# Distributional versus integrable derivatives, and delta functions*

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## Contents



# 1 Introduction

In one dimension, the delta function $\delta(x)$ is the classic example of a generalized function which is not locally Lebesgue integrable, i.e., not in $\mathrm{L}^1_{\mathrm{loc}}$, but is the distributional derivative of a generalized function $\theta(x)$, the Heaviside step function, which is locally integrable. The Heaviside function does have a continuous, hence $\mathrm{L}^1_{\mathrm{loc}}$ derivative, everywhere except at the origin. And this continuous almost everywhere derivative is almost everywhere equal to a function that is actually continuous everywhere, namely, zero.

This situation is described by saying that, although the Heaviside function has no continuous derivative at the origin, it does have a weak, distributional derivative.

In this note, we are interested in derivatives of inverse integral and fractional powers of the Euclidean norm $\|x\|^{-n}$, with $x \in \mathbb{R}^d$, and such powers multiplied by a step function or a logarithm, or occasional powers of a logarithm, and their role for integer $n$ in the traditional application of the $d$-dimensional Gauss-Ostrogradsky divergence theorem to arrive at the $d$-dimensional delta function and its derivatives. Good references for all of that are Schwartz [1], and, for one dimension, Lighthill [2].

For our purpose it is more interesting to ask whether derivatives of functions of $\|x\|$ fail to be locally integrable, rather than whether they fail to be continuous. In other words when making the distinction between ordinary and distributional derivatives, we consider locally integrable derivatives to be "ordinary". The notion of weak derivative in the theory of Sobolev spaces encapsulates exactly what is needed for this distinction, namely, the existence of the weak Sobolev derivative entails its local integrability.[1] With that distinction, the Heaviside function has a distributional derivative but not a weak Sobolev derivative.

The last section, on nonintegrable inverse powers of distance, raises the question of how to write integer inverse powers as invariant derivatives of locally integrable powers, or powers multiplied by a logarithm.

We use the *gradient form* of the divergence theorem in various proofs. It writes any component of the $d$-dimensional gradient of a function as a divergence with the same regularity: $\partial_i f = \partial \cdot (f \partial_i x)$.

## 2 Distributional versus weak Sobolev derivatives

Let $f \in \mathcal{D}(\mathbb{R}^d)$, the Schwartz space of smoooth functions with compact support, and $h \in \mathcal{D}'(\mathbb{R}^d)$, the Schwartz space of distributions. This discussion keeps a distinction between the two common notations for the linear functional defined by $h$ on $\mathcal{D}(\mathbb{R}^d)$, $\langle h,\ f\rangle$ and $\int hf\,\mathrm{d}x$. Namely, we use the second notation only when $hf$ is actually an absolutely integrable function, and the integral is a Lebesgue integral. The two coincide when $h \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, i.e., every $x \in \mathbb{R}^d$ has a neighborhood $U$ such that $h \in \mathrm{L}^1(U)$.

**Fact 1.** It is a basic property of distributions that $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$.

In particular we want to distinguish between the locally integrable derivative of a locally integrable function, which is the Sobolev weak derivative de-

[1] There are also strong Sobolev derivatives, defined as limits with Sobolev norms.

fined below, and the distributional derivative. The notation $\partial$ is used for the $d$-dimensional gradient or partial derivative operator on continuously differentiable functions, with $\partial_{\mathcal{D}'}$ and $\partial_{\mathrm{ws}}$ for the distributional and weak Sobolev versions, respectively.

**Definition 1.** Let $h$ belong to $\mathcal{D}'(\mathbb{R}^d)$. Its *distributional derivative* is the distribution $\partial_{\mathcal{D}'} h \in \mathcal{D}'(\mathbb{R}^d)$ defined by

$$\langle \partial_{\mathcal{D}'} h,\, f \rangle = -\langle h,\, \partial f \rangle\,. \tag{2.1}$$

**Definition 2.** Let $U \subset \mathbb{R}^d$ be open and nonempty. Let the function $h$ and all components of the vector $u$ belong to $\mathrm{L}^1(U)$. Then $u$ is defined as the *weak Sobolev derivative* of $h$ on $U$, $u = \partial_{\mathrm{ws}}\, h$, iff the following is a Lebesgue integral identity for all $f \in \mathcal{D}(\mathbb{R}^d)$ with support in $U$:

$$\int u f \,\mathrm{d}x = -\int h\, \partial f \,\mathrm{d}x\,, \qquad \mathrm{d}x \equiv \mathrm{d}x_1 \ldots \mathrm{d}x_d\,. \tag{2.2}$$

**Fact 2.** When the weak Sobolev derivative exists on $U$, and $h$ has a continuous derivative there, the two agree almost everywhere on $U$. When the weak Sobolev derivative exists on $\mathbb{R}^d$, it concides with the distributional derivative, whether or not it is continuous. The weak Sobolev derivative is unique up to sets of Lebesgue measure zero, and shares various properties with continuous derivatives, such as implying some form of absolute continuity of $h$.

In the following sections, we shorten "Lesbesgue measurable" to "measurable"; and "almost everywhere" refers to Lesesgue measure.

**Fact 3.** Any function almost everywhere continuous on $\mathbb{R}^d$ defines a measurable function on $\mathbb{R}^d$.

**Fact 4.** Any function continuous on $\mathbb{R}^d$ belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$.

**Fact 5.** Any function piecewise continuous[2] on $\mathbb{R}$ defines a measureable function that belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$.

The following basic fact is an immediate consequence of Lebesgue dominated convergence:

[2] *Piecewise continuous* means continuous on intervals with finite limits at endpoints.

**Fact 6.** Let $h \in L^1(\mathbb{R}^d)$. Then for any point $a \in \mathbb{R}^d$,

$$\int h \, dx = \lim_{\varepsilon \to 0} \int_{\|x-a\|>\varepsilon} h \, dx \,. \tag{2.3}$$

As a simple example of how a proof of Sobolev differentiability works, consider the following.

**Lemma 1.** In one dimension, the continuous function $|x|$ has the weak Sobolev derivative

$$\frac{d|x|}{dx} = \operatorname{sgn} x \,. \tag{2.4}$$

*Proof.* We need to prove that $h = |x|$ and $u = \operatorname{sgn} x$ satisfy the conditions in Definition 2 for $u$ to be the weak Sobolev derivative of $h$. According to Facts 4 and 5, both $h$ and $u$ satisfy the basic requirement of local integrability. Then for any $f \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, we calculate:

$$\int f \operatorname{sgn} x \, dx = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} f \, \frac{d|x|}{dx} \, dx \,, \tag{2.5a}$$

$$= \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \left[ \frac{d}{dx}(f \, |x|) - \frac{df}{dx} \, |x| \, dx \right] , \tag{2.5b}$$

$$= \lim_{\varepsilon \to 0} [-f(\varepsilon)\varepsilon + f(-\varepsilon)\varepsilon] - \int \frac{df}{dx} \, |x| \, dx \,, \tag{2.5c}$$

$$= - \int \frac{df}{dx} \, |x| \, dx \,. \tag{2.5d}$$

Equations (2.5a) and (2.5c) follow from the application of Fact 6 to the convergence of one-dimensional integrals excluding a small interval. ■

# 3 Derivatives of powers of distance

Important examples of the relationship between weak and distributional derivatives are provided by powers of the Euclidean distance $r \equiv \|x\| \geq 0$ from the origin in $\mathbb{R}^d$, and by powers of $r$ multiplied by powers of $\ln r$. In the following, $n$ and $m$ appearing in powers can be any real number, unless there is an explicit qualification.

**Definition 3.** The principal branch for real powers of the logarithm is defined by

$$\ln^m r = |\ln r|^m \left[\theta(r-1) + \theta(1-r)e^{im\pi}\right], \tag{3.1}$$

with the understanding that real powers of positive numbers are positive, and that $\ln^0 r = |\ln r|^0 = 1$.

**Fact 7.** Basic properties of $\ln^m r$:

i) The factor multiplying $|\ln r|^m$ is piecewise constant, so $\ln r$ is smooth on $\mathbb{R}^d$ when $r$ is neither zero nor one. The factor turns out to be removable in certain arguments.

ii) When $m > 0$, $\ln^m r$ is continuous on $\mathbb{R}^d$ except at $r = 0$, where it diverges.

iii) When $m < 0$, $\ln^m r$ is continuous on $\mathbb{R}^d$ except at $r = 1$, where it diverges. The direction of the divergence is discontinuous when $m$ is not an even, negative integer.

**Fact 8.** The functions defined almost everywhere by $\ln^m r/r^n$ at $r \neq 0$, and possibly at $r \neq 1$, are measurable on $\mathbb{R}^d$. So are all components of the unit vector $\hat{x} = x/r$ and of the vectors $\hat{x} \ln^m r/r^n$. Note that when $d = 1$, the single component of the vector $\hat{x}$ is $\operatorname{sgn} x$.

From now on, when we say that a vector has some property like being measurable, or integrable, or differentiable, we mean that every component has that property. For example, we might state the obvious fact that if $h$ is measureable on $\mathbb{R}^d$, then so is every component of $\hat{x}h$,[3] as "If $h$ is measurable, then so is $\hat{x}h$."

**Fact 9.** For real $n$ and $m$, the derivatives

$$\partial \frac{\ln^m r}{r^n} = \frac{\hat{x}}{r^{n+1}} \left(-n \ln^m r + m \ln^{m-1} r\right) \tag{3.2}$$

exist and are smooth on $\mathbb{R}^d$ for $r \neq 0$, and possibly $r \neq 1$. The r.h.s. defines measurable vectors on all of $\mathbb{R}^d$.

[3] Obvious because, according to Fact 8, every component of $\hat{x}$ is measurable, and the product of measurable functions is measurable.

It is a picky point that Fact 9 does not say that any of the *derivatives* is measurable at $r = 0$, or possisbly $r = 1$, not even when the measurable function defined almost everywhere by the r.h.s. is locally integrable there. We use that language only when the weak Sobolev derivative exists; and that remains to be proved or disproved, depending on $n$ and $m$, which is done in Theorem 2 at the end of this section. As mentioned in the *Introduction*, the case of the derivative of the step function in one dimension shows that the distinction is not empty; and that is amplified by the examples in Section 4, where the Laplacian and higher derivatives on certain functions of $r$ produce distributions with point support.

**Lemma 2.** The measureable functions and vectors $\ln^m r/r^n$ and $\hat{x}\ \ln^m r/r^n$ belong to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ if and only if $n < d$.

**Theorem 1.** The measureable functions and vectors $\ln^m r/r^n$ and $\hat{x}\ \ln^m r/r^n$ are locally integrable on $\mathbb{R}^d$ as follows:

i) at all $x$ with $r \neq 0$ and $r \neq 1$;

ii) at $x = 0$ iff $n < d$;

iii) at points with $r = 1$ iff $m > -1$.

*Proof.* The lemma for the vectors follows from that for the functions, because $\hat{x}h$ is locally integrable iff $\|\hat{x}h\| = |h|$ is. So the rest of the proof deals with $h = \ln^m r/r^n$.

At points excluding $r = 0$ and $r = 1$, the result follows from the smoothness of $\ln r$, Fact 7, and of $r$, together with the agreement of the continuous and weak Sobolev derivatives, Fact 2. That takes care of case i).

The proofs for $r = 0$ and $r = 1$ both involve explicit limits of integrals of continuous or piecewise continuous functions of $r$ over a bounded, spherically symmetric region in $\mathbb{R}^d$ which excludes a small neighorhood of $r = 0$ or $r = 1$, and does not include, respectively, $r = 1$ or $r = 0$. Namely, let $b < 1$, let $a$ be the radius zero or one, and consider the limit:

$$\int\limits_{b>|r-a|>\varepsilon} h\,\mathrm{d}x = \int\limits_{b>|r-a|>0} \theta(|r-a|-\varepsilon)\,h\,\mathrm{d}x\,, \tag{3.3a}$$

$$= \int\limits_{b>|r-a|>\varepsilon} h\,r^{d-1}\mathrm{d}r \int \mathrm{d}\Omega\,, \tag{3.3b}$$

$$= S_d \int\limits_{b>|r-a|>\varepsilon} h\,r^{d-1}\mathrm{d}r\,, \tag{3.3c}$$

$$S_d \equiv \int \mathrm{d}\Omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{3.3d}$$

To prove the lemma for the functions, we need only consider the integral in a spherical, $d$-dimensional volume of finite radius $b < 1$ centered at the origin, since the discontinuity in the derivative of the log powers at $r = 1$ is not problem for local integrability there. Let $h(r)$ be $|\ln r|^m/r^n$, which is equal to $\ln^m r/r^n$ up to a constant sign or complex factor, according to Eq. (3.1). The basic calculation is:

$$\int\limits_{b>r>\varepsilon} h\,\mathrm{d}x = \int\limits_{b>r>0} \theta(r-\varepsilon)\,h\,\mathrm{d}x\,, \tag{3.4a}$$

$$= \int_\varepsilon^b h\,r^{d-1}\mathrm{d}r \int \mathrm{d}\Omega\,, \tag{3.4b}$$

$$= S_d \int_\varepsilon^b h\,r^{d-1}\mathrm{d}r\,, \tag{3.4c}$$

$$S_d \equiv \int \mathrm{d}\Omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{3.4d}$$

In (3.4b), $\mathrm{d}\Omega$ is the solid angle element subtended by the hypersphere centered at the origin in $d$ dimensions, and the total solid angle in (3.4d) is the finite area of the unit hypersphere. The argument is to apply the Lebesgue monotone convergence theorem to a subsequence of the limit $\varepsilon \to 0$ on the r.h.s. of (3.4a). This is a limit of finite integrals over the sphere of a sequence of measurable functions $\theta(r-\varepsilon)\,h$, which is a pointwise, nonnegative, monotone increasing sequence when the functions are defined as zero on the set $r < \varepsilon$. Then $h$ is locally integrable at zero precisely when the limit is finite.

It remains to discuss the cases for the one-dimensional integral of $h$. Recall that $m$ is nonnegative.

i) For $n \neq d$ and integer $m$, the following,[4] which is the integral of $hr^{d-1}$ up to a sign, is by inspection convergent at $\varepsilon = 0$ for $n < d$, and divergent for $n > d$ :

$$\begin{aligned}\int_{\varepsilon}^{b} & r^{d-1-n} \ln^m r \, \mathrm{d}r \\ &= \frac{r^{d-n}}{m+1} \sum_{k=0}^{m} (-1)^k (m+1)(m) \ldots (m-k+1) \frac{\ln^{m-k} r}{(d-n)^{k+1}} \Bigg|_{\varepsilon}^{b}\end{aligned} \tag{3.5a}$$

ii) For $n < d$ and noninteger $m$, the integral on the l.h.s. of (3.5a) converges by Lebesgue dominated convergence, using any of the absolute values of the integrands with integer log powers larger than $m$ as the dominating function.

iii) For $n > d$ and noninteger $m$, the integral on the l.h.s. of (3.5a) diverges, because up to a constant factor the integrand is greater than that for any nonnegative integer less than $m$. That even works when the integer is zero, so $\ln^0 r = 1$, by choosing $b$ so that $|\ln^m r| \geq |\ln^m b| \geq 1$ in the range of integration.

iv) For $n = d$ and real $m$, the following integral,[5] whose integrand is positive up to a constant factor, is divergent at $\varepsilon = 0$ :

$$\int_{\varepsilon}^{b} \frac{\ln^m r}{r} \, \mathrm{d}r = \frac{\ln^{m+1} r}{m+1} \Bigg|_{\varepsilon}^{b} \tag{3.5b}$$

That covers all the cases, so $h$ is locally integrable at zero, and therefore on all of $\mathbb{R}^d$, if and only if $n < d$ . ■

The main result of this section is the following:

**Theorem 2.** Let $U \subset \mathbb{R}^d$ be open and nonempty, let $h$ be either $1/r^n$ or $\ln r / r^n$, and let $u$ be the corresponding r.h.s. of (3.2).

i) If $0 \notin U$, then $\partial_{\mathrm{ws}} h$ exists on $U$ for all $n$ and is almost everywhere equal to the continuous vector $u$.

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[4] Gradshteyn and Ryzik [3], 2.722, p. 203.

[5] *Ibid.*, 2.721.2, p. 203.

ii) If $0 \in U$ and $n+1 < d$, then $\partial_{\mathrm{ws}} h$ exists on $U$ and is almost every equal to the measurable vector $u$.

iii) If $0 \in U$ and $n+1 \geq d$, then $\partial_{\mathrm{ws}} h$ does not exist on $U$.

*Proof.*

i) Since $0 \notin U$, the proof is an exercise in the application of the Lebesgue integral integration by parts identity (2.2) as the essential definition of the weak Sobolev derivative, together with the gradient form of the $d$-dimensional divergence theorem, to differentiable functions with compact support. ■

ii) Since $n+1 < d$, Lemma 1 says that both $u$ and $h$ are locally integrable, so by Fact 6:

$$\int u f \,\mathrm{d}x = \lim_{\varepsilon \to 0} \int_{r>\varepsilon} u f \,\mathrm{d}x \,, \tag{3.6a}$$

$$= \lim_{\varepsilon \to 0} \int_{r>\varepsilon} (\partial h) f \,\mathrm{d}x \,, \tag{3.6b}$$

$$= \lim_{\varepsilon \to 0} \int_{r>\varepsilon} [\partial(hf) - h\partial f] \,\mathrm{d}x \,, \tag{3.6c}$$

$$= -\lim_{\varepsilon \to 0} \left[ h(\varepsilon)\varepsilon^{d-1} \int_{r=\varepsilon} \hat{x} f \,\mathrm{d}\Omega \right] - \int h\partial f \mathrm{d}x \,, \tag{3.6d}$$

$$= -\int h\partial f \mathrm{d}x \,. \tag{3.6e}$$

The minus sign in front of the limit in (3.6d) comes from the application of the vector form of the divergence theorem, with $\hat{x}$ the outward normal pointing into the hole made by exclusion of the small inner sphere. Of course $f$ vanishes on the surface of the outer sphere of radius $b$. The limit vanishes because the factor $h(\varepsilon)\,\varepsilon^{d-1}$ is either $\varepsilon^{d-1-n}$ or $\varepsilon^{d-1-n}\ln\varepsilon$, both of which go to zero; and the rest of the inner surface integral goes to $f(0)\int \hat{x}\,\mathrm{d}\Omega$, which is not only bounded, but vanishes from spherical symmetry. ■

iii) If $n+1 \geq d$ and the weak Sobolev derivative exists, it is almost everywhere equal to $u$, which is not locally integrable according to Lemma 1, a contradiction. ■

# 4 Distributions with point support

Without loss of generality, we consider only distributions with support at $x = 0$. It is well-known that any such distribution is a finite linear combination of the delta function and its derivatives.

Below we reproduce the classic proof of the classic formula for the delta function in $d > 0$ dimensions:

$$-\operatorname{sgn}(d-2) \triangle \frac{1}{r^{d-2}} = |d-2|\; S_d\, \delta(x), \qquad d \neq 2 \tag{4.1a}$$

$$-\triangle \ln \frac{1}{r} = S_2\, \delta(x)\,. \qquad d = 2 \tag{4.1b}$$

See the end of this section for a list with the coefficients spelled out through four dimensions, using Eq. (3.4d) for $S_d$.

We actually do the proof for arbitrary derivatives of the delta function, for which we use the Schwartz derivative monomial notation:

$$m = (m_1, ...m_d), \qquad |m| = m_1 + ... + m_d\,, \tag{4.2a}$$

$$D^m = \frac{\partial^{m_1}}{\partial {x_1}^{m_1}} \cdots \frac{\partial^{m_d}}{\partial {x_d}^{m_d}}. \tag{4.2b}$$

There should be no confusion with the use of $m$ as a power in Sec. 2, because $\ln^m r$ does not occur in this section.

In the statement and proof of the theorem we keep the pedantic weak Sobolev derivative notation, to make explicit what quantities are and are not $L^1_{\text{loc}}$. Often one does not care, and any special notation is left out, as in Eqs. (4.1a) and (4.1b), with the understanding that the weak derivative is intended. That's part of the charm of generalized functions—they can be differentiated without worry.

**Theorem 3.** For dimension $d > 0$ FIXME!!:

$$-\operatorname{sgn}(d-2)\, D_{\mathrm{w}}^{m}\, \partial_{\mathrm{w}} \cdot \partial \frac{1}{r^{d-2}} = |d-2|\ S_d\, D_{\mathrm{w}}^{m}\, \delta(x), \quad d \neq 2 \tag{4.3a}$$

$$-D_{\mathrm{w}}^{m}\, \partial_{\mathrm{w}} \cdot \partial\, \ln \frac{1}{r} = S_d\, D_{\mathrm{w}}^{m}\, \delta(x)\,. \qquad d = 2 \tag{4.3b}$$

*Proof.* First we need some convenience definitions:

$$h = \operatorname{sgn}(d-2) \frac{1}{r^{d-2}}\,, \qquad \partial h = -\,|d-2|\, \frac{x}{r^{d}}\,, \qquad d \neq 2 \tag{4.4a}$$

$$h = \ln \frac{1}{r}\,, \qquad \partial h = -\frac{x}{r^{2}}\,, \qquad d = 2 \tag{4.4b}$$

$$\partial h = -c_d\, \frac{x}{r^{d}}\,. \qquad d > 0 \tag{4.4c}$$

Note that in every dimension both $h$ and $\partial h$ are $\mathrm{L}^{1}_{\mathrm{loc}}$ and polynomial bounded. Furthermore, the divergence $\partial \cdot \partial h$ is defined, and vanishes at $x \neq 0$ because

$$\partial \cdot \frac{x}{r^{n}} = (d-n) \frac{1}{r^{n}}\,, \qquad x \neq 0\,. \tag{4.5}$$

Then we calculate:

$$\langle D_{\mathrm{w}}^{m}\, \partial_{\mathrm{w}} \cdot \partial\, h,\ f \rangle = (-1)^{|m|+1} \int (\partial\, h) \cdot \partial\, D^{m} f\, \mathrm{d}x\,, \tag{4.6a}$$

$$= (-1)^{|m|+1} \lim_{\varepsilon \to 0} \int_{r > \varepsilon} (\partial\, h) \cdot \partial\, D^{m} f\, \mathrm{d}x\,, \tag{4.6b}$$

$$= (-1)^{|m|+1} \lim_{\varepsilon \to 0} \int_{r > \varepsilon} \partial \cdot [(\partial\, h)\, D^{m} f]\ dx\,, \tag{4.6c}$$

$$= (-1)^{|m|} \lim_{\varepsilon \to 0} \int_{r = \varepsilon} \hat{x} \cdot (\partial\, h) D^{m} f\, \varepsilon^{d-1} \mathrm{d}\Omega\,, \tag{4.6d}$$

$$= -(-1)^{|m|} D^{m} f(0)\, c_d \int \mathrm{d}\Omega\,, \tag{4.6e}$$

$$= -c_d\, S_d\, \langle D_{\mathrm{w}}^{m}\, \delta,\ f \rangle. \tag{4.6f}$$

Here are the steps:

(4.6a) The definition of $D^m_{\mathrm{w}}\partial_{\mathrm{w}}$, and local integrability.

(4.6b) The dominated convergence property of distributions.

(4.6c) Integration by parts in (4.6b), and vanishing of the divergence of $\partial h$.

(4.6d) The $d$-dimensional divergence theorem.

(4.6e) Convergence in (4.6d) is dominated by $\sup |D^m f|$, the definition of $\partial h$ in (4.4c), and the fact that, at $r = \varepsilon$,

$$\frac{\hat{x}\cdot x}{r^d} = \frac{1}{r^{d-1}} = \frac{1}{\varepsilon^{d-1}}\,. \tag{4.7}$$

(4.6f) The definition of $D^m_{\mathrm{w}}\delta$.

(4.6f) Proves the theorem, by inspection of the definition of $c_d$ in Eqs. (4.4a–4.4c). ■

Here is the promised list of spelled-out formulas, with the weak derivative understood:

$$-\frac{\mathrm{d}^m}{\mathrm{d}x^m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(-|x|) = 2\frac{\mathrm{d}^m}{\mathrm{d}x^m}\delta(x), \qquad d = 1 \tag{4.8a}$$

$$-D^m \triangle \ln\frac{1}{r} = 2\pi\, D^m\delta(x), \qquad d = 2 \tag{4.8b}$$

$$-D^m \triangle \frac{1}{r} = 4\pi\, D^m\delta(x), \qquad d = 3 \tag{4.8c}$$

$$-D^m \triangle \frac{1}{r^2} = 2\pi^2 D^m\delta(x). \qquad d = 4 \tag{4.8d}$$

# 5 Nonintegrable inverse powers of distance

In one dimension, fractional and integral inverse powers are definable as distributions, which agree with the ordinary functions at $x \neq 0$, by taking derivatives of locally integrable functions:

$$|x|^\alpha = \frac{1}{(\alpha+1)...(\alpha+n)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(\operatorname{sgn} x)\ |x|^{\alpha+n}\right], \quad \alpha + n > -1\,, \tag{5.1a}$$

$$\frac{1}{|x|^n} = \frac{1}{(n-1)!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(-\operatorname{sgn} x)^n \ln \frac{1}{|x|}\right]. \tag{5.1b}$$

In Eq. (5.1a), $\alpha$ is not allowed to be a negative integer between $-1$ and $-n$.

Equation (5.1b) chooses a particular way of resolving the delta function derivative ambiguity in the definition of inverse integer powers. Namely, it obeys the following distributional identity:

**Lemma 3.** Let $|x|^{-n}$ be defined as a distribution for integer $n \geq 0$ by Eq. (5.1b). Then

$$x^n \frac{1}{|x|^n} = (\operatorname{sgn} x)^n\,. \tag{5.2}$$

*Proof.* This is easily proved by using the dominated convergence property and integration by parts on

$$\left\langle \varphi,\, x^n \frac{1}{|x|^n} \right\rangle = \frac{(-1)^n}{(n-1)!} \lim_{\varepsilon\to 0} \int\limits_{|x|>\varepsilon} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^n\varphi) \operatorname{sgn} x \ln \frac{1}{|x|^n}\, \mathrm{d}x\,. \ \blacksquare \tag{5.3a}$$

It is known that inverse integer powers of the $n$-dimensional distance $\|x\| = r$ are definable as distributions which coincide with the ordinary functions at $x \neq 0$, by the *partie finie* construction, which is laid out in Schwartz. It is also known that every distribution can be written as a finite sum of derivatives of locally integrable, even continuous functions. Since the distance is Euclidean invariant, it is not surprising that such a sum can be written in terms of invariant derivatives.

Some useful formulas for arbitrary real $\alpha$ at $x \neq 0$:

$$\partial\, r = \frac{x}{r^2}\,, \qquad \partial \ln r = \frac{x}{r^2}\,, \tag{5.4a}$$

$$\partial \frac{1}{r^\alpha} = -\frac{\alpha\, x}{r^{\alpha+2}}\,, \qquad \partial\left(\frac{1}{r^\alpha}\ln\frac{1}{r}\right) = -\frac{x}{r^{\alpha+2}}\left(1+\alpha\ln\frac{1}{r}\right), \tag{5.4b}$$

$$x\cdot\partial \frac{1}{r^\alpha} = -\frac{\alpha}{r^\alpha}\,, \qquad x\cdot\partial\left(\frac{1}{r^\alpha}\ln\frac{1}{r}\right) = -\frac{1}{r^\alpha}\left(1+\alpha\ln\frac{1}{r}\right), \tag{5.4c}$$

$$\triangle \frac{1}{r^\alpha} = \frac{\alpha\,(\alpha+2-d)}{r^{\alpha+2}}\,, \qquad \triangle\left(\frac{1}{r^\alpha}\ln\frac{1}{r}\right) = \frac{1}{r^{\alpha+2}}\left[2\alpha+2-d + \alpha(\alpha+2-d)\ln\frac{1}{r}\right]. \tag{5.4d}$$

The first equation in (5.4d) defines a distribution for any fractional power that is not locally integrable by iteration of the laplacian on an integrable power. It also serves for integer inverse powers with $\alpha+2 < d$.

The second equation in (5.4d) works for $\alpha+2 = d$:

$$\triangle\left(\frac{1}{r^{d-2}}\ln\frac{1}{r}\right) = \frac{d-2}{r^d}\,. \tag{5.5}$$

Now that $r^{-d}$ is defined as a distribution, the laplacian can be iterated on it to get the remaining inverse integer powers, via the first equation in (5.4d).

Again, the definition for inverse integer powers $n \geq d$ picks out a particular resolution of the delta function derivative ambiguity at $x = 0$.

**Lemma 4.** As defined above by derivatives, $\|x\|^{-n}$ for integer $n \geq d$ obeys the distributional identity:

$$x\,r^{n-1}\frac{1}{r^n} = \frac{x}{r}\,, \qquad n \text{ odd} \tag{5.6a}$$

$$r^n\frac{1}{r^n} = 1\,. \qquad n \text{ even} \tag{5.6b}$$

*Proof.* Apply the same argument as in the proof of Lemma 3. ■

# References


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