Raw notes on closed operators*

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1 Operators

Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$. Banach spaces are complete by definition. We use the term *normed linear space* when we want to consider normed spaces that are not necessarily complete.

Let $A : \mathcal{B} \to \mathcal{B}$ be a map with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$. We call A an *operator*. When A is linear, it is understood that $\mathcal{D}(A)$, and hence $\mathcal{R}(A)$, is linear.

The *graph* of A, denoted by $\Gamma(A)$, is the set of all pairs (x, Ax) with $x \in \mathcal{D}(A)$. A and its graph determine each other.

An *extension* of A is an operator on $\mathcal{B}$ whose graph contains the graph of A.

All of the discussion in these notes extends without difficulty to the situation where A is a mapping between different Banach spaces. The term *operator* is also commonly used for such mappings. Unless otherwise stated, all operators in these notes map within the same Banach space.

2 Closure

The *direct sum* $X_1 \oplus X_2$ of two normed linear spaces X_1 and X_2 is the set of pairs (x_1, x_2) of vectors in X_1 and X_2, equipped with the obvious linear structure and the norm $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$. The vectors x_1 and x_2 are called the *components* of (x_1, x_2).

The completion of the direct sum of two normed spaces is the direct sum of their completions.

The operator A is *closed* if its graph is closed in the Banach space $\mathcal{B} \oplus \mathcal{B}$.[1]

Equivalent definition: A is *closed* if, whenever both x_n and Ax_n are fundamental sequences, they obey $\lim x_n = x \in \mathcal{D}(A)$ and $\lim Ax_n = Ax$.

Lemma. If a closed operator A is either linear or antilinear, and if x_n is a fundamental sequence in $\mathcal{D}(A)$ such that Ax_n is also fundamental, then if x_n converges to zero, so does Ax_n.

Proof. Assume that both sequences are fundamental and that $x_n \in \mathcal{D}(A)$ converges to zero. By the sequential definition of being closed, zero is in $\mathcal{D}(A)$ and Ax_n converges to $A \cdot 0$, which is zero for linear or antilinear operators. □

[1] Reed and Simon, [4, p. 250].

An operator A is *closable* if the closure of its graph $\overline{\Gamma(A)}$ is the graph of an operator.[2] The corresponding operator, called the *closure* of A, is denoted by $\bar{A}$. Thus $\Gamma(\bar{A}) = \overline{\Gamma(A)}$.

Equivalent definition: A is *closable* if, whenever $x_n, y_n \in \mathcal{D}(A)$ are fundamental sequences with the same limit, and Ax_n and Ay_n are also fundamental, it follows that $\lim Ax_n = \lim Ay_n$.

If A is closable, $\bar{A}$ is its smallest closed extension.

The closure of a linear or antilinear operator is respectively a linear or antilinear operator.

Note that the sequence (x_n, y_n) is fundamental in $\mathcal{B} \oplus \mathcal{B}$ iff the component sequences x_n and y_n are both fundamental in $\mathcal{B}$.

If A is closed and bounded, then $\mathcal{D}(A)$ is closed, because if x_n is fundamental and A is bounded, then Ax_n is also fundamental, and since A is closed, $\lim(x_n, Ax_n) = (x, Ax)$ is in $\Gamma(A)$, so in particular $x \in \mathcal{D}(A)$.

The *closed graph theorem* states that if A is linear and $\mathcal{D}(A)$ is $\mathcal{B}$, then A is bounded iff A is closed.[3]

3 Inverse

An operator A is *injective* if, for $x, y \in \mathcal{D}(A)$, $Ax = Ay \implies x = y$, that is, $x \neq y \implies Ax \neq Ay$. An injective operator is said to be *one-to-one*, or 1-1.

If A is injective, its unique *inverse operator* A^{-1} is defined on $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$ by $A^{-1}y = x$ iff $y = Ax$. Note that A^{-1} is injective, with inverse A.

If A is not injective, it has no inverse.

The graph of the inverse of an injective operator A is the reverse of the graph of A. That is, $(y, x) \in \Gamma(A^{-1})$ iff $(x, y) \in \Gamma(A)$. The reversal operation is an isometry of $\mathcal{B} \oplus \mathcal{B}$ onto itself.

The inverse of a closed, injective operator is closed.

If A is closed and injective and its inverse is bounded, then $\mathcal{R}(A)$ is closed.

[2] This is the Wikipedia definition of closability. Reed and Simon [4, p. 250] define an operator as closable when it has a closed extension.

[3] Reed and Simon [4, p, 83] give a proof for the more general case where A is a linear map between Banach spaces.

4 Induced norm

In this section, we assume that A is either linear or antilinear, and that $\mathcal{D}(A)$, and hence $\mathcal{R}(A)$, is a linear space. That is, $A(x+y) = Ax+Ay$ for all $x, y \in \mathcal{D}(A)$; and for complex scalar multiplication, either $Acx = cAx$ for all c and all $x \in \mathcal{D}(A)$, or $Acx = \bar{c}Ax$ for all c and all $x \in \mathcal{D}(A)$.

When A is injective, its inverse is either linear or antilinear.

When A is injective and either linear or antilinear, the function defined on the linear space $\mathcal{D}(A)$ by $\|x\|_A \equiv \|Ax\|$ satisfies the axioms for a norm. It is called the *norm induced by* A. The obvious proof uses all of the mentioned properties.

If A is closed, the normed space defined by $\mathcal{D}(A)$ and the induced norm $\|\cdot\|_A$ need not be complete.

The norm defined on $\mathcal{D}(A)$ by $\|x\|_{\Gamma(A)} \equiv \|x\|+\|Ax\| = \|x\|+\|x\|_A$ is called the *graph norm* induced by A.

Two norms on the same linear space are said to be *compatible* if, whenever a sequence in the space is fundamental in both norms and converges to zero in one of the norms, it also converges to zero in the other.[4]

Theorem. If A is closed and injective, the norms $\|\cdot\|$ and $\|\cdot\|_A$ are compatible on $\mathcal{D}(A)$.

Proof. Suppose that x_n is fundamental in both $\|\cdot\|$ and $\|\cdot\|_A$. Then (x_n, Ax_n) is fundamental in $\mathcal{B} \oplus \mathcal{B}$.

Because (x_n, Ax_n) is fundamental and A is closed, if x_n converges to zero, then Ax_n converges to $A \cdot 0$, which is zero because A is linear or antilinear.

And if Ax_n converges to zero, then (x_n, Ax_n) converges to $(x, 0)$. Since A is not only closed, but injective, A^{-1} exists, and $(x_n, Ax_n) = (A^{-1}y_n, y_n)$, with $y_n \in \mathcal{D}(A^{-1}) = \mathcal{R}(A)$. Thus both y_n and $A^{-1}y_n$ are fundamental in $\|\cdot\|$. Since A^{-1} is closed, and $y_n = Ax_n$ converges to zero, it follows that $A^{-1}y_n$ converges to $x = A^{-1} \cdot 0$, which is zero because A^{-1} is linear or antilinear. □

If A is closed and injective, the norms $\|\cdot\|$ and $\|\cdot\|_{A^{-1}}$ are compatible on $\mathcal{R}(A) = \mathcal{D}(A^{-1})$, because A^{-1} is closed and injective.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the same linear space X are said to be *comparable*, with the first not stronger than the second and the second not weaker than the first, if $\|\cdot\|_1 \le C\|\cdot\|_2$ for some positive constant C.[5]

[4] Gelf'and and Shilov, [1, p. 13]. Perhaps they introduced this notion?

[5] Gel'fand and Shilov, [1, p. 12]. We follow their practice of relaxing the language by saying that the first is weaker and the second stronger.

Two comparable norms on the same linear space are *equivalent* if each is not weaker than the other. In that case they have the same fundamental sequences and generate the same topology.

If two norms on X are comparable, its completion X_2 in the stronger norm $\|\cdot\|_2$ has a *natural mapping* into its completion X_1 in the weaker norm $\|\cdot\|_1$, based on the fact that a fundamental sequence in the stronger norm is fundamental in the weaker [1, p. 13]. When restricted to X, this map is the same as simple inclusion; and the restricted map is thus injective. But the unrestricted map need not be one-to-one. Two points in the stronger completion X_2 may map into the same point in the weaker completion X_1. If that happens, at least one of the points in X_2 must be outside of X.

If the original space is complete in the stronger norm, the natural mapping into the weaker completion is the same as inclusion, and hence is injective.

If the original space is complete in both of two comparable norms, the norms are equivalent.[6]

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are comparable and also compatible, the natural mapping of the stronger into the weaker completion is injective [1, p. 14], and one can regard X, its stronger completion X_2 and its weaker completion X_1, to be related by inclusion: $X \subset X_2 \subset X_1$.

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are compatible but not comparable on X, the norms defined by

$$\|x\|_{\max} \equiv \max(\|x\|_1, \|x\|_2)$$

and

$$\|x\|_{\text{sum}} \equiv \|x\|_1 + \|x\|_2$$

are comparable to stronger than, and compatible with, each of $\|\cdot\|_1$ and $\|\cdot\|_2$. They are also equivalent: $\|\cdot\|_{\max} \leq \|\cdot\|_{\text{sum}}$ and $\|\cdot\|_{\text{sum}} \leq 2\,\|\cdot\|_{\max}$.

References

[1] I. M. Gel'fand and G. E. Shilov, *Spaces of Fundamental and Generalized Functions* (*Generalized Functions*, vol. 2), Academic Press, New York, 1968.

[6] Gel'fand and Shilov [1, p. 13] state that this is due to a "well-known theorem of Banach on the boundedness of an inverse operator", and refer to [2].

[2] L. A. Lyusternik and V. I. Sobolev, *Elements of Functional Analysis*, Ungar, New York, 1961.

[3] Edward Nelson, "Analytic Vectors", *Annals of Mathematics*, **70** (1959) 572–615.

[4] Michael Reed and Barry Simon, *Functional Analysis* (*Methods of Modern Mathematical Physics*, vol. I), Academic Press, New York, 1972.