Direct Integrals and Improper Dirac States*

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This sketch discusses the correspondence between some basic direct integral notions and the Dirac improper, continuum state formalism. The correspondence is simple for tensor product states, with the application to general states induced by linear extension and Hilbert space completion. The tensor product context restricts the direct integral to the case where it is defined over a single measure space, and the component Hilbert spaces are all the same (isometric).[1] The general case can be recovered by taking countable direct sums of tensor product spaces like the following, with the insertion of appropriate measure spaces. Let

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\,, \tag{1a}$$

$$\mathcal{H}_1 = \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}p)\,, \tag{1b}$$

$$\mathcal{H}_2 = \{\text{separable Hilbert space, possibly finite dimensional}\}. \tag{1c}$$

We are going to regard $\mathcal{H}$ as a direct integral over $\mathrm{d}p$.

To set that up, note that $\mathcal{H}$ is equivalent to $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}p, \mathcal{H}_2)$, with $\psi \in \mathcal{H}$ represented by Lesbesgue-measureable, vector-valued functions ψ_p on $\mathbb{R}^n$ with values in $\mathcal{H}_2$,[2] and with inner product

$$\langle \psi, \psi \rangle = \int \left\langle \psi_p, \psi_p \right\rangle_2 \,\mathrm{d}p\,. \tag{2}$$

[1] Reed and Simon [1, p. 280] call this a *constant fiber direct integral*.

[2] Weak measureability of vectors in $\mathcal{H}_2$ is defined as the measurability of $\left\langle \chi, \psi_p \right\rangle_2$ for all $\chi \in \mathcal{H}_2$. Reed and Simon [2, p. 115] have a discussion of the equivalence between weak and strong measureablity.

The Hilbert space $L^2(\mathbb{R}^n, dp, \mathcal{H}_2)$ is said to be the direct integral over dp of the Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{R}^n$, with $\mathcal{H}_p = \mathcal{H}_2$ for all p. The following is a direct integral notation for the identification $\mathcal{H} = L^2(\mathbb{R}^n, dp, \mathcal{H}_2)$:

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_p \, dp \,, \qquad \mathcal{H}_p = \mathcal{H}_2 \,, \tag{3a}$$

$$\psi = \int^{\oplus} \psi_p \, dp \in \mathcal{H} \,, \qquad \psi_p \in \mathcal{H}_2 \,. \tag{3b}$$

This is a continuous form of the notation for discrete direct sums of Hilbert spaces,

$$\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i \,, \tag{4a}$$

$$\psi = \bigoplus_{i=1}^{\infty} \psi_i \,, \qquad \psi_i \in \mathcal{H}_i \tag{4b}$$

$$\langle \psi, \psi \rangle = \sum_{i=1}^{\infty} \langle \psi_i, \psi_i \rangle_i \;, \tag{4c}$$

for the case where all of the $\mathcal{H}_i$'s are the same.

Now consider product states:

$$\psi = \phi \otimes \chi \in \mathcal{H} \,, \quad \phi \in \mathcal{H}_1 \,, \quad \chi \in \mathcal{H}_2 \,, \tag{5a}$$

$$\langle \psi, \psi \rangle = \langle \phi, \phi \rangle_1 \, \langle \chi, \chi \rangle_2 \;, \tag{5b}$$

$$\langle \phi, \phi \rangle_1 = \int |\phi(p)|^2 \, dp \,. \tag{5c}$$

Dirac improper state notation applied to $\mathcal{H}_1$ gives:

$$\langle p \,|\, \phi \,\rangle \equiv \phi(p) \,. \tag{6a}$$

$$\langle \phi, \phi \rangle_1 = \int \langle \phi \,|\, p \,\rangle dp \langle \, p \,|\, \phi \rangle, \tag{6b}$$

$$\mathbb{1}_1 = \int |\, p \,\rangle dp \langle \, p \,| \,. \tag{6c}$$

Statements like $\langle p \,|\, \psi \,\rangle \in \mathcal{H}_2$ for almost every p make sense for product states, with the definition

$$\langle p \,|\, \psi \,\rangle \equiv \langle p \,|\, \phi \,\rangle \, \chi = \phi(p) \, \chi \,. \tag{7}$$

For such states we have

$$\psi = \int |\, p \,\rangle \mathrm{d}p \langle\, p \,|\, \phi \rangle \chi \,. \tag{8}$$

If we identify

$$\psi_p = \langle p \,|\, \phi \,\rangle \, \chi \,, \tag{9}$$

we get the direct integral identity

$$\psi = \int^{\oplus} \psi_p \, \mathrm{d}p = \int |\, p \,\rangle \mathrm{d}p \langle\, p \,|\, \phi \rangle \, \chi \,. \tag{10}$$

Now, for general states $\psi \in \mathcal{H}$, we define $\langle p \,|\, \psi \,\rangle \in \mathcal{H}_2$, with improper $\langle\, p \,|$ corresponding to $\mathcal{H}_1$, by linear extension and Hilbert space completion from Eq. (7):

$$\psi = \sum_{i,j} \phi_i \otimes \chi_j \,, \tag{11a}$$

$$\langle p \,|\, \psi \,\rangle \equiv \sum_{i,j} \langle p \,|\, \phi_i \,\rangle \, \chi_j \in \mathcal{H}_2 \,. \tag{11b}$$

After identifying

$$\psi_p = \langle p \,|\, \psi \,\rangle \,, \tag{12}$$

we claim the direct integral identity:

$$\psi = \int^{\oplus} \psi_p \, \mathrm{d}p = \int |\, p \,\rangle \mathrm{d}p \langle\, p \,|\, \psi \rangle. \tag{13}$$

This justifies the notation:

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_p \, \mathrm{d}p = \int |\, p \,\rangle \mathrm{d}p \langle\, p \,|\, \mathcal{H} \rangle. \tag{14}$$

Finally, note that this decomposition corresponds to representations of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}$ by functions, respectively, vector-valued functions, on the simultaneous spectra of the n components of the momentum operators $\boldsymbol{P}$ on $\mathcal{H}_1$ and $\mathbb{P} = \boldsymbol{P} \otimes \mathbb{1}_2$ on $\mathcal{H}$, respectively. The action of an operator function $f(\boldsymbol{P})$ on $\mathcal{H}_1$ or $f(\mathbb{P})$ on $\mathcal{H}$ is the following:

$$[f(\boldsymbol{P})\phi](p) = \langle p \,|\, f(\boldsymbol{P})\, \phi \,\rangle = f(p)\, \phi(p)\,, \tag{15a}$$

$$f(\mathbb{P})\psi = \int^{\oplus} f(p)\, \psi_p \, \mathrm{d}p = \int f(p)\,|\, p\,\rangle \mathrm{d}p \langle\, p \,|\, \psi \rangle. \tag{15b}$$

The action of $|\, p\,\rangle \mathrm{d}p \langle\, p\,|$ on $\mathcal{H}_1$ could be extended to an action on $\mathcal{H}$, just as definition (11b) extends the action of the improper ket $\langle\, p\,|$. But it may be less confusing to introduce a separate notation, $E(\mathrm{d}p)$ for the improper operator on $\mathcal{H}_1$, and $\mathbb{E}(\mathrm{d}p)$ for the improper operator on $\mathcal{H}$:

$$E(\mathrm{d}p) \equiv |\, p\,\rangle \mathrm{d}p \langle\, p\,|\,, \tag{16a}$$

$$\mathbb{E}(\mathrm{d}p) \equiv E(\mathrm{d}p) \otimes \mathbb{1}_2\,. \tag{16b}$$

With this notation, when f is the characteristic function h of a measurable subset $U \subset \mathbb{R}^n$ of the simultaneous spectrum of $\boldsymbol{P}$ or $\mathbb{P}$, we get the following expressions for the spectral projection operators $E(U)$ and $\mathbb{E}(U)$:

$$E(U) = \int h(p) E(\mathrm{d}p) = \int_U E(\mathrm{d}p)\,, \tag{17a}$$

$$\mathbb{E}(U) = E(U) \otimes \mathbb{1}_2\,, \tag{17b}$$

$$= \int h(p)\, \mathbb{E}(\mathrm{d}p) = \int_U \mathbb{E}(\mathrm{d}p)\,. \tag{17c}$$

The corresponding operator direct integral expression is

$$\mathbb{E}(U) = \int^{\oplus} h(p)\, \mathbb{1}_p \, \mathrm{d}p = \int_U^{\oplus} \mathbb{1}_p \, \mathrm{d}p\,, \qquad \mathbb{1}_p = \mathbb{1}_2\,. \tag{18}$$

References

[1] Michael Reed and Barry Simon, *Analysis of Operators* (*Methods of Modern Mathematical Physics*, vol. IV), Academic Press, New York, 1978.

[2] Michael Reed and Barry Simon, *Functional Analysis* (*Methods of Modern Mathematical Physics*, vol. I), Academic Press, New York, 1972.