters of torus in the following sense. Let $f : G \to T$ be a homomorphism of algebraic groups over F, with T a torus. Let $f^* : {}^L T \to {}^L G$ the dual map. Let $\pi \in \Pi(G/F)$, $\chi \in \Pi(T/F)$. Then we have equivalence*

$$\mathcal{L}(\pi \otimes \chi \circ f) \cong \mathcal{L}(\pi) \cdot (f^* \circ \mathcal{L}(\chi)).$$

Proposition 3.4. *For any homomorphism $\sigma : {}^L G \to \mathrm{GL}(N)$ of complex algebraic groups, the representation*

$$\sigma \circ \mathcal{L}(\pi^\vee) : \mathcal{W}_F \times \mathrm{SL}_2 \xrightarrow{\mathcal{L}(\pi^\vee)} {}^L G \xrightarrow{\sigma} \mathrm{GL}(N)$$

is equivalent to the contragradient of

$$\sigma \circ \mathcal{L}(\pi) : \mathcal{W}_F \times \mathrm{SL}_2 \xrightarrow{\mathcal{L}(\pi)} {}^L G \xrightarrow{\sigma} \mathrm{GL}(N),$$

i.e., we have

$$\sigma \circ \mathcal{L}(\pi^\vee) \cong (\sigma \circ \mathcal{L}(\pi))^\vee.$$

Proposition 3.5. *Suppose $G = H_1 \times H_2$. Let*

$$\otimes : \Pi(H_1/F) \times \Pi(H_2/F) \to \Pi(G/F)$$

denote the map on L-parameters induced by the inclusing

$$^L(H_1 \times H_2) \to {}^LH_1 \times {}^LH_2$$

defined by the diagonal map $\mathcal{W}_F \to \mathcal{W}_F \times \mathcal{W}_F$. Then

$$\mathcal{L}_{G/F} = \otimes(\mathcal{L}_{H_1/F} \times \mathcal{L}_{H_2/F}).$$

Proposition 3.6. *Let $r : G \to G'$ be a central isogeny and let $r^* : {}^LG' \to {}^LG$ denote the dual map. Then if $\pi \in \Pi(G'/F)$,*

$$\mathcal{L}(\pi \circ r) = r^* \circ \mathcal{L}(\pi).$$

If $\pi \circ r$ is reducible, the left hand side refers to any of the irreducible constituents; in particular, they lie in a single L-packet.

Proposition 3.7. *Let F be a finite separable extension of E and $H = \mathrm{Res}_{F/E}\, G$ and let π be an irreduciblee representation of $H(E) = G(F)$. The parameters for WD_F with values in ${}^LG = \mathcal{W}_F \ltimes \hat{G}$ can be naturally identified with parameters for WD_E with values in ${}^LH = \mathcal{W}_E \ltimes \hat{G}^{[F:E]}$. With respect to this identification we have*

$$\mathcal{L}_{H/E}(\pi) = \mathcal{L}_{G/F}(\pi).$$

To state the next property of LLC, we need to make the following definition.

Definition. For $\pi \in \Pi(G/F)$, define the *semisimple Langlands parameter* of π to be the map

$$\mathcal{L}^{\mathrm{ss}} : \mathcal{W}_F \xrightarrow{\mathrm{id} \times \begin{pmatrix} |\cdot|^{\frac{1}{2}} & \\ & |\cdot|^{-\frac{1}{2}} \end{pmatrix}} \mathcal{W}_F \times \mathrm{SL}_2 \xrightarrow{\mathcal{L}(\pi)} {}^LG.$$

Proposition 3.8. *Let $P \subseteq G$ be a parabolic subgroup rational over F, with Levi quotient M. Let $i_M : {}^LM \to {}^LG$ be the inclusion as a Levi subgroup of the dual parabolic ${}^LP$. Let Ind_M^P denote normalized (using chosen square root of q) parabolic induction, which is independent of the the choice of P containing M. Suppose π is an irreducible subquotient of $\mathrm{Ind}_M^G(\tau)$. Then*

$$\mathcal{L}^{ss}(\pi) \cong i_M \circ \mathcal{L}^{ss}(\tau).$$

Proposition 3.8 tells us when $\tau \in \Pi(G/F)$ can be obtained by parabolic induction (the non-supercuspidal ones), how we can recover the parameter for τ from the parameter of the representation of the Levi. Complementary to Proposition 3.8, we can also say something about the Langlands parameter of supercuspidal representations.

Proposition 3.9. *If $\pi \in \Pi(G/F)$ belongs to the discrete series, the image of $\mathcal{L}(\pi)$ is not contained in any proper parabolic subgroup of ${}^LG$.*

3.3. **Automorphic properties.** The local Langlands correspondence is also expected to preserve certain invariants called local factors. These local factors has been defined for representations of the Weil groups, see [Tat79, §3.3]

3.3.1. *Local factors for Weil-Deligne representations.* Fix a finite dimensional algebraic representation $\sigma : {}^LG \to \mathrm{GL}(V)$. For a given L-parameter $\varphi : WD_F \to {}^LG$, let $N = d\varphi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Lie}({}^LG)$. Notice that σ induces a map $d\sigma : \mathrm{Lie}({}^LG) \to \mathfrak{gl}(V) = \mathrm{End}(V)$ at the level of Lie algebras. So N can be thought of as an operator acting on V. Let $V_N := \ker(N)$ be the kernel of N on V. Since the action of N commutes with the action of $\mathcal{W}_F$, and $\mathcal{I}_F$ is normal in $\mathcal{W}_F$, the invariant space $V_N^{\mathcal{I}_F}$ are preserved by $\varphi(\Phi_F)$.

Definition. The *local L-factor* of (φ, σ) is

$$L(s, \varphi, \sigma) = \det(1 - q^{-s}\varphi(\Phi_F)\varphi\begin{pmatrix} q^{-\frac{1}{2}} & \\ & q^{\frac{1}{2}} \end{pmatrix})|V_N^{\mathcal{I}})^{-1}.$$

Definition. Let $\psi : F \to \mathbb{C}^\times$ be a smooth additive character. The *local ϵ-factor* of (φ, σ, ψ) is

$$\epsilon(s, \varphi, \sigma, \psi) = \epsilon((\sigma \circ \varphi) \otimes |\cdot|^s, \psi, dx) \cdot \det(-q^{-s}\varphi(\Phi_F)\varphi\begin{pmatrix} q^{-\frac{1}{2}} & \\ & q^{\frac{1}{2}} \end{pmatrix}|V^{\mathcal{I}}/V_N^{\mathcal{I}}),$$

where

- dx is the Haar measure on F normailzed so that the biggest fractional ideal contained in $\ker(\psi)$ has volume 1;
- the first term on the right hand side, $\epsilon(\sigma \otimes |\cdot|^s, \psi, dx)$ is the local ϵ-factor for representations of Weil group.

3.3.2. *Automorphic L-functions.* Now we want to define the corresponding local factors on the automorphic side. There is explicit formula to do so in general, we can summarize the expected properties as below.

Definition. Let $\sigma : {}^L G \to \mathrm{GL}(V)$ be an algebraic representation, $\mathcal{A}$ be a class of irreducible representations of $G(F)$. A *theory of automorphic L-functions* for G over F, $\mathcal{A}$, and σ consists of the following.

(1) For any $\pi \in \mathcal{A}$, and any additive character $\psi : F \to \mathbb{C}^\times$,
   - a meromorphic function $s \mapsto L(s, \pi, \sigma)$ that is holomorphic in a right half-plane, of the form $P(q^{-s})$ where P is a polynomial of degree at most $\dim \sigma$ and constant term 1, and
   - an entire function $s \mapsto \epsilon(s, \pi, \sigma, \psi)$, which is a constant multiple of an exponential of the form $q^{\alpha s}$ for some complex number s.

(2) If π is shperical (has a nonzero vector fixed by maximal special compact open) with parameter $\mathcal{L}(\pi)$, then $L(s, \pi, \sigma) = L(s, \mathcal{L}(\pi), \sigma)$. Moreover, if $\ker(\psi) = \mathcal{O}_F$, then $\epsilon(s, \pi, \sigma, \psi) = 1$.

(3) Let K be a global field with a place v such that $F \cong K_v$, and $\mathcal{G}$ a connected reductive group over K with $\mathcal{G}(K_v) \cong G(F)$. There is a class $\mathcal{A}_K$ of cuspidal automorphic representations of $\mathcal{G}(\mathbb{A}_K)$ such that any $\pi \in \mathcal{A}$ can be realized by as the local component of some $\Pi \in \mathcal{A}_K$. It is assumed that for every place w of K and every local component Π_w of $\Pi \in \mathcal{A}_K$ there are local factors $L(s, \Pi_w, \sigma)$ and $\epsilon(s, \Pi_w, \sigma, \psi_w)$ satisfying (1) and (2), with the local factors already defined when $w = v$, For any unitary cuspidal automorphic representation $\Pi \in \mathcal{A}_K$ with $\Pi_v \cong \pi$, the formal product

$$L(s, \Pi, \sigma) := \prod_w L(s, \Pi_w, \sigma)$$

converges aboslutely in a right-half plane, and extends to a meromorphic function of $\mathbb{C}$ that satisfies the functional equation

$$L(s, \Pi, \sigma) = \epsilon(s, \Pi, \sigma) \cdot L(1 - s, \Pi^\vee, \sigma)$$

where

$$\epsilon(s, \Pi, \sigma) := \prod_w \epsilon(s, \Pi, \sigma, \psi_w)$$

if $\prod_w \psi_w$ defines a character of $\mathbb{A}_K/K$.

Example 3.1. Rankin-Selberg, Godement-Jacquet, etc.

Proposition 3.10. *Suppose there is a theory of automorphic L-functions for G over F and σ. Then for any $\pi \in \Pi(G/F)$ and $\psi : F \to \mathbb{C}^\times$.*

$$L(s, \pi, \sigma) = L(s, \mathcal{L}(\pi), \sigma); \epsilon(s, \pi, \sigma, \psi) = \epsilon(s, \mathcal{L}(\pi), \sigma, \psi).$$

Another important property is the *endoscopic character identities*. It predicts the relationshops between the L-packets of G and its endoscopic groups. One can find more details in [Kal16, §1, §2.3]. We will discuss more in Section 5

4. Local Langlands correspondence for GL_2

Local Langlands correspondence has been solved for GL_n by Harris-Taylor and Henniart. In this section, we will concentrate on the first non-abelian case, GL_2, and see how LLC works explicitly in this case following [BH06]. In this section we assume p to be odd.

4.1. Irreducible admissible representations of GL_2.

4.1.1. *Non-supercuspidal representations.* Up to conjugacy, $G = \mathrm{GL}_2$ has only one proper parabolic, $B = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$ with Levi $T = \left(\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right)$ and unipotent radical $U = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$. Moreover, we know that any irreducible representations of T has the form $\left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$, with χ_1 and χ_2 admissible characters of $F^\times$. So by Proposition 1.2, any non-supercuspidal irreducible representation is a constituent of $\mathrm{Ind}_T^G \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$. There are three cases.

- *Principal series (infinite dimensional).* If $\chi_1/\chi_2 \neq |\cdot|$ or $|\cdot|^{-1}$, then $\mathrm{Ind}_T^G \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ is an admissible irreducible infinite-dimensional representation.
- *Principal series (one-dimensional).* If $\chi_1/\chi_2 = |\cdot|$ or $|\cdot|^{-1}$, $\mathrm{Ind}_T^G \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ has length 2 and fits into exact sequences
$$1 \to \chi \circ \det \to \mathrm{Ind}_B^G \begin{pmatrix} \chi|\cdot|^{-\frac{1}{2}} & \\ & \chi|\cdot|^{\frac{1}{2}} \end{pmatrix} \to \mathrm{St}(\chi) \to 1$$
or
$$1 \to \mathrm{St}(\chi) \to \mathrm{Ind}_B^G \begin{pmatrix} \chi|\cdot|^{\frac{1}{2}} & \\ & \chi|\cdot|^{-\frac{1}{2}} \end{pmatrix} \to \chi \circ \det \to 1$$
The irreducible admissible character $\chi \circ \det$ is called the one-dimensional principal series.
- *Steinberg representations.* For each admissible $\chi : F^\times \to \mathbb{C}^\times$, we can obtain an admissible irreducible infinite-dimensional $\mathrm{St}(\chi)$, called the Steinberg representation.

4.1.2. *Supercuspidal representations.* By Proposition 1.2, these representations cannot be obtained by parabolic induction. However, the idea is very close. Roughly speaking, we obtain the non-supercuspidal representations in the following way:

(1) Start with a character of a maximal torus (a split one).
(2) Extend the character to a larger group (the Borel subgroup)
(3) Induce the character to obatin of representation of $G(F)$.

The basic idea to construct a supercuspidal cuspidal representation is to induce from a character of a "compact torus".

(1) Start with a character of a maximal torus (a compact one).
(2) Extend the character as much as possible (by looking at the stabilizer of the character).
(3) (Compactly) induce the character to obtain a representation of $G(F)$.

Here are the details.

Definition. Let H be a closed subgroup of $G(F)$. Let (τ, W) be a smooth representation of H. We define the *compactly induced representation* $\mathrm{cInd}_H^G \tau$ to be space of functions $f : G(F) \to W$ such that

(1) $f(hg) = \tau(h) f(g)$, $h \in H, g \in G(F)$,
(2) there exists a compact open subgroup $K \subseteq G(F)$ such that $f(gk) = f(g)$ for all $g \in G(F)$, $k \in K$, and
(3) f is compactly supported modulo H.

G acts on $\mathrm{cInd}\,\tau$ be right translation, i.e., $(g \cdot f)(g') = f(g'g)$.

Set $K = \mathrm{GL}(\mathcal{O}_F) \subseteq \mathrm{GL}_2(F)$ be the maximal compact open subgroup, and Z the center of $\mathrm{GL}_2(F)$.

Let E/F be a quadartic field extension (since $p > 2$, E/F is tamely ramified). Fixing an F-basis for E we get an identification

$$E \cong F \oplus F, \quad \mathrm{Aut}_F(E) \cong \mathrm{GL}_2(F).$$

$E^\times$ acts on E by multiplication, which gives an embedding

$$E^\times \hookrightarrow \mathrm{GL}_2(F)$$

which is unique up to conjugacy. We can normalize it so $\mathcal{O}_E \hookrightarrow \mathrm{GL}_2(\mathcal{O}_F)$.

Example 4.1. If $E = F(\sqrt{\varpi})$, then we can take $a + b\sqrt{\varpi} \in E^\times \mapsto \left(\begin{smallmatrix} a & b \\ b\varpi & a \end{smallmatrix}\right)$.

Let E/F be quadratic as above and let $\chi : E^\times \to \mathbb{C}^\times$ a smooth character.

Definition. The **level** of χ is the smallest positive integer $n \geq 0$ such that $\chi(1 + \mathfrak{p}_E^{n+1}) = 1$.

Let $\mathrm{Gal}(E/F) = \{1, \sigma\}$, and define

$$\begin{aligned} \chi^\sigma : E^\times &\to \mathbb{C}^\times \\ x &\mapsto \chi(\sigma(x)). \end{aligned}$$

Definition. A pair $(E/F, \chi)$ is called *admissible* if

(1) χ does not factor as $\chi = \psi \circ N_{E/F}$, where $N_{E/F} : E^\times \to F^\times$ is the norm map, and $\phi : F^\times \to \mathbb{C}^\times$ is a smooth character, and
(2) if $\chi|_{1+\mathfrak{p}_E}$ does factor through $N_{E/F}$, then E/F is unramified.

Definition. Let E/F be a quadratic extension as above.

(1) We say $(E/F, \chi)$ is equivalent to $(E/F, \chi')$ if $\chi = \chi'$ or χ^σ.
(2) The pair $(E/F, \chi)$ is **minimal** if χ is of level n and $\chi|_{1+\mathfrak{p}_E^n}$ does not factor through $N_{E/F}$.

Our goal is to attach $(E/F, \chi)$ a irreducible, supercuspidal representation.

(1) Suppose first that $(E/F, \chi)$ is minimal.
   (a) Level of χ is 0.
   By definition, E/F is unramified and $\chi|_{\mathcal{O}_E^\times}$ is inflated from a regular (i.e., not equal to its Galois conjugate) character of $k_E^\times$. So we get a cuspidal representation W_χ of $\mathrm{GL}_2(k_F)$ (from the Weil representations). Extend this to KZ by letting $\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)$ act by the scalar $\chi(\varpi)$. Set $\mathrm{SC}(E/F, \chi) := \mathrm{cInd}_{KZ}^G(W_\chi)$.
   (b) Level of χ is $2m + 1 \geq 1$.
      - If E/F is unramified, set

$$J_{m+1} = K_{m+1} = \begin{pmatrix} 1 + \mathfrak{p}^{m+1} & \mathfrak{p}^{m+1} \\ \mathfrak{p}^{m+1} & 1 + \mathfrak{p}^{m+1} \end{pmatrix}$$

      - If E/F is ramified, set

$$J_{m+1} = \begin{pmatrix} 1 + \mathfrak{p}^{\lfloor \frac{m}{2} \rfloor + 1} & \mathfrak{p}^{\lfloor \frac{m+1}{2} \rfloor} \\ \mathfrak{p}^{\lfloor \frac{m+1}{2} \rfloor + 1} & 1 + \mathfrak{p}^{\lfloor \frac{m}{2} \rfloor + 1} \end{pmatrix}$$

   Then the character $\chi : E^\times \to \mathbb{C}^\times$ extends to a character

$$\Lambda : E^\times \cdot J_{m+1} \to \mathbb{C}^\times.$$

   Set $\mathrm{SC}(E/F, \chi) = \mathrm{cInd}_{E^\times J_{m+1}}^G(\Lambda)$.
   (a) Level of χ is $2m \geq 2$.
   By minimality, eveness implies that E/F must be unramified. So there exists a representation Λ of the group $E^\times K_m$ which is "determined by χ" with the properties

- $\Lambda|_{F^\times} = \chi|_{F^\times}^{\oplus q}$
- $\mathrm{tr}(\Lambda(\zeta)) = -\chi(\zeta)$, for $\zeta \in k_E^\times \backslash k_F^\times$
- $\cdots$

Set $\mathrm{SC}(E/F,\chi) = \mathrm{cInd}_{E^\times K_m}^G(\Lambda)$.

(2) For an arbitrary admissible pair $(E/F,\chi)$, we can write $\chi = \chi' \cdot (\psi \circ N_{E/F})$ where $(E/F,\chi')$ is minimal and $\psi : F^\times \to \mathbb{C}^\times$. Set $\mathrm{SC}(E/F,\chi) := \mathrm{SC}(E/F,\chi') \otimes (\psi \circ \det)$.

Theorem 4.1. *For $p > 2$, the map $(E/F,\chi) \mapsto \mathrm{SC}(E/F,\chi)$ gives a bijection between equivalence classes of admissible pairs and isomorphism classes of irreducible supercuspidal representations of $\mathrm{GL}_2(F)$.*

4.2. **Weil-Deligne representations of** GL_2**.** By definition, the Weil-Deligne representations of GL_2 is an admissible homomorphism

$$\varphi : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{GL}_2 = \mathcal{W}_F \ltimes \mathrm{GL}_2(\mathbb{C}).$$

Since GL_2 is split over F, the $\mathcal{W}_F$-action is trivial. Therefore, it suffices to find the equivalent classes of admissible homomorphisms

$$\varphi : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C}).$$

4.3. **Local factors.**

4.3.1. *The Godement-Jacquet local constants.* The local factors on the automorphic side comes from a theory of automorphic L-functions called Godement-Jacquet theory. Instead of defining these automorphic L-functions in details, we give an explicit description of the local L-factors and ϵ-factors. Note that in this case, we can simply choose the algebraic representation $\sigma : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(V)$ to be the identity map.

Definition. For irreducible admissible representations π of $\mathrm{GL}_2(F)$, we define the *local L-factor* $L(s,\pi)$ as follows.

(1) For irreducible principal series, realized as the irreducible quotient of $\mathrm{Ind}_B^G\begin{pmatrix}\chi_1 & \\ & \chi_2\end{pmatrix}$, we set

$$L(s,\pi) = L(s,\chi_1)L(s,\chi_2) = \frac{1}{(1-\alpha_1 q^{-s})(1-\alpha_2 q^{-s})},$$

where $\alpha_i = \chi_i(\varpi)$ if χ_i is unramfied, and $\alpha_i = 0$ otherwise.

(2) For Steinberg representation $\mathrm{St}(\chi)$, realized as the irreducible quotient of $\mathrm{Ind}_B^G\begin{pmatrix}\chi|\cdot|^{-\frac{1}{2}} & \\ & \chi|\cdot|^{\frac{1}{2}}\end{pmatrix}$, we set

$$L(s,\pi) = \frac{1}{1-\alpha q^{-s}},$$

where $\alpha = q^{-\frac{1}{2}}\chi(\varpi)$ if $\chi|\cdot|^{\frac{1}{2}}$ is unramfied, and $\alpha = 0$ otherwise.

(3) For π supercuspidal, we set

$$L(s,\pi) = 1.$$

Definition. For irreducible representations π of $\mathrm{GL}_2(F)$, we define the *local ϵ-factor* by

$$\epsilon(s,\pi,\psi) = \frac{\zeta(\frac{3}{2}-s,\hat{\Phi},f^\vee)}{\zeta(\frac{1}{2}+s,\Phi,f)} \cdot \frac{L(\pi,s)}{L(\pi^\vee,1-s)}.$$

where

- $\zeta(s,\Phi,f) = \int_{\mathrm{GL}_2(F)} \Phi(g)f(g)|\det g|^s dg$ is the *zeta integral* for some Haar measure dg on $\mathrm{GL}_2(F)$, f a matrix coefficient (Definition 1.3) of π, $\Phi \in C_c^\infty(M_2(F))$.
- $\hat{\Phi}(x) = \int_{M_2(F)} \Phi(y)(\psi\circ\mathrm{Tr})(xy)dy$ is the *Fourier transform* where dy is the unique self-dual Haar measure on $M_2(A)$ such that $\hat{\hat{\Phi}}(x) = \Phi(-x)$ holds for all $\Phi \in C_c^\infty(M_2(F))$.
- $f^\vee(g) = f(g^{-1})$ is a matrix coefficient of $\pi^\vee$.

4.3.2. *The Langlands-Deligne local constants.* We have already defined these local factors for Weil-Deligne representations (see Definition 3.3.1 and Definition 3.3.1). Again, here we simply choose the algebraic representation $\sigma : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2$.

Instead of giving an explicit formula for them, here we present serveral properties of the L-factor.

Proposition 4.1. *Let φ be a L-parameter of GL_2.*

(1) If φ is irreducible, then

$$L(s, \varphi) = 1.$$

(2) If $\varphi = \varphi_1 \oplus \varphi_2$ is semisimple (so φ_i are characters of $\mathcal{W}_F$, or equivalently $\mathcal{W}_F^{ab} = F^\times$), then

$$L(s, \varphi_1 \oplus \varphi_2) = L(s, \varphi_1)L(s, \varphi_2),$$

where $L(s, \varphi_i)$'s are the local L-factors given in Tate's thesis.

Remark. Recall from Definition 4.3.1 that we also have

(1) If π is supercuspidal, then $L(s, \pi) = 1$, and
(2) $L(s, \mathrm{Ind}_B^G \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}) = L(s, \chi_1)L(s, \chi_2)$.

Comparing with Proposition 4.1(1)(2), if we believe that local Langlands correspondence matches the local factors, it's natural to make the following guess.

(1) The supercuspidal representations correspond to irreducible Weil-Deligne representations, and
(2) $\mathrm{Ind}_B^G \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$ or its irreducible quotient correspondes to $\chi_1 \oplus \chi_2$.

4.4. **Local Langlands correspondnence.** Now we can write down the local Langlands coorespondence $\mathcal{L} : \Pi(\mathrm{GL}_2 / F) \to \mathrm{LP}(\mathrm{GL}_2 / F)$ explicitly.

- For the 1-dimensional principal series, $\pi = \chi \circ \det$, realized as the irreducible quotient of $\mathrm{Ind}_B^G \begin{pmatrix} \chi|\cdot|^{\frac{1}{2}} & \\ & \chi|\cdot|^{-\frac{1}{2}} \end{pmatrix}$, we have
$$\mathcal{L}(\pi) = (\chi|\cdot|^{\frac{1}{2}} \oplus \chi|\cdot|^{-\frac{1}{2}}) \times \mathrm{triv} : \mathcal{W}_F \times \mathrm{SL}_2 \to \mathrm{GL}_2(\mathbb{C}).$$
- For the infinite dimensional principal series, $\pi = \mathrm{Ind}_B^G \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$, we have
$$\mathcal{L}(\pi) = (\chi_1 \oplus \chi_2) \times \mathrm{triv} : \mathcal{W}_F \times \mathrm{SL}_2 \to \mathrm{GL}_2(\mathbb{C})$$
Here $\chi_1/\chi_2 \neq |\cdot|$ or $|\cdot|^{-1}$.
- To the Steinberg representation, $\pi = \mathrm{St}(\chi)$, realized as the irreducible quotient of $\mathrm{Ind}_B^G \begin{pmatrix} \chi|\cdot|^{-\frac{1}{2}} & \\ & \chi|\cdot|^{\frac{1}{2}} \end{pmatrix}$, we have
$$\mathcal{L}(\pi) = (\chi \oplus \chi) \times i : \mathcal{W}_F \times \mathrm{SL}_2 \to \mathrm{GL}_2(\mathbb{C})$$
where $i : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ is the natural embedding. Notice that in this case $\mathcal{L}(\pi)(\mathcal{W}_F) \subseteq Z$ is indeed centralized $\mathcal{L}(\pi)(\mathrm{SL}_2)$ even though the latter is not trivial as in the other cases.
- For the supercuspidal representation, $\pi = \mathrm{SC}(E/F, \chi)$, we have
$$\mathcal{L}(\pi) : \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\chi) \times \mathrm{triv} : \mathcal{W}_F \times \mathrm{SL}_2 \to \mathrm{GL}_2(\mathbb{C}).$$
In fact, $\mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \chi$ is irreducible.

Remark. (1) To summarize, principal series correspond to semisimple reducible parameters with trivial SL_2 action; Steinberg representations correspond to semisimple reducible parameters with non-trivial SL_2 action; supercuspidal representations correspond to irreducible parameters.

(2) We can also tell which of these representations have tempered parameters (i.e., the image of $\mathcal{W}_F \times \mathrm{SU}_2$ is bounded). They are infinite dimensional principal series $\mathrm{Ind}_B^G \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$ with χ_1, χ_2 unitary, Steinberg representations $\mathrm{St}(\chi) = \mathrm{St} \otimes \chi$ with χ unitary and supercuspidal representations. It is known that these representations turn out to be the tempered representations of $\mathrm{GL}_2(F)$ (Definition 1.3).
(3) Following Property 3.9, we can also make a guess about which representations are in (essentially) discrete series. We expect the image of $\mathcal{L}(\pi)$ to be contained in no proper parabolics. It turns out that these representations are the Steinberg representations and supercuspidal representations. The principal series, on the other hand, always have L-parameters with image lying in the diagonal torus and trivial SL_2-action, which justify Property 3.8. Also notice that the Steinberg representations are irreducible subquotient of $\mathrm{Ind}_T^G \tau$, $\mathcal{L}^{\mathrm{ss}}(\pi)$ lies in the diagonal torus while $\mathcal{L}(\pi)$ does not. Also, if we choose $\tau = \begin{pmatrix} \chi|\cdot|^{\frac{1}{2}} & \\ & \chi|\cdot|^{-\frac{1}{2}} \end{pmatrix}$, then $\mathrm{Ind}_T^G(\tau)$ has two irreducible subquotients $\chi \circ \det$ and $\mathrm{St}(\chi)$, even though $\mathcal{L}(\chi \circ \det) \neq \mathcal{L}(\mathrm{St}(\chi))$, we have $\mathcal{L}^{\mathrm{ss}}(\chi \circ \det) = \mathcal{L}^{\mathrm{ss}}(\mathrm{St}(\chi)) = \chi|\cdot|^{\frac{1}{2}} \oplus \chi|\cdot|^{-\frac{1}{2}}$. This is a good illustration of Property 3.8 (both its vaildity and the importance of the usage of $\mathcal{L}^{\mathrm{ss}}$ instead of $\mathcal{L}$),

Theorem 4.2 (Local Langlands correspondence for $\mathrm{GL}_2(F)$). *The map $\mathcal{L} : \Pi(\mathrm{GL}_2 / F) \to \mathrm{LP}(\mathrm{GL}_2 / F)$ given by above is a bijection such that*

$$L(s, (\chi \circ \det)\pi) = L(s, \mathcal{L}(\pi) \otimes \chi)$$
$$\epsilon(s, (\chi \circ \det)\pi, \psi) = \epsilon(s, \mathcal{L}(\pi) \otimes \chi, \psi)$$

for all $\pi \in \Pi(\mathrm{GL}_2 / F)$, all smooth characters $\chi : F^\times \to \mathbb{C}^\times$ and all smooth additive characters $\psi : F \to \mathbb{C}^\times$.

5. Endoscopy

Let G be a quasi-split connected reductive group over a local field F, and $\Gamma = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group.

The theory of endoscopy starts with the functorial aspect of the Langlands program. Let's consider another quasi-split connected reductive group H, together with an admissible embedding ${}^L\eta : L^H \to L^G$. With this embedding, there is a way to transfer a L-parameter of H to one of G: for $\varphi^H : WD_F \to {}^LG$, we simply get $\varphi = {}^L\eta \circ \varphi^H : WD_F \to {}^LH \to {}^LG$. The Langlands functoriality conjecture expects such transfer to be compatible with a "good" tranfer of representations on the automorphic side. In particular, one may wanna ask, how are the two L-packets related? There is an expected answer called endoscopic character identity. To introduce this conjectured property, we need the theory of endoscopy.

The study of Langlands functoriality often leads to correspondence that are defined up to stable conjugacy. The theory of endoscopy is a series of technique invented to investigate rational conjugacy from the inside of stable conjugacy classes, justifying its name (it originally means a procedure in which an instrument is introduced into the body to give a view of its internal parts.)

5.1. Group cohomology.

Definition. We define the functor $H^i(G, -) : G\text{-Mod} \to \mathrm{Ab}$ to be the i-th right derived functor of the (left exact) G-invariance functor $G\text{-Mod} \to \mathrm{Ab}$ given by $M \mapsto M^G$.

Low dimensional cohomology can be generalized to non-abelian groups.

Definition. Let X be a group together with an action of G compatible with its group structure. We can define $H^0(G, X) = X^G$ (which is a group) and $H^i(G, X) = Z^1(G, X)/\sim$ where

- $Z^1(G, X)$ consists of *1-cocycles*, i.e., maps $a : G \to X$ such that $a(g_1 g_2) = a(g_1)a(g_2)^{g_1}$;
- $a \sim a'$ if $a'(g) = xa(g)(x^g)^{-1}$.

Note that this conincides with Definition 5.1 when X is abelian.

5.2. **Stable conjugacy.**

Definition. Two elements $\sigma, \sigma' \in G(F)$ are *stably conjugate* if there exists $g \in G(\bar{F})$ such that

(1) $g\delta g^{-1} = \delta'$, and
(2) for every $\sigma \in \Gamma$ the element $g^{-1}\sigma(g)$ belongs to $G^\circ_{\sigma_s}$, the connected component of the centralizer the semisimple part of δ.

Remark. Note that we automatically have $g\delta_s g^{-s} = \delta'_s$ by functoriality of Jordan decomposition, and therefore $g^{-1}\sigma(g) \in G_{\delta_s}$. So stable conjugacy coincide with $G(\bar{F})$-conjugacy for elements whose semisimple parts have connected centralizers. Moreover, by a theorem of Steinberg, if G is simply connected, G_{δ_s} is connected. So for simply-connected groupo stable conjugacy is the same as $\bar{F}$-conjugacy.

Example 5.1. In $G = \mathrm{SL}_2$ over $\mathbb{R}$ (which is simply-connected),

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are stably conjugate (namely by $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$), but not conjugate by $\mathrm{SL}_2(\mathbb{R})$.

Definition. An element $\sigma \in G(F)$ is *regular semisimple* (resp. *strongly regular semisimple*) if G°_δ (resp. G_δ) is a torus.

Proposition 5.1. *Let $t \in G(F)$ be a semisimple element, so $t \in T$ for some maximal torus T of G.*

(1) t is regular iff t is not in the kernel of any absolute associated to T, i.e., the eigenvalues of t are distinct.
(2) t is strongly regular iff $\mathrm{Stab}_{\Phi(T)}(t)$ is trivial.

Remark. It follows immediately that

(1) In the strongly regular semisimple case, G_δ is in fact a maximal torus, which will be denoted by T_δ.
(2) Both regulariry and Strong regularity is a Zariski dense open condition.

Example 5.2. Here is an example of regular but not strongly regular semisimple element. Let $G = \mathrm{PGL}_2$ over $\mathbb{C}$. Consider

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly it has distinct eigenvalues ± 1, but the Weyl element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ takes

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = g$$

So by Proposition 5.1, g is regular but not strongly regular. Indeed,

$$G_g = \left\{ \begin{pmatrix} * & \\ & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} & * \\ 1 & \end{pmatrix} \right\} \cong \mathbb{Z}_2 \ltimes \mathbb{G}_m$$

is not a torus but has a tours as its connected component. If one think of $G = \mathrm{PGL}_2$ as the automorphism group of $\mathbb{C}P^1$ given by Möbius transform, G_δ consists of those preserving $\{0, \infty\}$, i.e., $z \mapsto \alpha z$ or $z \mapsto \frac{\alpha}{z}$.

Proposition 5.2. *Let $\delta, \delta' \in G(F)$ be strongly regular semisimple elements with centralizers $T_\delta, T_{\delta'}$. For any g such that $\mathrm{Ad}(g)(\delta) = \delta'$, we have isomorphism $\mathrm{Ad}(g) : T_\delta \to T_{\delta'}$.*

(1) *The isomorphism* $Ad(g) : T_\delta \to T_{\delta'}$ *only depends on* δ, δ' *but not* g. *It is defined over* F. *We shall call in* $\varphi_{\delta,\delta'}$.
(2) $\sigma \mapsto g^{-1}\sigma(g)$ *belongs to* $Z^1(\Gamma, T_\delta)$ *and its cohomology class is independent of the choice of* g. *We shall call it* $\mathrm{inv}(\delta, \delta')$.

5.3. **Endoscopic data.**

Definition. An *endoscopic triple* (H, s, η) consists of

(1) a quasi-split connected reductive group H,
(2) an embedding $\eta : \hat{H} \to \hat{G}$ of complex algebraic groups;
(3) an element $s \in (Z(\hat{H})/Z(\hat{G}))^\Gamma$;

such that

(1) η identify $\hat{H}$ with $\mathrm{Cent}(\eta(s), \hat{G})^\circ$;
(2) the $\hat{G}$-conjugacy class of η is stable under the action of Γ that is defined by $\eta^\sigma = \sigma_{\hat{G}} \circ \eta \circ \sigma_{\hat{H}}^{-1}$;
(3) s lifts to $Z(\hat{H})^\Gamma$.

An isomorphism of endoscopic triples $(H_1, s_1, \eta_1) \to (H_2, s_2, \eta_2)$ is an isomorphism of algebraic groups $f : H_1 \to H_2$ defined over F subject to the conditions

(1) $\eta_1 \circ \hat{f}$ and η_2 are $\hat{G}$-conjugate;
(2) the images of $\hat{f}(s_2)$ and s_1 in $\pi_0((Z(\hat{H}_1)/Z(\hat{G}))^\Gamma)$ coincide.

Definition. An endoscopy triple (H, s, η) is called *elliptic* if $Z(\hat{H})^{\Gamma,\circ} = Z(\hat{G})^{\Gamma,\circ}$.

Remark. If H is endoscopic in G (i.e., $\hat{H}$ is a centralizer in $\hat{G}$), one may wanna ask if H is a subgroup of G. In many cases, $\hat{H}$ is a Levi in $\hat{G}$, so one can realize H as the dual Levi in G. But in general this is false. Here are some examples.

(1) Consider $\hat{H} = \mathrm{SL}_2 \times \mathrm{SL}_2 \subseteq \hat{G} = \mathrm{Sp}_4$ given by decomposing a 4-dimsional symplectic space as the direct sum of two 2-dimensional ones. Apparently $\hat{H}$ is the centralizer of a product of (different) central elements in SL_2. However, there is no embedding of $H = \mathrm{PGL}_2 \times \mathrm{PGL}_2$ in $G = \mathrm{SO}_5$.
(2) More generally we have $H = SO_{2p+1} \times SO_{2q+1}$ endoscopic in $G = \mathrm{SO}_{2p+2q+1}$ with $\hat{H} = \mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q} \subseteq \hat{G} = \mathrm{Sp}_{2p+2q}$.
(3) Another example is $H = \mathrm{SO}_{2p} \times \mathrm{Sp}_{2q}$ endoscopic in $G = \mathrm{Sp}_{2p+2q}$ with $\hat{H} = \mathrm{SO}_{2p} \times \mathrm{SO}_{2q+1} \subseteq \hat{G} = \mathrm{SO}_{2p+2q+1}$.

Definition. A *Whittaker datum* for G is a $G(F)$-conjugacy class of pairs $w = (B, \psi)$, where B is a F-rational Borel subgroup of G, and $\psi : B_u(F) \to \mathbb{C}^\times$ is a generic character (whose restriction to each relative simple root subgroup is non-trivial).

Definition. An *extended endoscopic triple* $e = (H, s, {}^L\eta)$ consists of an endoscopic triple and an extension of $\eta : \hat{H} \to \hat{G}$ to an embedding of L-groups ${}^L\eta : {}^LH \to {}^LG$.

5.4. **Admissible isomorphisms.** Let (H, s, η) be an endoscopic triple. Let $T^H \subseteq H$ and $T \subseteq T$ be maximal tori.

Definition. An isomorphism $i : T^H \to T$ is *admissible* if the following diagram commutes (up to conjugacy)

$$\begin{array}{ccc} \hat{T}^H & \xleftarrow[\hat{i}]{\sim} & \hat{T} \\ \big\downarrow & & \big\downarrow \\ \hat{H} & \xhookrightarrow{\eta} & \hat{G} \end{array}.$$

Remark. This is a generalization of the isomorphism $\mathrm{Ad}(g) : T_\delta \to T_{\delta'}$ we have in Proposition 5.2.

Definition. Two strongly regular semisimple elements $\gamma \in H(F)$ and $\delta \in G(F)$ are called *related* if there exists an admissible isomorphism $\varphi : T^H \to T$ such that $\varphi(\gamma) = \delta$.

Proposition 5.3. *If λ, δ are strongly regular then φ is unique if it exists and will be called $\varphi_{\gamma,\delta} : T_\gamma \to T_\delta$.*

5.5. **Transfer factors.** To make sure that the transfer function is smooth, we need to introduce auxiliary local term. These are called transfer factors.

Given extended endoscopic $e = (H, s, {}^L\eta)$ and a Whittaker datum $w = (B, \psi)$, there is a function

$$\Delta : H_{\text{sr}}(F) \times G_{\text{sr}}(F) \to \mathbb{C}$$

called the *Langlands-Shelstad transfer factor.* If $\gamma \in H(F)$ and $\delta \in G(F)$ are not related, $\Delta(\gamma, \delta) = 0$. Now we assume γ and δ are related, so by Proposition 5.3, there exists an admissible isomorphism $\varphi_{\gamma,\delta} : T_\gamma \to T_\delta$ that maps γ to δ. In this case, $\Delta(\gamma.\delta)$ is a product

$$\epsilon \cdot \Delta_1^{-1} \cdot \Delta_2 \cdot \Delta_3 \cdot \Delta_4. \tag{5.1}$$

5.5.1. *Auxiliary data.* To construct the local transfer factor, we need some auxiliary data.

Definition. Let $\alpha \in R(T_\sigma, G)$. Define $\Gamma_\alpha = \text{Stab}_\Gamma(\alpha)$, $\Gamma_{\pm\alpha} = \text{Stab}_\Gamma(\{\pm\alpha\})$, $F_\alpha = \bar{F}^{\Gamma_\alpha}, \bar{F}_{\pm\alpha} = \bar{F}^{\Gamma_{\pm\alpha}}$. Call α *symmetric* if $[F_\alpha : F_{\pm\alpha}] = 2$ and otherwise *asymmetric.*

Definition. A set of *a-data* for $R(T_{\delta,G})$ is a set $\{\alpha_a \in \bar{F}^\times \mid \alpha \in R(T_{\delta,G})\}$ such that

(1) $a_{\sigma\delta} = \delta(a_\alpha)$ for all $\sigma \in \Gamma$;
(2) $a_{-\alpha} = -a_\alpha$.

Definition. A set of *χ-data* for $R(T_\delta, G)$ is a set $\{\chi_\alpha \mid \alpha \in R(T_\delta, G)\}$ such that

(1) $\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ is a continuous character;
(2) $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$;
(3) $\chi_{-\alpha} = \chi_\alpha^{-1}$;
(4) if α is symmetric, then $\chi_\alpha | F_{\pm\alpha}^\times$ is the quadratic character $F_{\pm\alpha}^\times \to \{\pm 1\}$ associated to the quadratic extension $F_\alpha / F_{\pm\alpha}$ by local class field theory.

5.5.2. *Tate-Nakayama duality.* The original statement ot Tate-Nakayama is that for any torus defined over F, the pairing

$$H^1(\Gamma, T) \otimes H^1(T, X^*(T)) \to H^2(\Gamma, \mathbb{G}_m) \xrightarrow{\text{CFT}} \mathbb{Q}/\mathbb{Z} \tag{5.2}$$

is perfect.

Langlands provided a reinterpretation of (5.2). Tensoring with $X^*(T)$ the exponential sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \to 1$$

and using $X^*(T) = X_*(\hat{T})$, we get an exact sequence

$$0 \to X_*(\hat{T}) \to \text{Lie}(\hat{T}) \xrightarrow{\exp} \hat{T} \to 1.$$

Take the long exact sequence of group cohomology (associated to Γ-invariance), we get isomorphism

$$\pi_0(\hat{T}^\Gamma) = \text{cok}(\text{Lie} \xrightarrow{exp} \hat{T}^\Gamma) \to H^1(\Gamma, X^*(T)).$$

We use again the exponential map to obtain the embedding $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$, and so a perfect pairing

$$H^1(\Gamma, T) \otimes \pi_0(\hat{T}^\Gamma) \to \mathbb{C}^\times. \tag{5.3}$$

Now we can describe the factors in (5.1). Fix a pinning $(T_0, B_0, \{X_\alpha\})$ of G and a non-trivial additive character $\Gamma : F \to \mathbb{C}^\times$.

5.5.3. ϵ. The complex number ϵ is given by

$$\epsilon = \epsilon(\frac{1}{2}, X_*(T_0)_{\mathbb{C}} - X_*(T_0^H), \Gamma)$$

where $T_0^H \subseteq H$ is a maximally split maximal tori.

5.5.4. Δ_1. The complex number Δ_1 is given by

$$\Delta_1 = \langle \lambda, \hat{\varphi}_{\gamma,\delta}(s) \rangle.$$

Here $\langle -, - \rangle$ is given by Tate-Nakayama duality (5.3), $\hat{\varphi}_{\gamma,\delta}(s) \in [\hat{T}_\delta / Z(\hat{G})]^\Gamma$ viewed as an element in $\pi_0(T_{\delta,\mathrm{sc}}^\Gamma)$. λ is the *Langlands-Shelstad splitting invariant*, constructed as follows. Choose $g \in G_{\mathrm{sc}}(\bar{F})$ such that $gT_0g = T_\delta$, so that $g^{-1}\delta(g) \in Z^1(F, N(T_0, G_{\mathrm{sc}}))$. Let w_σ be its image in $N(T_0, \Omega(T_0))$, where $\Omega(T_0)$ is the Weyl group of T_0. Let $\dot{w}_\sigma \in N(T_0, G_{\mathrm{sc}})$ be the Tits lift of w_σ associated to the fixed pinning. However, the Tits section $\Omega(T_0) \to N(T_0, G_{\mathrm{sc}})$ is not multiplicative in general, $\sigma \mapsto \hat{w}_\sigma$ is generally not a 1-cocycle. To fix this, define

$$x_\sigma := \prod_{\substack{\alpha > 0 \\ \sigma^{-1}\alpha < 0}} \alpha^\vee(a_\alpha) \in T_{\delta,\mathrm{sc}}(\bar{F})$$

where positivity is associated to gB_0g^{-1}. Then $g^{-1}x_\sigma g \cdot \dot{w}_\sigma$ is another element of $Z^1(F, N(T_0, G_{\mathrm{sc}}))$, whose image in $Z^1(F, \Omega(T_0))$ conincides with that of $g^{-1}\sigma(g)$. Therefore the product $(g^{-1}x_\sigma g \cdot \dot{w}_\sigma) \cdot (g^{-1}\sigma(g))^{-1}$ takes values in $T_{0,\mathrm{sc}}(\bar{F})$ and moreover its image under $\mathrm{Ad}(g) : T_{0,\mathrm{sc}} \to T_{\delta,\mathrm{sc}}$, i.e., the element $x_\sigma g \dot{w}_\sigma \sigma(g)^{-1}$, belongs to $Z^1(F, T_{\sigma,\mathrm{sc}})$. Its class is independent of g will be denoted by λ.

5.5.5. Δ_2. The complex number Δ_2 is given by

$$\Delta_2 = \prod_\alpha \chi_\alpha \left(\frac{a(\sigma) - 1}{a_\alpha} \right)$$

where the product is taken over the Γ-orbits in $R(T_\sigma, G) \backslash R(T_\gamma, H)$. Here we identify $R(T_\gamma, H)$ with a subset of $R(T_\sigma, G)$ using isomorphism $\varphi_{\gamma,\delta} : T_\gamma \to T_\delta$.

5.5.6. Δ_3. $\Delta_3 = \theta(\delta)$, where $\theta : T_\delta(F) \to \mathbb{C}^\times$ is a character constructed as follows. The chosen χ-data for $R(T_\delta, G)$ leads to a $\hat{G}$-conjugacy class of L-embeddings ${}^LT_\delta \to {}^LG$. We can use $\varphi : T_\gamma \to T$ to transport the choice to $R(T_\delta, H)$ obtaining an $\hat{H}$-conjugacy class of L-embeddings ${}^LT_\gamma \to {}^LH$. These are the vertical arrows in the diagram

$$\begin{array}{ccc} {}^LT_\gamma & \xrightarrow{{}^L\varphi_{\gamma,\delta}} & {}^LT_\delta \\ \big\downarrow & & \big\downarrow \\ {}^lH & \xhookrightarrow{{}^L\eta} & {}^LG \end{array} .$$

In general the diagram fails to commute, and the failure is measured by an element in $H^1(\mathcal{W}_F, \hat{T}_\delta)$, which by the local correspondence for tori gives a character $\theta : T_\delta(F) \to \mathbb{C}^\times$.

5.5.7. Δ_4. Δ_4 is defined by

$$\Delta_4 = \prod_\alpha |\alpha(\delta) - 1|^{\frac{1}{2}},$$

where the product is over $R(T_\delta, G) \backslash R(T_\gamma, H)$.

5.5.8. *κ-behaviour.* The following is a fundamental property of the local factor, which we may call its *κ-behaviour*:

Proposition 5.4. *If δ, δ' are stably conjugate, then*

$$\Delta(\gamma, \delta') = \Delta(\gamma, \delta) \cdot \langle \mathrm{inv}(\delta, \delta'), \hat{\varphi}_{\gamma,\delta}^{-1}(s) \rangle$$

where $\mathrm{inv}(\delta, \delta') \in H^1(\Gamma, T_\delta)$ *is defined in Proposition 5.2(2).*

We need to explain the second term $\langle \mathrm{inv}(\delta, \delta'), \hat{\varphi}_{\gamma,\delta}(s) \rangle$ on the right hand side. This is given by the Tate-Nakayama duality (5.3), but it's a little bit tricky. Here, we think of $\hat{\varphi}_{\gamma,\delta}(s)$ as an element in

$$[Z(\hat{H})/Z(\hat{G})]^\Gamma \hookrightarrow [\hat{T}_\gamma/Z(\hat{G})]^\Gamma \xrightarrow{\hat{\varphi}_{\gamma,\delta}^{-1}} [\hat{T}_\delta/Z(\hat{G})]^\Gamma$$

(notice that the first map depends on an embedding $\hat{T}_\gamma \hookrightarrow \hat{H} \xrightarrow{\eta} \hat{G}$ which is only well-defined up to $\hat{H}$-conjugacy, but fortunately quotient by the center doesn't remember the conjugacy). But the last torus is the adjoint form of $\hat{T}_\delta$, which is the dual of the simply-connected form of T_δ. So we need to lift $\mathrm{inv}(\delta, \delta')$ to the simply-connected cover. But we have different lifts. Will they give us different pairing? No! The reason is that s lifts to an element in $Z(\hat{H})^\Gamma$ as in Definition 5.3.

5.6. **Local transfer.**

5.6.1. *Orbital integral.*

Definition. For $f \in C_c^\infty(G(F))$ and a strongly regular semisimple element $\delta \in G(F)$, let $O_\delta(f)$ denote the *orbital integral*

$$O_\delta(f) = \int_{T_\delta(F)\backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dt}$$

where dg and dt are choices of Haar measure on $G(F)$ and $T_\delta(F)$.

Note that orbital integral only depends on the rational conjugacy class of δ.
We can also define a stable analog:

Definition. The *stable orbital integral* at δ is

$$SO_\delta(f) := \sum_{\delta'} O_{\delta'}(f),$$

where the sum is taken over (a set of representatives for) the set of rational conjugacy classes inside the stable conjugacy class of γ.

Definition. A G-conjugate invariant distribution I ($I(f) = I(f^g)$ where $f^g(x) = f(g^{-1}xg)$) is called *stably invariant* if $I(f) = 0$ for all $f \in C_c^\infty(G(F))$ which satisfy $SO_\delta(f) = 0$ for all strongly regular semisimple $\delta \in G(F)$.

5.6.2. *Matching.* Again work with a extended endoscopic triple $(H, s, {}^L\eta)$ and a Whittaker datum $w = (B, \psi)$. This defines a transfer factor $\Delta : H_{\mathrm{sr}}(F) \to G_{\mathrm{sr}}(F) \to \mathbb{C}$.

Definition. The functions $f \in C_c^\infty(G(F))$ and $f^H \in C_c(H(F))$ are Δ-*matching* if

$$SO_\gamma(f^H) = \sum_\delta \Delta(\gamma, \delta) O_\delta(f), \forall \gamma \in H_{\mathrm{sr}}(F),$$

where the sum runs over the set of rational conjugacy classes of strongly regular semisimple elements.

Remark. The sum on the right hand side is a finite sum, since γ is only related to finitely many δ.

Theorem 5.1 (fundamental Lemma)**.** *With the above setting,*

(1) *For every f there exists a matching f^H.*
(2) *If G and $(H, s, {}^L\eta)$ are unramified (i.e., admit integral models $\mathcal{G}, \mathcal{H}$ with generic fibers $\mathcal{G}_F = G, \mathcal{H}_F = H$), and f is the characteristic function of a hyperspecial maximal compact subgroup, then f^H can be taken as the characteristic function of a hyperspecial maximal compact subgroup, i.e.,*

$$f = 1_{G(\mathcal{O}_F)} \Rightarrow f^H = 1_{H(\mathcal{O}_F)}.$$

Remark. Theorem 5.1 is fundamental in the following sense. Recall that the goal of Langlands functoriality conjecture is to find the way to transfer representations between G and H. To do so, we have a powerful tool called trace formula.

The trace formula is an identity between

- the geometric side, orbital integrals;
- the spectral side, characters of representations.

One thing we can do is to compare trace forumlas for different groups. Theorem 5.1 tells us how to compare the geometric side, so we also know how to compare the spectral side. But these characters determines representations, indicating how to transfer representations.

5.6.3. *Character identity.* Now we can connect the theory of endoscopy with the local Langlands correspondence, and refine the local Langlands conjecture. Here ”Proposition” stands for conjectured property of the local Langlands correspondence.

Definition. Let $\varphi : WD_F \to {}^L G$ be a Langlands parameter, define $S_\varphi = \mathrm{Cent}(\varphi, \hat{G})$ and $\bar{S}_\varphi = S_\varphi / Z(\hat{G})^\Gamma$.

Proposition 5.5. *For a fixed choice of Whittaker datum $w = (B, \psi)$, there exists a map $\rho_w : \Pi_\varphi(G/F) \to \mathrm{Irr}(\pi_0(\bar{S}_\varphi))$, which is bijective when F is non-archimedean , and injective when F is archimedean.*

Proposition 5.6. *When φ is tempered, there is a unique (B, ψ)-generic constituent of $\Pi_\varphi(G/F)$ (π is (B,ψ)-generic if $\mathrm{Hom}_{B_u(F)}(\pi.\psi) \neq 0$), and it is mapped to the trivial representation by ρ_w.*

Proposition 5.7. *When φ is tempered, the distribution*

$$S\Theta_\varphi := \sum_{\pi \in \Pi_\varphi(G)} \mathrm{Tr}\, \rho_w(\pi)(1) \cdot \Theta_\pi$$

is stable, where $\Theta_\pi : C_c^\infty \to \mathbb{C}$ is the Harish-Chandra distribution given by

$$\Theta_\pi(f) = \mathrm{Tr}\, \pi(f) = \mathrm{Tr} \int_{G(F)} f(x)\pi(x)dx.$$

Proposition 5.8. *Let $e = (H, s, {}^L\eta)$ be an extended endoscopic triple, $\varphi^H : WD_F \to {}^L H$ be a tempered Langlands parameter, $\varphi = {}^L\eta \circ \varphi^H$. If f and f^H are $\Delta_{e,w}$-matching, then*

$$\sum_{\pi \in \Pi_\varphi(G)} \mathrm{Tr}\, \rho_w(\pi)(s) \cdot \Theta_\pi(f) = \sum_{\tau \in \Pi_{\varphi^H}(H)} \mathrm{Tr}\, \rho(\tau)(1) \cdot \Theta_\tau(f^H).$$

Remark. The left hand side depends on a choice of Whittaker datum (B, ψ) of G. So what's the Whittaker datum we are going to use for H?

The right hand side can be written as

$$\sum_{\tau \in \Pi_{\varphi^H}(H)} \dim \rho(\tau) \cdot \Theta_\tau(f^H).$$

Notice that different choice of Whittaker datum will just change $\rho(\tau)$ by a twist of a character, so the right hand side is in fact independent of the choice of a Whittaker datum, and we can pick any one we like.

References

[BH06] Colin J Bushnell and Guy Henniart. *The local Langlands conjecture for GL (2)*, volume 335. Springer Science & Business Media, 2006.

[Kal16] Tasho Kaletha. The local langlands conjectures for non-quasi-split groups. In *Families of automorphic forms and the trace formula*, pages 217–257. Springer, 2016.

[Tat79] John Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*, volume 2, pages 3–26, 1979.