Notes on C^∞ and analytic vectors, and unitary representations of the Poincaré group*

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WORK IN PROGRESS!!

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1 Introduction

These notes review the rigorous justification for the calculus of infinitesimal generators of unitary representations of the Poincaré group, that is, iSL$(2,\mathrm{C})$. As physicists, we learn early that the self-adjointness of unbounded, Hermitian operators (unbounded observables), is nontrivial, that having a dense domain of definition is not enough for a unique self-adjoint extension, i.e., not enough for essential self-adjointness, but that any different extensions are interesting because they have physical interpretations, etc. Mostly, we are able to understand the issues, but even some of us on the mathematical physics spectrum are happy to rely on assurances by the mathematicians about the legitimacy of particular calculations, if we think it could matter.

In the case of unbounded observables belonging to Lie algebras of infinitesimal generators of continuous groups, the situation has actually been rather good for a long time. Since the elegant work of Gårding in 1947 [4], we have been free to operate on Gårding domains of C^∞ vectors, formed by smearing the unitary representation acting on any dense set of vectors with smooth functions on the group, with the assurance of essential self-adjointness. And Nelson's definitive, comprehensive, and dense but quite pedagogical work of 1959 [9] asserts the existence of common domains of essential self-adjointness consisting of analytic vectors.

Nelson mentions [9, p. 592] that the replacement of smooth by analytic smearing over the group in Gårding's construction gives analytic vectors,[1] and shows by extending a result of Gelfand [5] for bounded representations of a one-dimensional group of operators that smearing with the fundamental solution of the heat equation on the group produces a dense domain of analytic vectors for Banach space representations of the group. His theorem for Hilbert space representations uses instead analytic dominance by an elliptic operator in the enveloping algebra of the Lie algebra representation, with analyticity of the vectors defined by strong power series expansion of the action of the operators. Flato, Simon, Snellman, and Sternheimer [3] give an interesting condition for the common essential self-adjointness of the infinitesimal generators, and hence their integrability to one-parameter subgroup representations, called the "FS^3 criterion".

The Gårding construction is simple. It does use group manifold concepts that are not that common among physicists, but they are not that hard to understand. Physical interpretation of the resulting vectors is, however, at best indirect.

[1] He refers to Cartier and Dixmier [2] and Harish-Chandra [8] as sources for this remark.

The Nelson theorems for analytic vectors are in the category of full-blown "hard analysis", and the condition for common essential self-adjointness in the Hilbert space version is mathematically technical. The FS^3 criterion aims to be more practical, but remains technical.

For those unitary representations of the Poincaré group that describe elementary particles, the treatment below, starting with Section 6, gives conditions for C^∞ and analytic domains of essential self-adjointness characterized by regularity and growth properties of group-invariant domains of momentum space wave functions; still technical, but more in the sense of "soft analysis". It shares with the Gårding method the key simplification of invariance of the domains under the group representation, rather than just under the representation of the Lie algebra, which avoids Nelson's famous counter example.[2] And like the Gårding approach, it deals directly with analyticity of the vectors without power series expansion. As a natural development of ideas that have been around for a long time, we would not be surprised to learn that these results are already known. But since we have not been able to find references in the literature, it is possible that they appear here for the first time.

Section 2 reviews the standard theorems on unitary representations of continuous Abelian groups and the self-adjointness of their infinitesimal generators. Section 3 reviews the Gårding construction and its application to C^∞ vectors in Hilbert space. The Nelson theorem for analytic vectors in Hilbert space is reviewed in Section 4.

Sections 6 through 8 discuss applications to unitary representations of the Poincaré group that have physical energy-momentum. Cases with zero mass particles are a bit tricky.

The last section is a bilbiography. The most helpful references have been Reed and Simon, for self-adjointness and direct integrals, the Gårding and Nelson articles on, respectively, C^r and analytic vectors, and the wonderful Gel'fand and Shilov books on generalized functions, for spaces of analytic functions. The comprehensive book on group representations by Barut and Raczka covers, among many other things, group manifold concepts, the Gårding construction of C^∞ vectors, Nelson's analytic vector theorems, the Gårding-Nelson construction of analytic vectors based on the group heat equation, and the FS^3 criterion. We found the book's exposition of these topics to be quite good, and it provided a useful overall perspective.[3]

[2] Reed and Simon [12, 273]. See the end of Section 2 below for a brief description.

[3] Unfortunately legal copies of the book are not readily available in the United States.

2 Self-adjoint and essentially self-adjoint operators

The theorems and definitions in this section are quoted from Reed and Simon [12]. Unless otherwise stated, domains for linear operators are taken to be linear subspaces.

Theorem 1 (VIII.7 [12, 265])**.** Let A be a self-adjoint operator and define $U(t) = e^{itA}$. Then

(a) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.

(b) If $\varphi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$.

(c) For $\psi \in D(A)$, $\dfrac{U(t)\psi - \psi}{t} \to iA\psi$ as $t \to 0$.

(d) If $\lim\limits_{t\to 0} \dfrac{U(t)\psi - \psi}{t}$ exists, then $\psi \in D(a)$.

Theorem 2 (VIII.10 [12, 269])**.** Suppose that $U(t)$ is a strongly continuous one-parameter, unitary group. Let D be a dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly differentiable. Then i^{-1} times the strong derivative of $U(t)$ is essentially self-adjoint on D and its closure is the infinitesimal generator of $U(t)$.

The first part of the following theorem constructs a C^∞ Gårding [4] domain for the n-dimensional translation group. The second part is, as far as I know, the SNAG theorem.

Theorem 3 (VIII.12 [12, 270])**.** Let $\boldsymbol{t} \to U(\boldsymbol{t}) = U(t_1 \dots t_n)$ be a strongly continuous map of $\mathbb{R}^n$ into the unitary operators on a separable Hilbert space $\mathcal{H}$ satisfying $U(\boldsymbol{t}+\boldsymbol{s}) = U(\boldsymbol{t})U(\boldsymbol{s})$ and $U(0) = 1$. Let D be the set of finite linear combinations of vectors of the form

$$\varphi_f = \int_{\mathbb{R}^n} f(\boldsymbol{t})U(\boldsymbol{t})\,\varphi\,\mathrm{d}t\,, \qquad \varphi \in \mathcal{H}, \quad f \in C_0^\infty(\mathbb{R}^n)\,.$$

Then D is a domain of essential self-adjointness for each of the generators A_j of the one-parameter groups $U(0, 0, \dots, t_j, \dots, 0)$, each $A_j : D \to D$ and the A_j

commute, $j = 1, \ldots, n$. Furthermore, there is a projection-valued measure P_Ω on $\mathbb{R}^n$ so that

$$(\varphi, U(\boldsymbol{t})\,\psi) = \int_{\mathbb{R}^n} e^{it\cdot\lambda}\, \mathrm{d}(\varphi, P_\lambda \psi)$$

for all $\varphi, \psi \in \mathcal{H}$.

I assume that the subscript Ω in Theorem VIII.12 above refers to Borel sets of $\mathbb{R}^n$, by analogy with Reed and Simon's definition below for a single, bounded operator. In the definition, χ_Ω is the characteristic function of the set Ω.

Definition 1 ([12, 234])**.** Let A be a bounded self-adjoint operator and Ω a Borel set of $\mathbb{R}$. $P_\Omega \equiv \chi_\Omega(A)$ is called the *spectral projection* of A.

Reed and Simon's proof uses part (c) of Theorem VIII.13, quoted below, which is equivalent to a definition of commutation that is stronger than just commuting on a common domain of essential self-adjointness. Here are the definition and the theorem:

Definition 2 ([12, 271])**.** Two possibly unbounded self-adjoint operators A and B are said to *commute* if and only if all the projections in their associated projection-valued measures commute.

Theorem 4 (VIII.13 [12, 271])**.** Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$. Then the following three statements are equivalent:

(a) A and B commute.

(b) If $\operatorname{Im}\lambda$ and $\operatorname{Im}\mu$ are nonzero, then $R_\lambda(A)\, R_\mu(B) = R_\mu(B)\, R_\lambda(A)$.

(c) For all $s, t \in \mathbb{R}$, $e^{itA}\, e^{isB} = e^{isB}\, e^{itB}$.

In the theorem, $R_\lambda(A)$ and $R_\mu(B)$ are the resolvent operators for A and B, e.g., $R_\lambda(A) = (\lambda I - A)^{-1}$, which are bounded.

Reed and Simon [12, 273] give a counter example due to Nelson which shows that commutation on a common domain of essential self-adjointness is weaker, even with the requirement of invariance of the domain under both operators. Specifically, he constructs two operators A and B that are essentially self-adjoint and commuting on the same dense, invariant domain D, but such that the one-parameter groups e^{itA} and e^{isB} do *not* commute.

3 Gårding domains

The only formal theorem in Gårding's very short article [4] is about continuous Lie group representations by bounded operators on Banach spaces, without mentioning the special case of unitary representations on Hilbert spaces. The *Encyclopedia of Mathematics* currently has no article on Gårding domains. *Wikipedia* does have an article, which defines a Gårding domain as corresponding to a strongly continuous unitary representation of a topological group on a separable Hilbert space, where it is taken to be any common linear domain of essential self-adjointness for all infinitesimal generators of unitary representations of one-parameter subgroups which is invariant under the subgroup representations and their generators. This is different from the usage of the term in works by Nelson [9], Nelson and Stinespring [10], and others [3]. We shall return to this point at the end of this section.

Below is a paraphrase of the original Gårding theorem, except for part (d), which follows by a trivial extension of Gårding's proof.

Theorem 5 (Gårding [4])**.** Let G be an analytic Lie group with elements $g \in G$ represented continuously by bounded operators $T(g)$ on a Banach space $\mathcal{B}$. Let $\mu(\Omega) = \int_\Omega \mathrm{d}g$ be a left Haar measure on G, and let $\mathrm{C}_0^r(G)$ be the set of real continuous functions on the group manifold with continuous derivatives up to order $r \geq 0$, or all orders in case $r = \infty$, and with compact support. Let A be the infinitesimal generator[4] of any one-parameter subgroup of G, and let $D(A) \subset \mathcal{B}$ be its dense domain. Let $\mathcal{B}_r \subset \mathcal{B}$ be the set of vectors of the form

$$x(f) = \int_G f(g) T(g)\, x \,\mathrm{d}g\,, \quad f \in \mathrm{C}_0^r, \quad x \in \mathcal{B}\,.$$

Then

(a) $\mathcal{B}_{r+1}$ is dense in $\mathcal{B}$;

(b) $\mathcal{B}_{r+1} \subset D(A)$;

(c) $A\mathcal{B}_{r+1} \subset \mathcal{B}_r$.

(d) $T(g)\, \mathcal{B}_r = \mathcal{B}_r$ for all $g \in G$.

[4]An infinitesimal generator A for a strongly continuous representation of a one-parameter subgroup $h(t)$ may be defined up to a constant by the $t \to 0$ limit of $[T(h(t)) - I]x/t$ on the set $D(A)$ of $x \in \mathcal{B}$ for which the limit exists.

Proof. I'm not sure why part (a) of the theorem is not stated for r instead of $r+1$, since $\mathcal{B}_{r+1} \subset \mathcal{B}_r$, and the denseness of the smaller set implies that of the larger. Gårding's proof of (a) uses a standard C^∞ sequence of positive functions δ_n on the group manifold with unit Haar integral having compact support shrinking around the identity, for which $x(\delta_n) \to x$. The proof of (b) and (c) is based on the calculation,

$$T(h)\, x(f) = x(f^h)\,, \quad f^h(g) \equiv f(h^{-1}g)\,, \quad h \in G\,,$$

with h eventually restricted to one-parameter subgroups. Part (d) is immediate from the same calculation, because multiplication by a group element is a smooth map of the group manifold onto itself, so f^h is in $C_0^r(G)$ whenever f is. □

Theorem 6. Let $\mathcal{B}$ in Theorem 5 be a separable Hilbert space $\mathcal{H}$, and let $T(g)$ be a strongly continuous, unitary representation $U(g)$ of the group G. Then the symmetric infinitesimal generators of its one-parameter subgroups are self-adjoint, and have $\mathcal{B}_{r+1}$ as a common domain of essential self-adjointness for each $r \geq 0$.

Proof. Apply Theorem 2 to $\mathcal{B}_{r+1}$ and the unitary one-parameter groups, which inherit the strong continuity of $U(g)$. □

The only requirements for the proof of Theorem 6 are differentiability of the one-parameter subgroup representations, denseness of the domains of differentiability, and invariance of those domains under the one-parameter subgroups.

Definition 3. Let r be a nonnegative integer. A vector x in a Banach space $\mathcal{B}$ is said to be a *C^r vector* for a bounded continuous representation $T(g)$ of an analytic Lie group G on $\mathcal{B}$ if $T(g)x$ is strongly differentiable on the group manifold for all orders $\leq r$, where $r = 0$ denotes continuity, and $r = \infty$ denotes differentiability to all orders.

Theorem 7. Let $\mathcal{C}_r$ be the set of C^r vectors for the representation $T(g)$ with infinitesimal generators A in Theorem 5. Then

(a) $T(g)\, \mathcal{C}_r \subset \mathcal{C}_r$ for all $g \in G$;

(b) $\mathcal{C}_r$ is dense in $\mathcal{B}$;

(c) $\mathcal{C}_{r+1} \subset D(A)$;

(d) $A\mathcal{C}_{r+1} \subset \mathcal{C}_r$.

Proof. Part (a) follows from the group law of the representation and the fact that g acts as a smooth map of the group manifold onto itself. Part (b) follows from Theorem 5 because $\mathcal{B}^r \subset \mathcal{C}_r$. Parts (c) and (d) follow from differentiability of the one-parameter subgroups and the definition of C^r vectors. To be explicit, these results include $\mathcal{C}_0 = \mathcal{B}$ and $\mathcal{C}_\infty = \cap_r \mathcal{C}_r$. □

Theorem 8. Let $\mathcal{B}$ in Theorem 7 be a separable Hilbert space $\mathcal{H}$, and let $T(g)$ be a strongly continuous, unitary representation $U(g)$ of the group G. Then the symmetric infinitesimal generators of its one-parameter subgroups are self-adjoint, and have $\mathcal{C}_{r+1}$ as a common domain of essential self-adjointness for each $r \geq 0$.

Proof. Given Theorem 7, apply Theorem 2 to the restriction of $U(g)$ to the one-parameter subgroups of G. □

So, what domains should we call Gårding domains? From Theorems 6 and 8, the requirements of essential self-adjointness and invariance under the generators in the Wikipedia definition for Hilbert spaces appear superfluous. That definition requires differentiability, but does not mention Gårding's elegant construction of differentiable vectors by smearing over the group. Nelson [9, 589] calls the C^∞ case of Gårding's construction a *Gårding space*. We follow that style, restricting the *Gårding domain* terminology to C^r domains constructed by group smearing, as in Theorems 5 and 6, and using the terminology *dense, group-invariant* C^r *domain* for Wikipedia-style domains, like those in Theorems 7 and 8. The domains $\mathcal{C}_r$ are the largest dense, group-invariant C^r domains, automatically dense by Theorem 7; but Theorem 8 extends to smaller domains in the following way:

Theorem 9. Let $U(g)$ be a strongly continuous unitary representation of an analytic Lie group G on a separable Hilbert space $\mathcal{H}$. Let D be a dense $U(G)$-invariant C^{r+1} domain. The any symmetric infinitesimal generator A of a one-particle subgroup of G is essentially self-adjoint on D and $AD \subset \mathcal{C}_r$.

4 Analytic vectors

The short article on analytic vectors in *Encyclopedia of Mathematics* has a list of references. Among those, the "FS^3 criterion" paper by Flato, Simon, Snellman, and Sternheimer [3] is interesting and clarifying.

Nelson's original article, is explanatory and illuminating, but dense. In particular, we have found his calculus of absolute values, used for the efficient expression of statements and results, an unwanted conceptual load, which we have had to confront repeatedly when we have consulted the paper because it hasn't stuck. And it's really not that difficult.

commutative free monoid

free monoid

5 Poincaré Group Preliminaries

The irreducible representation for nonnegative mass and half-integer spin is the essential component for all three classes of representations of the Poincaré group for which we discuss C^∞ and analytic vectors in the following sections. The basic cases are the irreducible representation itself, finite tensor products of irreducible representations. Fock representations are countable direct sums of these.

and the Clebsch-Gordan reduction of finite tensor products.

6 Unitary irreducible representations of iSL$(2, \mathrm{C})$

The Lie group iSL$(2, \mathrm{C})$ is the simply connected covering group of the connected part of the inhomogeneous Lorentz group, known to physicists as the *Poincaré group*. The Poincaré group together with the discrete, homegeneous inversions for space, time, and total reflection is called the *extended Poincaré group*. The homogenous part of iSL$(2, \mathrm{C})$ is the set of unimodular 2×2 matrices, SL$(2, \mathrm{C})$; and the homogeneous part of the Poincaré group is the set of proper, orthochroous Lorentz transformations, $\mathrm{L}_+^\uparrow$.

The group manifolds for $\mathrm{L}_+^\uparrow$ and SL$(2, \mathrm{C})$ are, respectively,

$$\mathcal{M}_\mathrm{L} = \left\{\Lambda \in \mathrm{GL}(4, \mathrm{R}) : \Lambda^\mathrm{T} G \Lambda = I, \quad \Lambda^0{}_0 > 0\right\} \tag{1a}$$

(1b)

the Poincaré group

6.1 nonzero mass

6.2 zero mass

7 Fock representations of iSL$(2, \mathrm{C})$

7.1 with mass gap

7.2 without mass gap

8 Physical spectrum representations of iSL$(2, \mathbf{C})$

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