Funct/ spectral approach to dynamics

Area	Functional analysis . . .	Dynamics / Geometry
Object	Operator	Map + Space
Characteristic	Spectrum ⟵ trace f-la ⟶	Periodic orbits / Mixing

Number theory

Dynamical zeta fns

Zeros

coincide w/ discrete part of the spectrum – Ruelle sp. resonances.

~~[illegible]~~ given as infinite sum over ~~periodic~~ primitive orbits

~~[illegible]~~

Applications:

- counting periodic orbits ⟶ via the spectrum
- dynamical zeta fn Σ…
  realize ↓ as spectral determinant
- mixing

But **Idea** spectrum depends on domain of operator. Space.

Idea: find some weird space where spec is (relatively) simple.!

$\subset$ anisotropic Sobolev spaces

Part I. Matrix case (Finite dimensional).

$\mathcal{L} = (\mathcal{L}_{ij}) : \mathbb{C}^N \to \mathbb{C}^N$

$\mathcal{L}_{ij} \in \mathbb{C}$

$\mathcal{L} = P J P^{-1}$

Jordan normal form

def $\{$eigenvalues of $\mathcal{L}\} = \{$e.v. of $J\}$ on diag. $= \{z_j, j = 1,..,N, |z_j| \geq |z_{j+1}|\}$ order

$\mathrel{\|}$

$\mathrm{Spec}(\mathcal{L})$

def Spec determinant (char. poly) of $\mathcal{L}$

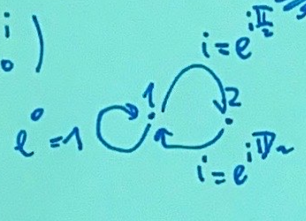
$$d(z) := \det(z \cdot \mathrm{Id} - \mathcal{L}) = \det(z \cdot \mathrm{Id} - J) = \prod_{j=1}^{N} (z - z_j)$$

$$\operatorname{tr} \mathcal{L}^n = \operatorname{tr} J^n = \sum_{j=1}^{n} z_j^n$$
$$\overset{\uparrow}{\mathcal{L}^n = P J^n P^{-1}}$$
trace is invariant

def if $\mathcal{L}_{ij} \neq 0$, $\mathcal{L}_{ij} = e^{V_{ij}}$ ↑ potential fn

associated graph $\mathcal{L}_{ji} \neq 0 \Rightarrow \exists$ oriented edge $i \to j$

ex $\mathcal{L} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$



def marked periodic orbit of period n starting at i_0
is $w := (i_0 i_1 \dots i_{n-1})$ s.t. $\mathcal{L}_{i_{k+1} i_k} \neq 0$ $k = 0, \dots, n-1$
$|w| = n$ period $i_n = i_0$

W_n - set of marked per. orbits of period n

$W := \bigcup_{n \geq 1} W_n$

$V_w := \sum_{k=0}^{n-1} V_{i_{k+1} i_k}$ Birkhoff sum of V along w

def primitive periodic orbit of period m
is $\gamma := (i_0 i_1 \dots i_{m-1})$ s.t. $\mathcal{L}_{i_{k+1} i_k} \neq 0$, $k = 0, \dots, m-1$
(γ is identified w/ its circular permutation) $|\gamma| = m$

Γ set of prim. per. orbits

$V_\gamma := \sum_{k=0}^{m-1} V_{i_{k+1} i_k}$

def primitive per. orbit
no marked pt
no smaller periodic orbit
~~(no repetition)~~
is not shorter traversed more then once

ex
$N(1) = 1$ $\pi(1) = 1$ (1)
$N(2) = 3$ $\pi(2) = 1$ (12)
$N(3) = 4$ $\pi(3) = 1$ (112)
$N(4) = 7$ $\pi(4) = 1$ (1112)
$\pi(5) = 2$ (11112) and $(1\,1212)$

Rem $w = \underbrace{(\gamma \dots \gamma)}_{K \text{ times}}$

$|w| = n = K \cdot |\gamma|$

$e^{V_w} = e^{K V_\gamma}$

<u>Prop</u> (trace formula for matrices)

$$\sum_{w \in W_n} e^{V_w} = \sum_{j=1}^{N} z_j^n \qquad \text{and} \qquad \sum_{w \in W_n} e^{V_w} = \sum_{\gamma \in \Gamma} |\gamma| \sum_{k \geq 1} e^{k V_\gamma} \delta_{n = k|\gamma|}$$

Pf

$$\sum_{j=1}^{N} z_j^n = \operatorname{tr} h^n = \sum_{i_0, i_1, \dots, i_n = i_0} h_{i_n i_{n-1}} h_{i_{n-1} i_{n-2}} \cdots h_{i_1 i_0} =$$

$$= \sum_{i_0, \dots, i_n = i_0} e^{V_{i_n, i_{n-1}}} \cdot \ \dots \cdot e^{V_{i_1 i_0}} = \sum_{w \in W_n} e^{V_w} \qquad \square$$

<u>Prop</u> (Bowen-Lanford zeta fn for matrices)

$$\zeta_{BL}(z) := \exp\left(- \sum_{n \geq 1} \frac{1}{n z^n} \sum_{w \in W_n} e^{V_w} \right) \overset{!}{=} \prod_{\gamma \in \Gamma} \left(1 - e^{V_\gamma - |\gamma| \ln z} \right)$$

converges for $|z| > |z_1|$ (analytic non-zero for $|z| > |z_1|$)

$$\zeta_{BL}(z) = \underbrace{z^{-N} d(z)}_{\text{extension on } \mathbb{C} \setminus \{0\}}$$

zeros of ζ_{BL} are e.v. of h.

Pf $z \neq 0$

$$d(z) = \prod_{j=1}^{N} (z - z_j) = z^N \prod_{j=1}^{N} \left(1 - \frac{z_j}{z}\right) = z^N \exp\left(\log \prod_{j=1}^{N} \left(1 - \frac{z_j}{z}\right) \right) =$$

$$= z^N \exp\left(\sum_{j=1}^{N} \log\left(1 - \frac{z_j}{z}\right) \right) \underset{\uparrow}{=} z^N \exp\left(- \sum_{j=1}^{N} \sum_{n \geq 1} \frac{1}{n} \left(\frac{z_j}{z}\right)^n \right) =$$

$\left|\frac{z_j}{z}\right| < 1 \ \forall j$

$\log(1-x) = -\sum_{n \geq 1} \frac{x^n}{n}$

$$= z^N \exp\left(- \sum_{n \geq 1} \frac{1}{n z^n} \operatorname{tr} h^n \right) = z^N \exp\left(- \sum_{n \geq 1} \frac{1}{n z^n} \cdot \sum_{w \in W_n} e^{V_w} \right)$$

converges when $\left|\frac{z_j}{z}\right| < 1 \ \forall j \to |z| > |z_1|$

$$-\sum_{n\geq1}\frac{1}{nz^n}\sum e^{V_w} = -\sum_{n\geq1}\frac{1}{nz^n}\sum_{\gamma\in\Gamma}|\gamma|\sum_{\substack{k\geq1\\ \text{s.t. } \cancel{n}=k|\gamma|}} e^{kV_\gamma} =$$

$$= -\sum_{\gamma\in\Gamma}\sum_{k\geq1}\frac{\cancel{|\gamma|}}{k\cancel{|\gamma|}z^{k|\gamma|}}e^{kV_\gamma} = -\sum_{\gamma\in\Gamma}\sum_{k\geq1}\frac{e^{kV_\gamma-k(\ln z)|\gamma|}}{k} = -\sum_{\gamma\in\Gamma}\sum_{k\geq1}\frac{\left(e^{V_\gamma-|\gamma|\ln z}\right)^k}{k} =$$

$$= \sum_{\gamma\in\Gamma}\log\left(1-e^{V_\gamma-|\gamma|\ln z}\right) = \log\prod_{\gamma\in\Gamma}\left(1-e^{V_\gamma-|\gamma|\ln z}\right)$$

$\square$

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assume: $\boxed{\text{end of day 1}}$

h adjacency matrix : $h_{ij} = \begin{cases}0\\1\end{cases}$ $\quad (V_{ij}=0)$

$\Rightarrow$ $\operatorname{tr} h^n = \# W_n =: N(n)$ number of marked per. pts

$\pi(n) = \#$ primitive per. orbits

and $N(n) = \sum_{k|n}\frac{n}{k}\pi\left(\frac{n}{k}\right)$ $\quad \left(= \sum_{\gamma\in\Gamma}|\gamma|\cdot\sum_{\substack{k \text{ s.t.}\\ n=k\cdot|\gamma|}}1 \right.$

$$= \underbrace{n\cdot\pi(n)}_{k=1} + \underbrace{\frac{n}{2}\pi\left(\frac{n}{2}\right)}_{k=2} + \ldots$$

$$\left. = \sum_{k|n}\frac{n}{k}\pi\left(\frac{n}{k}\right)\right)$$

assume:

h is mixing (def) $\exists\, n>0$ s.t. $(h^n)_{ij} > 0 \ \forall ij$

Perron-Frobenius thm

$\Rightarrow$ $\exists\, z_1 > 1$ simple, dominant

$$\frac{|z_j|}{z_1} \leq \varkappa < 1$$

spectral gap

Prop $\quad N(n) = e^{nh_{top}}\left(1+O(\varkappa^n)\right)$, $n\gg1$

top. entropy $h_{top} := \ln z_1 > 0$

$$\pi(n) = \frac{e^{nh_{top}}}{n}\left(1+O(\varkappa^n)+O\left(e^{-\frac{n}{2}h_{top}}\right)\right) \quad , n\gg1$$

⑤

Last time:

any matrix

$\boxed{\text{trace f-la}}$

$\operatorname{tr} h^n \; \forall n = \sum_{j=1}^{N} z_j^n = \sum_{w\in W_n} e^{V_w} = \sum_{\gamma\in\Gamma} |\gamma| \sum_{k \text{ s.t. } k|\gamma|=n} e^{kV_\gamma}$

$$\begin{tikzcd}
\text{matrix } N\times N,\; h_{ij}=e^{V_{ij}} \arrow[rr, leftrightarrow, bend left] \arrow[dr, leftrightarrow] & & \text{marked periodic orbits, primitive per. orbits} \arrow[dl, leftrightarrow] \\
& \text{Bowen-Lanford zeta fn} &
\end{tikzcd}$$

e^{V_w}

$w, \; W_n, \; W, \; N(n)$

$\gamma, \; \Gamma_n, \; \Gamma, \; \pi(n)$

e^{V_γ}

$\zeta_{BL}(z) := \exp\left(-\sum_{n\geq 1} \frac{1}{nz^n} \sum_{w\in W_n} e^{V_w}\right)$

$\boxed{\zeta_{BL}(z) = \prod_{\gamma\in\Gamma}\left(1-e^{V_\gamma - |\gamma|\ln z}\right)}$

$d(z) := \det(z\cdot Id - h)$

$z^{-N} d(z) = \zeta_{BL}(z)$

$\boxed{\text{e.v. of } h \text{ are zeros of } \zeta_{BL}}$

Now: $h_{ij} = \begin{cases}1\\0\end{cases}$ adjacency matrix

ex.

$\begin{pmatrix}0&0\\1&1\end{pmatrix}$

$\boxed{\text{trace f-la}}$

$\operatorname{tr} h^n = N(n) = \sum_{k|n} \frac{n}{k}\pi\left(\frac{n}{k}\right)$

$$\begin{tikzcd}
h_{ij}=\begin{cases}1\\0\end{cases} \text{ matrix} \arrow[rr, leftrightarrow, bend left] \arrow[dr, leftrightarrow] & & \text{marked per. orb, primitive per. orb} \arrow[dl, leftrightarrow] \\
& \text{Bowen Lanford zeta fn} &
\end{tikzcd}$$

$z^{-N} d(z) = \zeta_{BL}(z)$

$\zeta_{BL}(z) := \exp\left(-\sum_{n\geq 1}\frac{N(n)}{nz^n}\right)$

$\zeta_{BL}(z) = \prod_{\gamma\in\Gamma}\left(1-z^{-|\gamma|}\right)$

Idea: [e.v. of h are zeros of ζ_{BL}]

Prop

$\pi(n) = \frac{1}{n}\sum_{k|n}\mu(k)N\left(\frac{n}{k}\right)$

$\mu : \mathbb{N}\to\{-1,0,1\}$ Möbius fn

$\mu(1) = 1$

$\sum_{k|n}\mu(k) = 0$ if $n\geq 2$

$\mu(1) = 1$
$\mu(2) = -1$
$\mu(3) = -1$
$\mu(4) = 0$
$\dots$

Pf $P(n) := \frac{1}{n}\sum_{k|n}\mu(k)N\left(\frac{n}{k}\right)$

WTS: $P \equiv \pi$, i.e. check: $N(n) = \sum_{d|n}\frac{n}{d}P\left(\frac{n}{d}\right)$

$$\sum_{d|n}\frac{n}{d}P\left(\frac{n}{d}\right) = \sum_{d|n}\cancel{\frac{n}{d}}\cdot\cancel{\frac{d}{n}}\sum_{k|\frac{n}{d}}\mu(k)N\left(\frac{n}{\boxed{dk}}\right) = \sum_{m|n}\underbrace{\sum_{k|m} N\left(\frac{n}{m}\right)\mu(k)}_{\to\begin{cases}0, & m\geq 2\\ 1, & m=1\end{cases}} = N(n)$$

$k|\frac{n}{d} \to kd | n$, $= m$

□

Prop (Asymptotics of $N(n)$ and $\pi(n)$ for large n)

$$N(n) = e^{n h_{top}} (1 + O(\varkappa^n)) \quad , \quad n \gg 1$$

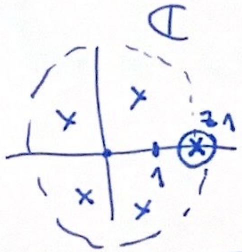
$$\pi(n) = \frac{e^{n h_{top}}}{n} (1 + O(\varkappa^n) + O(e^{-\frac{n}{2} h_{top}})) \quad , \quad n \gg 1.$$

Assume h is mixing

$\exists n > 0 : (h^n)_{ij} > 0 \quad \forall i,j = 1, \dots, N$

Perron-Frobenius thm

h mixing $\Rightarrow \exists z_1 > 1$ simple dominant e.v.



$$\left|\frac{z_j}{z_1}\right| \leq \varkappa < 1 \quad \forall j = 2, \dots, N$$

spectral gap.

Pf $\quad N(n) \overset{\text{trace fla}}{=} \sum_{j=1}^{N} z_j^n = z_1^n \left(1 + \sum_{j=2}^{N} \left(\frac{z_j}{z_1}\right)^n\right) = z_1^n (1 + O(\varkappa^n)) = e^{n h_{top}} (1 + O(\varkappa^n))$

$$\pi(n) = \frac{1}{n}\left(N(n) + O\left(e^{\frac{n h_{top}}{2}} (1 + O(\varkappa^n))\right)\right) = \frac{e^{n h_{top}}}{n} (1 + O(\varkappa^n) + O(e^{-\frac{n}{2} h_{top}}))$$

$\square$

Part II Finite rank perturbation of the shift operator

(first non-trivial example – not trace class)

def right Shift operator

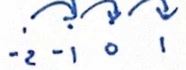
$\mathcal{L}_0 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $\ell^2(\mathbb{Z}) = \{ u = \{u_j\}_{j\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |u_k|^2 < \infty \}$

$(\mathcal{L}_0 u)_k = u_{k-1}$

$$\begin{matrix} u_{-2} & u_{-1} & u_0 & u_1 \\ \cdot & \cdot & \cdot & \cdot \\ -2 & -1 & 0 & 1 \end{matrix} \xrightarrow{\mathcal{L}_0} \begin{matrix} u_{-3} & u_{-2} & u_{-1} & u_0 \\ \cdot & \cdot & \cdot & \cdot \\ -2 & -1 & 0 & 1 \end{matrix} \quad \cdots$$

$$\mathcal{L}_0 = \begin{pmatrix} \ddots & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & \boxed{0} & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \\ & & & & & \ddots \end{pmatrix}$$

ass. graph



-2 -1 0 1

Rem generalized e.f. $u_k = \lambda^k$, $(\mathcal{L}_0 u)_k = u_{k-1} = \lambda \cdot u_k$

def $\operatorname{spec}(\mathcal{L}_0) := \{ \lambda \in \mathbb{C} : (\mathcal{L}_0 - \lambda\cdot \mathrm{Id})$ ~~doesn't have~~ is not invertible (bdd) inverse$\}$

inj, surj, linear op.

Prop $\operatorname{spec}(\mathcal{L}_0) = S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$

Pf

Claim 3 ~~inj~~ (no e.v.)

$(\mathcal{L}_0 u)_k = \lambda u_k$

$u_{k-1} = \lambda u_k$

$|\lambda| = 1$

$$\Rightarrow \quad u = \begin{pmatrix} \vdots \\ \lambda^2 u_0 \\ \lambda u_0 \\ u_0 \\ \frac{u_0}{\lambda} \\ u_0/\lambda^2 \\ \vdots \end{pmatrix} \notin \ell^2(\mathbb{Z})$$

$\sum_{k\in\mathbb{Z}} |u_k|^2 = \sum_{k\in\mathbb{Z}} |u_0|^2 = \infty .$ ↯

Claim 1 $\mathcal{L}_0$ is unitary operator

$\mathcal{L}_0^* = \mathcal{L}_0^{-1}$

$\mathcal{L}_0^{-1}$ = left shift , $(\mathcal{L}_0^{-1} u)_k = u_{k+1}$

$(\mathcal{L}_0 u, v) = (u, \mathcal{L}_0^* v)$

$(u,v) := \sum_{k\in\mathbb{Z}} u_k \overline{v_k}$ $\qquad$ $\mathcal{L}_0^* v = v_{\ell+1}$

$(\mathcal{L}_0 u, v) = \sum_{k\in\mathbb{Z}} u_{k-1}\overline{v_k} \underset{k-1=\ell}{=} \sum_{\ell\in\mathbb{Z}} u_\ell \overline{v_{\ell+1}} = \sum_{\ell\in\mathbb{Z}} u_\ell \overline{(\mathcal{L}_0^* v)_\ell}$

$= (u, \mathcal{L}_0^* v)$

Claim 2 $\operatorname{spec}(\text{unitary op}) \subset S^1$

$|\lambda| < 1 \Rightarrow U - \lambda\cdot\mathrm{Id} = U(\mathrm{Id} - \underbrace{\lambda\cdot U^{-1}}_{\|\cdot\| < 1})$

$\|U\| = \|U^{-1}\| = 1$

→ not in spec $\qquad$ invertible

$|\lambda| > 1 \Rightarrow U - \lambda\cdot\mathrm{Id} = -\lambda(\mathrm{Id} - \underbrace{\tfrac{1}{\lambda}U}_{\|\cdot\| < 1})$

→ not in spec.

end of day 2

Claim 4 $\operatorname{spec}(\mathcal{L}_0) \neq \emptyset$

(i.e. $\exists \lambda \in S^1$ s.t. $\lambda \in \operatorname{spec}(\mathcal{L}_0)$)

$(\mathcal{L}_0 - \lambda\cdot\mathrm{Id})^{-1}$ exists $\forall \lambda \in \mathbb{C}$

(it's analytic)

$\lambda \to \infty \quad \to 0 \quad \xrightarrow{\text{Liouville thm.}} \quad (\mathcal{L}_0 - \lambda\cdot\mathrm{Id})^{-1} \equiv 0$ ↯

Claim 5 fix $\lambda \in S^1$ $\quad \mathcal{L}_0 = U^{-1}(\lambda \mathcal{L}_0) U$

$(Uu)_k := \lambda^k u_k$

$\boxed{U \mathcal{L}_0 U^{-1} = \lambda \mathcal{L}_0}$, U is unitary. $\Rightarrow \operatorname{spec}(\mathcal{L}_0) = \operatorname{spec}(\lambda \mathcal{L}_0)$

spec. is invariant

$\lambda_0 \in \operatorname{spec} \Rightarrow \lambda\cdot\lambda_0 \in \operatorname{spec}$

⑧

$$\left(U^{-1}(\lambda h_0)Uu\right)_k = \lambda^{-k}(\lambda h_0 Uu)_k = \lambda^{-k+1}(Uu)_{k-1} = \lambda^{1-k}\cdot\lambda^{k-1}\cdot u_{k-1}$$

$$= u_{k-1} = (h_0 u)_k$$

$\forall \lambda \in S^1 \Rightarrow S^1 \subset \mathrm{spec} \Rightarrow \mathrm{spec}(h_0) = S^1.$

☐

What happens if we take $\lambda = e^{-r} < 1$, $r>0$?

(not unitary anymore) : $\ell^2 \to \ell^2$

$$U = \begin{pmatrix} \ddots & & & & \\ & \lambda^{-2} & & & \\ & & \lambda^{-1} & & \\ & & & \boxed{1} & \\ & & & & \lambda \\ & & & & & \lambda^2 \\ & & & & & & \ddots \end{pmatrix}$$

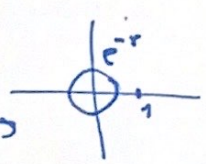
but $U h_0 U^{-1} = \lambda h_0$ – in $\ell^2(\mathbb{Z})$.

$h_0^U :=$

$U^{-1}u \notin \ell^2$ but where?

$\mathrm{spec}(h_0^U)$ in ℓ^2 $= \lambda \cdot S^1 = e^{-r} S^1$

smaller radius.



$$\begin{tikzcd} \ell^2(\mathbb{Z}) \arrow[r, "h_0^U := U h_0 U^{-1}"] & \ell^2(\mathbb{Z}) \\ U^{-1}\ell^2(\mathbb{Z}) =: \mathcal{H}^U(\mathbb{Z}) \arrow[u, "U"] \arrow[r, "h_0"] & \mathcal{H}^U(\mathbb{Z}) := U^{-1}\ell^2 \arrow[u, "U"'] \end{tikzcd}$$

allow sequences to have right heavy tail.

$$\boxed{\begin{matrix} \mathrm{spec}(h_0) & = \mathrm{spec}(h_0^U) \\ \text{in } \mathcal{H}^U(\mathbb{Z}) & \text{in } \ell^2(\mathbb{Z}) \end{matrix}}$$

U: isometry

$$\|u\|_{\mathcal{H}^U(\mathbb{Z})} = \|Uu\|_{\ell^2(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |\lambda|^{2k} |\mu_k|^2\right)^{1/2} = \left(\sum_{k\in\mathbb{Z}} e^{-rk2} |\mu_k|^2\right)^{1/2}$$

$\mathcal{H}^U$ anisotropic Sobolev space $\left(\frac{U_{j+1\,j+1}}{U_{jj}} = \frac{\lambda^{j+1}}{\lambda^j} = \lambda = e^{-r} < 1\right)$

U decays along trajectories of h_0 – right shift

escape fn

(Lyapunov fn)

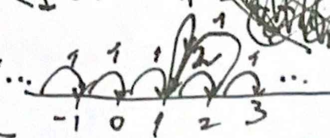
Perturbation of h_0

$$\mathcal{L} := h_0 + \overset{\downarrow}{M}$$

↳ $N\times N$ matrix

$M_{ij} \neq 0$, $i,j = 1\dots N$

ex: $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$



$$\mathcal{L} = \begin{pmatrix} \ddots & & \\ & 1 & 0 \\ & 1 & \boxed{\begin{matrix} 1 & 1 \\ 2 & 0 \end{matrix}} \\ & & & \ddots \end{pmatrix}$$

Problem : $\mathrm{spec}\,\mathcal{L} \neq \mathrm{spec}\,\mathcal{L}_0 \cup \mathrm{spec}\,M$

☹ even if $\mathcal{L}$ is $N\times N$ matrix

ex $\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 1$

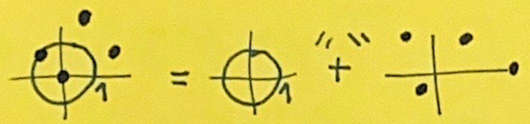
$= \lambda^2 - 3\lambda + 1$

$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda_\pm = \pm 1$ $\quad \lambda_\pm = \frac{3\pm\sqrt{5}}{2}$

But

Thm Finite rank perturbation doesn't change the essential spectrum. (spec \ discrete — { isolated e.v. of finite multiplicities})

$\mathcal{L} := \mathcal{L}_0 + M$

spec 

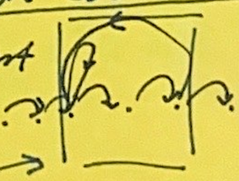
(i.e. $S^1 \subseteq \operatorname{spec}(\mathcal{L})$)

def flat trace
$\operatorname{tr}^\flat(\mathcal{L}) := \sum_{i\in\mathbb{Z}} \mathcal{L}_{ii}$ (sum over diagonal)

Prop (trace f-la for perturbed shift operator)

$$\sum_{w\in W_n} e^{V_w} = \sum_{j=1}^{N} (z_j)^n \qquad z_j \text{ are e.v. of } \mathcal{L} \text{ (not } M\text{!)}$$

Rem. You can easily prove it directly but we want to illustrate the method of anisotropic Sobolev spaces.
↳ everything is defined by finite part



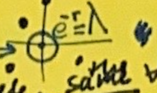
Pf $\operatorname{tr}^\flat \mathcal{L}^n = \sum_{w\in W_n} e^{V_w}$ ← done in Prop for finite matrices

WTS: $\boxed{\operatorname{tr}^\flat \mathcal{L}^n = \sum_{j=1}^{N} z_j^n}$ ← not true in general

Rem $\mathcal{L}$ is <u>not</u> trace class

Consider $\mathcal{L}$: $\ell^2_U = U^{-1}\ell^2(\mathbb{Z})$ $\quad (U^{-1}u)_k = \lambda^k u_k$

$\mathcal{L}^U$ ⟺ $\boxed{U\mathcal{L}U^{-1} \text{ on } \ell^2(\mathbb{Z})}$ $\quad e^{-r} = |\lambda| < 1$.

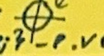
$\operatorname{spec}(U\mathcal{L}U^{-1}) =$ 

take it small

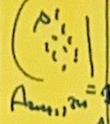
discrete part is same $\forall \lambda$

Spectral decomposition:
(Riesz contour integral)

$\mathcal{L} = \tilde{\mathcal{L}}_0 + \tilde{M}$, $\tilde{\mathcal{L}}_0\tilde{M} = \tilde{M}\tilde{\mathcal{L}}_0 = 0$

s.t. $\operatorname{spec}(\tilde{\mathcal{L}}_0) =$ 

$\operatorname{spec}(\tilde{M}) = \{z_j\}$ – e.v. of $\mathcal{L}$



$A_{2n+1,2n} = 1$
$A_{2n,2n+1} = 1$

<u>very strong result</u>: true for trace class operators

<u>Lidskii theorem</u>:

$$\sum_{i=1}^{\infty} z_j^n = \operatorname{tr} \mathcal{L}^n$$

finite rank ⊂ trace class ⊂ compact ⊂ bdd

id is not trace class $\sum 1 = \infty$

trace depends on basis

<u>ex</u>. $Ae_i = \begin{cases} e_{2n+1}, & i = 2n \\ e_{2n}, & i = 2n+1 \end{cases}$

$A_{ij} = (Ae_i, e_j)$

[illegible]

bounded linear $\mathcal{L}$ is trace class if ~~compact~~ and [illegible]

$$\sum_{j=1}^{\infty} (\sqrt{\mathcal{L}^*\mathcal{L}}\, e_j, e_j) < \infty$$

$$\Downarrow$$

$$\operatorname{tr} \mathcal{L} := \sum_{j=1}^{\infty} (\mathcal{L} e_j, e_j)$$

$\langle Ae_n, e_n\rangle = 0$ but $\langle Af_n, f_n\rangle = 1$ $f_n = 2^{-\frac{1}{2}}(e_{2n} + e_{2n+1})$

$$\mathcal{L}^n = (\mathcal{L}_0 + \hat{M})^n = (\tilde{\mathcal{L}}_0 + \tilde{M})^n = \tilde{\mathcal{L}}_0^n + \hat{M}^n$$

$$\|\tilde{\mathcal{L}}_0^n\| \le \|\tilde{\mathcal{L}}_0\|^n = e^{-r\cdot n}$$

~~$\tilde{M}^n$ doesn't depend~~

$\mathrm{tr}\,\hat{M}^n$ doesn't depend on $\lambda = e^{-r}$

$$\mathrm{tr}\,\tilde{M}^n = \mathrm{tr}^\flat \tilde{M}^n = \sum_{j=1}^{N} z_j^n$$

W.T.S. $|\mathrm{tr}^\flat \tilde{\mathcal{L}}_0^n| < n \cdot e^{-r\cdot n}$

$$\Rightarrow \mathrm{tr}^\flat(\underbrace{\tilde{\mathcal{L}}_0^n + \hat{M}^n}_{\mathcal{L}^n}) = \underbrace{\mathrm{tr}^\flat \tilde{\mathcal{L}}_0^n}_{< n e^{-r\cdot n} \xrightarrow[r\to\infty]{} 0} + \mathrm{tr}\,\tilde{M}^n = \sum_{j=1}^{N} z_j^n$$

$$\tilde{\mathcal{L}}_0 = \mathcal{L} - \tilde{M} = \mathcal{L}_0 + (M - \tilde{M})$$

$\mathrm{tr}^\flat \mathcal{L}_0^n = 0$, $\|\tilde{\mathcal{L}}_0^n\| \le e^{-r\cdot n}$, $\mathrm{tr}(M-\tilde{M}) \le C$, $\|\mathcal{L}_0^n\| \le e^{-r\cdot n}$

$$\tilde{\mathcal{L}}_0^n = \mathcal{L}_0^n + \sum_{k=0}^{n-1} \mathcal{L}_0^k \cdot (M-\tilde{M}) \cdot \tilde{\mathcal{L}}_0^{n-k}$$

$$|\mathrm{tr}^\flat(\tilde{\mathcal{L}}_0^n)| \le \underset{0}{\underset{\|}{|\mathrm{tr}^\flat \mathcal{L}_0^n|}} + \sum_{k=0}^{n-1} \underbrace{\|\mathcal{L}_0^k\|}_{\le e^{-rk}} \cdot \mathrm{tr}(M-\tilde{M}) \cdot \underbrace{\|\tilde{\mathcal{L}}_0\|}_{\le e^{-r(n-k)}}$$

$$\le n \cdot \underbrace{\mathrm{tr}(M-\tilde{M})}_{\le C} \cdot e^{-rn}$$

$n=1$: $\tilde{\mathcal{L}}_0 = \mathcal{L}_0 + (M-\tilde{M})$

$$\tilde{\mathcal{L}}_0^{n+1} = \tilde{\mathcal{L}}_0^n \cdot \tilde{\mathcal{L}}_0 = \Big(\mathcal{L}_0^n + \sum_{k=0}^{n-1} \mathcal{L}_0^k (M-\tilde{M}) \tilde{\mathcal{L}}_0^{n-k}\Big)(\mathcal{L}_0 + (M-\tilde{M}))$$

$$= \mathcal{L}_0^{n+1} + \mathcal{L}_0^n (M-\tilde{M}) + \sum_{k=0}^{n-1} \mathcal{L}_0^k (M-\tilde{M}) \tilde{\mathcal{L}}_0^{n-k+1}$$

$$= \mathcal{L}_0^{n+1} + \sum_{k=1}^{n} \mathcal{L}_0^k (M-\tilde{M}) \tilde{\mathcal{L}}_0^{n-k}$$

□

$\boxed{\text{end of day 3}}$

Prop zeta fn.

$$\zeta(z) := \exp\Big(-\sum_{n\ge1} \frac{1}{n} z^n \sum_{w\in W_n} e^{V_w}\Big) = \prod_{\gamma\in P} \left(1 - e^{V_\gamma - |\gamma| \ln z}\right)$$

zeros of ζ on $\mathbb{C}\setminus\{0\}$ are e.v. of $\mathcal{L}$ — called Ruelle resonances (Ruelle spectrum).

(≠ spec. determinant)

zeta fn is analog of spec. det.

kyno spectral determinant!

map: shift + grammar

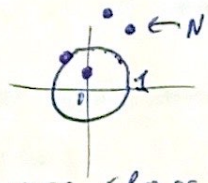
orbit ⇔ $(\dots i_0 i_1 \dots i_{n-1})$

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d, leftrightarrow, "\text{zeros are e.v. of } \mathcal{L}"'] & \text{periodic orbits on graph} \arrow[dl, leftrightarrow] \\ \text{zeta fn} & \end{tikzcd}$$

Last time

$$\mathcal{L} = \underset{\substack{\uparrow \\ \text{rightshift}}}{\mathcal{L}_0} + \underset{\substack{\uparrow \\ N\times N}}{M}$$

spec($\mathcal{L}$) =



$\mathcal{L}$ is <u>not</u> trace class operator $\Rightarrow$ <u>NO</u> Lidskii theorem

$\forall n \; \sum_{i=1}^{\infty} z_i^n = tr\,\mathcal{L}^n$

$d(z) := \det(z \cdot Id - \mathcal{L}) = \prod_{i=1}^{\infty}(z - z_i)$

but we can prove :

<u>trace fla</u> $\sum_{i=1}^{N} z_i^n = tr^b \mathcal{L}^n = \sum_{w \in W_n} e^{V_w} = \sum_{\gamma \in \Gamma} |\gamma| \sum_{\substack{k \\ k|\gamma| = n}} e^{kV_\gamma}$

$tr^b \mathcal{L} = \sum_{i\in\mathbb{Z}} \mathcal{L}_{ii}$

<u>how?</u> (we want to get rid of essential spectrum (S^1)).

use anisotropic Sobolev space

$\mathcal{H}^U := U^{-1} \ell^2(\mathbb{Z})$

$(Uu)_k = \lambda^k u_k$, $\lambda = e^{-r} < 1$.

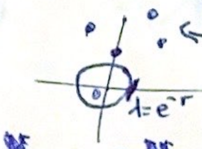
$$\begin{tikzcd} \mathcal{H}^U \arrow[r, "\mathcal{L}"] \arrow[d, "U"'] & \mathcal{H}^U \arrow[d, "U"] \\ \ell^2(\mathbb{Z}) \arrow[r, "\mathcal{L}^U"] & \ell^2(\mathbb{Z}) \end{tikzcd}$$

$\mathcal{L}^U := U \mathcal{L} U^{-1}$

study $\mathcal{L}^U$ on $\ell^2(\mathbb{Z}) \Leftrightarrow$ study $\mathcal{L}$ on $\mathcal{H}^U$

$\underset{\text{on } \ell^2}{\text{spec}(\mathcal{L}^U)} = \underset{\text{on } \mathcal{H}^U}{\text{spec}(\mathcal{L})}$

=



$\leftarrow$ N e.v. doesn't depend on λ (space $\mathcal{H}^U$)

spectral decomposition: $\mathcal{L} = \tilde{\mathcal{L}}_0 + \tilde{M}$ on $\mathcal{H}^U$

spec $\tilde{\mathcal{L}}_0 = \oint_{e^{-r}}$

spec $\tilde{M}$ = e.v. of $\mathcal{L}$

$\tilde{\mathcal{L}}_0 \tilde{M} = \tilde{M} \tilde{\mathcal{L}}_0 = 0$

if one of the e.v. is zero? $\tilde{M} = (N-1)\times(N-1)$

what if all e.v are zeros?

$\Rightarrow \tilde{M} \equiv 0.$

$\mathcal{L} = \tilde{\mathcal{L}}_0$

$tr^b(\mathcal{L}) = tr^b(\tilde{\mathcal{L}}_0 + \tilde{M})^n = tr^b \tilde{\mathcal{L}}_0^n + tr^b \tilde{M}^n$

$\|\tilde{\mathcal{L}}_0^n\| \le e^{-r \cdot n}$

$tr^b \tilde{M}^n = tr\,\tilde{M}^n = \sum_{j=1}^{N} z_j^n$

WTS $|tr^b \tilde{\mathcal{L}}_0^n| \le n \cdot e^{-rn} \xrightarrow[r\to\infty]{} 0$ $\Rightarrow$ $tr^b \mathcal{L}^n = \sum_{j=1}^{N} z_j^n$

$\square$

Prop zeta fn (analog of spec. det).

$$\zeta(z) := \exp\Big(-\sum_{n\geq 1}\frac{1}{n z^n}\sum_{w\in W_n} e^{V_w}\Big) = \prod_{\gamma\in\Gamma}\big(1-e^{V_\gamma - |\gamma|\ln z}\big)$$

zeros of ζ $\overset{\text{on } \mathbb{C}\setminus 0 ?}{\text{are}}$ e.v. of $\mathcal{L}$ — called Ruelle resonances (Ruelle spectrum)

Rem ↗ no spec. determinant $d(z)$

Part III Expanding flow $\overset{/\text{diffeo}}{\text{on}}$ $\mathbb{R}$. / expanding map on $\mathbb{R}$.

flow $\varphi^t(x) := e^t x$, $t>0$, $x\in\mathbb{R}$

exp map (time-1 flow) $\varphi^e(x) = e\cdot x$, $|e|>1$.

$\mathcal{L}^e u(x) = u(\varphi^{-1}x) = u(\frac{x}{e})$

n times $\varphi^n(x) = e^n x$

everything below works for this model (just take $t = n\in\mathbb{Z}_+$)

corresponding linear vector field

$Y = x\frac{d}{dx}$ on $\mathcal{S}(\mathbb{R})$ Schwartz class

$$\begin{cases} \frac{d}{dt}\varphi^t(x) = Y\varphi^t(x) \\ \varphi^t(x)\big|_{t=0} = x \end{cases}$$

$\frac{d}{dt}\varphi^t(x) = \frac{d}{dt}(e^t x) = e^t x = x\cdot\frac{d}{dx}e^t x = Y\varphi^t(x)$

$\varphi^0(x) = e^0\cdot x = x$

$-\frac{du(e^{-t}x)}{dt} = x\frac{d}{dx}u(e^{-t}x)$

~~$-\frac{d^k}{dt^k}u(e^{-t}x)\big|_{t=0} = \ldots$~~

transfer operator

~~$(e^{-tY}u)(x) = \sum_{k\geq 0}\frac{(-tx\frac{d}{dx})^k}{k!}u(x) = \sum_{k\geq 0}\frac{(-t)^k}{k!}(x\frac{d}{dx})^k u(x)$~~

~~$(e^{-tY}u)(x) = \ldots$~~ [illegible]

$-\frac{du(y)}{dy}\Big|_{y=e^{-t}x}\cdot\frac{dy}{dt} = +x\cdot e^{-t}\frac{du}{dy}$ ✓

$(\mathcal{L}^t u)(x) := u(\varphi^{-t}x) = u(e^{-t}x)$ $\Longrightarrow$ $\mathcal{L}^t = e^{-tY}$

$x\frac{d}{dx}u(e^{-t}x) = x\frac{d}{dy}u(y)\Big|_{y=e^{-t}x}\cdot\boxed{\frac{dy}{dx}}$ e^{-t}

Correlation fn

$\langle v|\mathcal{L}^t u\rangle_{L^2(\mathbb{R})} := \int_{\mathbb{R}} u(e^{-t}x)\bar{v}(x)\,dx$

$\parallel$

$(\mathcal{L}^t u, v)_{L^2(\mathbb{R})}$

$\Rightarrow u(e^{-t}x) = e^{-tx\frac{d}{dx}}u(x) = e^{-tY}u(x)$

$\langle\delta'|\varphi\rangle = -\varphi'(0)$

Naive approach (intuition)

$$\langle v|\mathcal{L}^t u\rangle = \int_{\mathbb{R}} u(e^{-t}x)\bar{v}(x)\,dx = \int_{\mathbb{R}}\sum_{k\geq 0}\frac{u^{(k)}(0)}{k!}\cdot(e^{-t}x)^k\bar{v}(x)\,dx = \sum_{k\geq 0} e^{-tk}\frac{u^{(k)}(0)}{k!}\langle v|x^k\rangle =$$

$$= \sum_{k\geq 0} e^{-tk}\langle v|\underbrace{x^k\rangle\langle\frac{(-1)^k\delta^{(k)}}{k!}}_{=:\Pi_k}|u\rangle = \sum_{k\geq 0} e^{-tk}\langle v|\Pi_k u\rangle = \langle v|\sum_{k\geq 0}e^{-tk}\Pi_k u\rangle$$

Rem $\{x^k\}$ dual to $\{\frac{(-1)^k\delta^{(k)}}{k!}\}$

$\langle\frac{(-1)^k}{k!}\delta^{(k)}|x^\ell\rangle_{L^2} = \delta_{k=\ell}$

$\Pi_k(f) = \Big(\int_{\mathbb{R}}\frac{(-1)^k\delta^{(k)}(x)}{k!}u(x)\,dx\Big)x^k$

$\overset{?}{\Longrightarrow}$ $\mathcal{L}^t = \sum_{k\geq 0}e^{-tk}\Pi_k$ spectral decomposition

Π_k is a projector on the eigenspace corresponding to eigenfn x^k

$$\Pi_k \circ \Pi_\ell = \delta_{k\ell}\,\Pi_k$$

$\mathcal{L}^t = \sum e^{-tk}\,\Pi_k \Rightarrow \underset{k\geq 0}{e^{-tk}}$ are e.v. of $\mathcal{L}^t$

$\underset{k\geq 0}{e^{-k}}$ are e.v of $\mathcal{L}$ | $\mathcal{L} = e^{-X}$

$k \geq 0$ are e.v of

Y

not surprisizing:

$x\frac{d}{dx}x^k = \underline{\underline{k}}\,x^k$

But $x^k \notin L^2(\mathbb{R})$

$$\frac{(-1)^k\delta^{(k)}}{k!} \notin \left(L^2(\mathbb{R})\right)^* = L^2(\mathbb{R})$$

<u>Idea</u>: Find a space where it is true!

Before that ~~exposition~~ ~~(Atiyah-Bott trace fla)~~

[illegible]

$$\mathcal{L}^t u(x) = u(e^{-t}x) = \int_{\mathbb{R}} \underbrace{\delta(y - e^{-t}x)}_{=:K_t(x,y)}\,u(y)\,dy \quad \text{– kernel of } \mathcal{L}^t$$

flat trace

$$\operatorname{tr}^\flat \mathcal{L}^t := \int_{\mathbb{R}} K_t(x,x)\,dx$$

<u>Prop</u> (Atiyah-Bott trace fla)

$$\operatorname{tr}^\flat \mathcal{L}^t = \frac{1}{1-e^{-t}} = \sum_{k\geq 0} e^{-kt} \qquad \left(\overset{?}{=} \text{ sum of eigenvalues?}\right)$$

<u>Pf</u>

$$\operatorname{tr}^\flat \mathcal{L}^t = \int_{\mathbb{R}} \delta(x - e^{-t}x)\,dx = \int_{\mathbb{R}} \delta(f(x)) = \int_{\mathbb{R}} \frac{dy\,\delta(y)}{1-e^{-t}} = \frac{1}{1-e^{-t}}$$

$$y = f(x) = x - e^{-t}x$$

$$dy = (1-e^{-t})\,dx$$

$\square$

<u>Rem</u> for Anosov diffeo

$$\mathcal{L}u(x) := e^{V(x)}\,u(\varphi^{-1}x)$$

$$\operatorname{tr}^\flat \mathcal{L}^n = \sum_{\substack{x\in M\\ \text{s.t.}\\ \varphi^n x = x}} \frac{e^{V_n(x)}}{|\det(1 - D_x\varphi^{-n}x)|}, \quad V_n(x) := \sum_{k=0}^{n-1} V(\varphi^{-k}x)$$

$\boxed{\text{end of day 4}}$

In our case $V\equiv 0$ the only periodic orbit is fixed pt $x=0$! $\varphi^{-n}x = e^{-n}x$

$\operatorname{tr}^\flat \mathcal{L}^n = \frac{1}{1-e^{-n}}$ $\quad D_x\varphi^n x\big|_{x=0} = e^{-n}$

Last time Part III Exp. flow / map on $\mathbb{R}$ ⑭

$$\varphi^t x := e^t x \quad , x \in \mathbb{R} \quad , t > 0$$

transfer op $\mathcal{L}^t u(x) := u(e^{-t}x)$

corr. fn $:= (\mathcal{L}^t u, v) = \int_{\mathbb{R}} \mathcal{L}^t u(x) \bar{v}(x)\, dx = \int_{\mathbb{R}} u(e^{-t}x) \bar{v}(x)\, dx =$

$$\text{"="} \sum_{k\geq 0} e^{-kt} (\Pi_k u, v) \quad , \Pi_k u := \left(u, \frac{(-1)^k \delta^{(k)}}{k!}\right) x^k$$

$$\text{"}\Rightarrow\text{"} \quad \mathcal{L}^t = \sum_{k \geq 0} e^{-kt} \Pi_k \quad \text{spectral decomposition}$$

e^{-kt} are e.v of $\mathcal{L}^t$

x^k are e.f.

But in which space? Definitely not in $L^2(\mathbb{R})$.

Prop 1) $(\mathcal{L}^t)^* u(x) = e^t u(e^t x)$ Perron-Frobenius operator ~~$(\mathcal{L}^t)^* \neq (\mathcal{L}^t)^{-1}$ but $\sqrt{e^t}$~~ (~~not unitary~~)

2) $(\mathcal{L}^t)^{-1} = \mathcal{L}^{-t}$

on $L^2(\mathbb{R})$ 3) $\mathcal{L}^t$ is not unitary

4) $\sqrt{e^{-t}}\mathcal{L}^t$ is unitary

5) $\operatorname{spec}(\sqrt{e^{-t}}\mathcal{L}^t) = S^1$

6) $\operatorname{spec}(\mathcal{L}^t) = \sqrt{e^t}\, S^1$

7) $\mathcal{L}^t$ is not trace class

Rem: $\mathcal{L}^t = e^{-tY}$, $Y = x\frac{d}{dx}$

$\operatorname{spec}(Y) = -\frac{1}{2} + i\mathbb{R}$

Pf 1) $(\mathcal{L}^t u, v) = \int_{\mathbb{R}} u(e^{-t}x)\bar{v}(x)\,dx = \int_{\mathbb{R}} u(y) \bar{v}(e^t y) \frac{dy}{e^{-t}} = (u, \mathcal{L}^{t*} v)$

$\mathcal{L}^* v(y) = e^t v(e^t y)$

2) obv.

1) & 2) $\Rightarrow$ 3)

4) obv.

5) do the same as for shift: use $Uu(x) := \lambda^{\log|x|} u(x)$.

$U\mathcal{L}^t U^{-1} = \lambda^t \mathcal{L}^t$

6) obv. $e \leftarrow K_t(x,x) = e^t$

7) check $\mathcal{L}^{t*}\mathcal{L}^t = e^t \mathrm{id} \notin$ trace class. □

What is Sobolev space

$$H^1 = \{f, f' \in L^2\}$$

p derivatives ← p

weak derivative

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty$$

$$\int_{\mathbb{R}} |f'(x)|^2 dx < \infty$$

int. by parts

$$(f', \varphi) := (-f, \varphi')$$

$$\varphi \in S(\mathbb{R})$$

$$\widehat{f'}(\xi) = \int_{\mathbb{R}} e^{-i\xi y} f'(y)\, dy = -\int_{\mathbb{R}} (e^{-i\xi y})'_x f(x)\, dx = +i\xi \int_{\mathbb{R}} e^{-i\xi y} f(y)\, dx$$

$$= +i\xi \hat{f}(\xi)$$

$$\|f'\|_{L^2} = \|\widehat{f'}\|_{L^2} = \int_{\mathbb{R}} |\widehat{f'}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\xi|^2 |\hat{f}|^2 d\xi$$

$$= \left(\int_{\mathbb{R}} |f'|^2 dx\right)^{1/2}$$

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2} = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$$

$$\|f\|_{H^1} = \sqrt{\|f\|^2_{L^2} + \|f'\|^2_{L^2}} = \left(\int_{\mathbb{R}} (|\xi|^2 + 1) |\hat{f}|^2 d\xi \right)$$

$$= \int_{\mathbb{R}} \left(\sqrt{|\xi|^2+1}\, |\hat{f}| \right)^2 d\xi$$

$$H^1 = \{ f : \hat{f}(\xi) \underbrace{\sqrt{|\xi|^2+1}}_{=: A(\xi)} \in L^2(\mathbb{R}) \} = \{ f : (\hat{f} A(\xi))^\vee \in L^2(\mathbb{R}) \}$$

$$(O_p A) f =:$$

~~$H^1 = O_p(\sqrt{|\xi|^2+1}) L^2(\mathbb{R})$~~

~~$((O_p A) u)(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} A(\xi) e^{-i\xi y} u(y)\, d\xi\, dy$~~

$$((O_p A) f)(x) := \int_{\mathbb{R}} \frac{e^{i\xi x}}{2\pi} \left(\int_{\mathbb{R}} e^{-i\xi y} f(y)\, dy \right) \underbrace{\sqrt{|\xi|^2+1}}_{A}\, d\xi$$

$$= \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}}}_{= T^*\mathbb{R}} e^{i\xi(x-y)} A(\xi) f(y)\, d\xi\, dy$$

$$H^1 := (O_p A)^{-1} L^2(\mathbb{R})$$

k derivatives → $H^k \subseteq H^{k'} \quad k \geqslant k'$

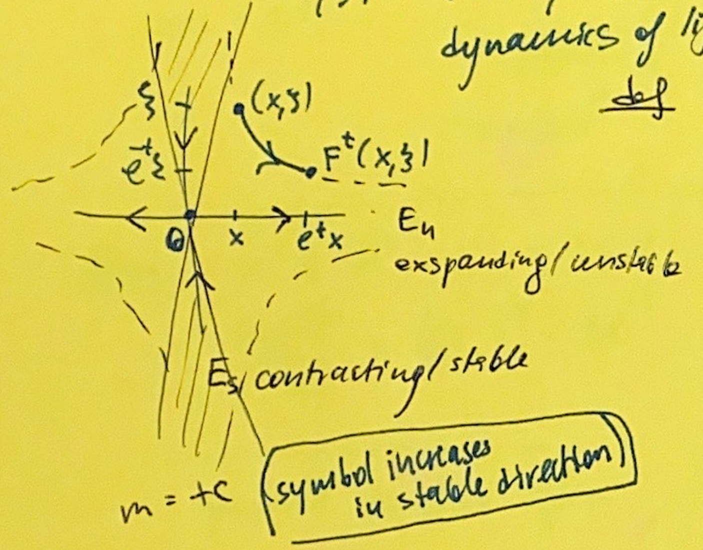
$df_x(X_x) = X_x(f)$, $X_x \in T_xM$, $f \in C^\infty(M)$, $df_x \in T^*_xM$ | $\varphi \circlearrowleft M$, T^*M | $f: M \to N$, $f_*: T_xM \to T_xN$, X_x; $f^*: T^*_{f(x)}N \to T^*_xM$, θ; $(f^*\theta)(X_x) = \theta(f_*X_x)$; $f_*(X)(\varphi) = X(\varphi f)$

Idea: go to $T^*\mathbb{R} = \mathbb{R}^2$

lift of $\varphi^t x = e^t x$, $x \in \mathbb{R}$

is $F^t(x,\xi) = (e^t x, e^{-t}\xi)$

dynamics of lift is scattering on trapped set $= (0,0)$



def $C > 0$.

Lyapunov (escape) fn

$$A_c(x,\xi) = \sqrt{1+|x|^2+|\xi|^2}^{\,m(x,\xi)}$$

order fn

$m \in C^\infty(\mathbb{R})$

- $m(\lambda(x,\xi)) = m(x,\xi)$ $\forall\, |(x,\xi)| \geq 1,\ \lambda \geq 1$
- $m(x,\xi) = \begin{cases} +C & \text{in conic of } x = 0 \text{ (stable)} \\ -C & \text{in conic of } \xi = 0 \text{ (unstable)} \end{cases}$
- $m(F^t(x,\xi)) \leq m(x,\xi)$ $\forall\, |z| \geq 1$.

Claim
$$\frac{A_c(F^t(x,\xi))}{A_c(x,\xi)} \leq Be^{-tC}$$

- stable, $x = 0$, ξ large

$$\frac{A_c(F^t(x,\xi))}{A_c(x,\xi)} \leq B.\frac{e^{-tC}|\xi|^C}{|\xi|^C} = Be^{-tC}$$

Pseudo-differential operator A_c is symbol

$$(Op\,A)u(x) := \frac{1}{2\pi}\iint_{T^*\mathbb{R}} e^{i\xi(x-y)} A_c(x,\xi)\, u(y)\, d\xi\, dy$$

$$\begin{tikzcd} L^2(\mathbb{R}) \arrow[r, "Q^t"] & L^2(\mathbb{R}) \\ \mathcal{H}_c \arrow[u, "OpA_c"] \arrow[r, "\mathcal{L}^t"] & \mathcal{H}_c \arrow[u, "OpA_c"'] \end{tikzcd}$$

$Q^t := OpA_c \circ \mathcal{L}^t \circ OpA_c^{-1}$

spec Q^t = spec $\mathcal{L}^t$
on L^2 on $\mathcal{H}_c$

$\mathcal{H}_c := OpA_c^{-1} L^2(\mathbb{R})$

$x^k \in \mathcal{H}_c$, $k < C$

$$Q^t := OpA_c \circ \mathcal{L}^t \circ OpA_c^{-1} = \mathcal{L}^t \circ \underbrace{\mathcal{L}^{-t} \circ OpA_c \circ \mathcal{L}^t} \circ OpA_c^{-1} = \mathcal{L}^t \circ \underbrace{Op(A_c \circ F^t) \circ OpA_c^{-1}} =$$

$$\underset{\text{Egorov thm}}{=} Op(A_c \circ F^t) + \underbrace{\text{smaller order}}_{\text{regular}} \underset{\substack{\text{composition}\\ \text{thm}}}{=} Op\,\frac{A_c \circ F^t}{A_c} + \underbrace{\text{smaller}}_{\text{regular}}$$

$$= \mathcal{L}^t \cdot \underbrace{Op\,\frac{A_c \circ F^t}{A_c}}_{L^2 \text{cont. thm} = P + \text{smaller}}, \quad \|P\| \leq \sup\left|\frac{A_c \circ F^t}{A_c}\right| \leq Be^{-tC}$$

$$Q^t = \underbrace{\mathcal{L}^t P}_{} + \underbrace{\overset{\text{regular}}{\text{smaller}}}_{\text{compact}}$$

$$\|.\| \leq \tilde{B} e^{-tc}$$

spec (Q^t) =

Rem $x^k \in \mathcal{H}^c$, $k < c-1$ Π^k is bdd op

$$\frac{(-1)\delta^{(k)}}{k!} \in (\mathcal{H}^c)^* = \mathcal{H}^{-c}$$

H^{-c} unstable

$\varphi \in \mathcal{H}_c$ $/\!/ = H^c$ stable

φ is regular in stable direction

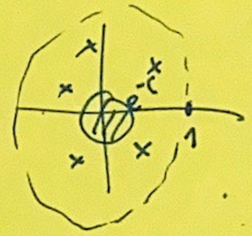
irregular in unstable

$x^2 |\xi|^{c} \in L^2$ stable

$|x|^{-c} \in L^2$ unstable

---

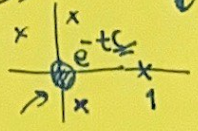
general Anosov diffeo



$\mathcal{L}$

|Wrap up|

Anosov diffeo / flow

	$\varphi \circlearrowright M$ $x = \varphi x$ $x \in M$	$x = \varphi^t x$ $x \in M$
transfer op (pullback)	$\mathcal{L}u(x) = u(\varphi^{-1}x)$	$\mathcal{L}^t u(x) = u(\varphi^{-t}x)$
lift of φ:	$F : T^*M \circlearrowleft$ $\xi \in T^*M$	$F^t : T^*M \circlearrowleft$
spec		spec $(\mathcal{L}^t)$ = 
		construct $\mathcal{H}_c$ wrt dynamics on T^*M F^t
		use microlocal an.

instead of evolution of individual trajectories you study evolution of functions

Rem we don't know the spec.

But if $\lambda = 1$ simple dominant $\iff$ mixing

in space $\mathcal{H}_c$

if diffeo meas. pres ~~⇒ simple, dom.~~ ⇒ (unitary) $\mathcal{L}$ Anosov ~~ergodic~~ ⇒ then mixing

smooth meas. pres ⇒ $\lambda = 1$ simple dom.

conj: C^∞ Anosov diffeo on conn. comp mfld M is mixing