Formulas for classical electrodynamics*

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Revised: January 8, 2024

Contents

This is a LaTeX version of formulas from a handwritten notebook entitled *Classical Electrodynamics*, which goes back to at least the early 1970's. The selection of formulas is of course personal. It is expanded a bit to allay the uncertain recall of details by an aging brain.

1 Conventions

We follow Jackson's special relativity and Gaussian unit conventions:

$$\mu, \nu = 0, 1, 2, 3 \qquad i, j = 1, 2, 3$$

$$x^\mu = (ct, \boldsymbol{x}) \qquad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{1.1a}$$

$$g_{00} = 1 \qquad g_{0i} = g_{i0} = 0 \qquad g_{ij} = -\delta_{ij} \tag{1.1b}$$

$$\epsilon_{0123} = -1 \qquad \epsilon_{123} = 1 \tag{1.1c}$$

Our notation for three-dimensional vectors and tensors is the following:

(i) Up and down three-dimensional tensor indices are related by the three-dimensional negative identity part of the Lorentz metric, consistent with the restriction of a four-vector index to its spatial components.

(ii) The summation convention for repeated four-dimensional indices is the usual one; they are written only as relatively up and down. But repeated three-dimensional indices are summed even when both are up or both down.

(iii) Physical three-vectors have a natural definition as fundamentally up or down, as indicated by the spatial components of the four-tuples for x^μ and ∂_μ in Eq. (1.1a).

(iv) When three-vector notation is used in dot or cross products, the natural definitions are understood. For example: $\boldsymbol{x} \cdot \boldsymbol{x} = x^i \, x^i$ and $\boldsymbol{x} \cdot \boldsymbol{\nabla} = x^i \, \boldsymbol{\nabla}_i$.

(v) The three-dimensional Kronecker delta symbol is naturally defined with two lower indices, as indicated in Eq. (1.1b). As a rule we mever write it with an upper index. We never write a four-dimensional Kronecker delta at all, which would be defined naturally as $\delta_{\mu\nu} = g^\mu{}_\nu$, and which would not be Lorentz covariant. The three-dimensional alternating symbol is naturally defined with all indices down, as indicated in Eq. (1.1c).

(vi) Dot and double dot products involving tensors are defined as might be expected. For example:

$$(F \cdot F)^{\mu\nu} \equiv F^\mu{}_\lambda \, F^{\lambda\nu}, \quad F : F \equiv F_{\mu\nu} \, F^{\mu\nu}, \quad (F \cdot j)^\mu \equiv F^{\mu\nu} j_\nu \,.$$

2 Maxwell's equations

In Gaussian units, the dimensions of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and of the charge and current densities ρ and $\boldsymbol{j}$ are the following:

$$\dim \boldsymbol{E} = \dim \boldsymbol{B} = \mathrm{QL}^{-2} \qquad \dim \rho = \mathrm{QL}^{-3} \qquad \dim \boldsymbol{j} = \mathrm{QL}^{-2}\mathrm{T}^{-1} \tag{2.1}$$

The natural three-vector indices are taken to be upper:

$$E^i = (E_x, E_y, E_z) \qquad B^i = (B_x, B_y, B_z) \qquad j^i = (j_x, j_y, j_z) \tag{2.2}$$

Here are the vacuum Maxwell equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho \qquad \boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j} \tag{2.3a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \qquad \boldsymbol{\nabla}\times\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{2.3b}$$

Note that the inhomogeneous equations (2.3a) imply the continuity equation for ρ and $\boldsymbol{j}$:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \tag{2.3c}$$

The antisymmetric electromagnetic field tensor $F^{\mu\nu}$, its dual $F_{\mathrm{D}}^{\mu\nu}$, and the charge four-current density j^μ express the electrodynamic quantities in Lorentz covariant form:

$$j^\mu = (c\rho, \boldsymbol{j}) \tag{2.4a}$$

$$F_{\mathrm{D}}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\,F_{\lambda\rho} \qquad F^{\mu\nu} = -F_{\mathrm{DD}}^{\mu\nu} \tag{2.4b}$$

$$F^{0\,i} = -E^i = -F^{i\,0} \qquad F^{i\,j} = -\epsilon_{i\,j\,k}\,B^k \tag{2.4c}$$

$$F_{\mathrm{D}}^{0\,i} = -B^i = -F_{\mathrm{D}}^{i\,0} \qquad F_{\mathrm{D}}^{i\,j} = \epsilon_{i\,j\,k}\,E^k \tag{2.4d}$$

These definitions lead to the covariant form of Maxwell's equations:

$$\partial\cdot F = \frac{4\pi}{c}j \quad \Rightarrow \quad \partial\cdot j = 0 \tag{2.5a}$$

$$\partial\cdot F_{\mathrm{D}} = 0 \quad \Longleftrightarrow \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{2.5b}$$

Equations (2.5a) and (2.5b) are equivalent to Eqs. (2.3a) and (2.3b), respectively. The logical equivalence in (2.5b) follows from the partial derivative identity

$$(\partial\cdot F_{\mathrm{D}})^{\mu} = -\tfrac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}\,\partial_{\nu}F_{\lambda\rho} \tag{2.6}$$

together with the antisymmetrization identity obeyed by the contraction of two alternating symbols on one index. The vanishing of $\partial\cdot F_{\mathrm{D}}$, especially in the form on the r.h.s. of the equivalence, is an electromagnetic Bianchi identity. It is not satisfied when there is a magnetic charge.

Lorentz covariant quadratic combinations:

$$F : F = -F_{\mathrm{D}} : F_{\mathrm{D}} = 2\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right) \tag{2.7a}$$

$$F : F_{\mathrm{D}} = -4\,\boldsymbol{E}\cdot\boldsymbol{B} \tag{2.7b}$$

$$(F\cdot F)^{\mu\nu} = (F\cdot F)^{\nu\mu} \tag{2.7c}$$

$$(F\cdot F)^{00} = \boldsymbol{E}^2 \qquad (F\cdot F)^{0\,i} = \epsilon_{i\,j\,k}\,E^j B^k = (\boldsymbol{E}\times\boldsymbol{B})^i$$

$$(F\cdot F)^{i\,j} = -\left(E^i E^j + B^i B^j\right) + \delta_{i\,j}\,\boldsymbol{B}^2 \tag{2.7d}$$

$$F\cdot F_{\mathrm{D}} = F_{\mathrm{D}}\cdot F = -\tfrac{1}{4}\,g\,F : F \tag{2.7e}$$

$$F_{\mathrm{D}}\cdot F_{\mathrm{D}} = F\cdot F + \tfrac{1}{2}\,g\,F : F \tag{2.7f}$$

It turns out that the traceless part of the symmetric tensor $F\cdot F$ is proportional to the electromagnetic energy-momentum tensor.

If a magnetic charge four-current density $j^{\mu}_{\mathrm{m}} = (c\rho_{\mathrm{m}}, \boldsymbol{j}_{\mathrm{m}})$ were to be introduced as a source for F_{D},

$$\partial\cdot F_{\mathrm{D}} = \frac{4\pi}{c} j_{\mathrm{m}} \qquad \Rightarrow \qquad \partial\cdot j_{\mathrm{m}} = 0 \tag{2.8}$$

then the homogeneous Maxwell equations in (2.3b) would become:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 4\pi\,\rho_{\mathrm{m}} \qquad -\boldsymbol{\nabla}\times\boldsymbol{E} - \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}_{\mathrm{m}} \tag{2.9}$$

This is the dual of Eq. (2.3a) under the replacements $\boldsymbol{E}\to\boldsymbol{B}$, $\boldsymbol{B}\to-\boldsymbol{E}$, $\rho\to\rho_{\mathrm{m}}$, and $\boldsymbol{j}\to\boldsymbol{j}_{\mathrm{m}}$.

3 Vector potential

The vector potential A^μ is assumed to have vanishing four-divergence, i.e., to belong to the Lorentz gauge class.

$$A^\mu = (\phi, \boldsymbol{A}) \tag{3.1a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu \equiv [\partial A]_{\mu\nu} \tag{3.1b}$$

$$\Box A = \partial{\cdot}\partial A = \frac{4\pi}{c} j \qquad \partial{\cdot}A = 0 \tag{3.1c}$$

Equations (3.1b) and (3.1c) define a solution of the Maxwell equations. All solutions can be written this way, with the vector potential unique in the Lorentz gauge class up to the four-gradient of a scalar function that obeys the homogeneous wave equation. Equation (3.1b) implies that $\partial \cdot F_{\mathrm{D}}$ vanishes, because it makes the Bianchi identity for F automatic.

Putting together Eqs. (2.4c) and (3.1b) gives expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of A:

$$F^{0\,i} = -E^i = \frac{1}{c}\frac{\partial A^i}{\partial t} + \partial_i \phi \qquad\qquad \boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{3.2a}$$

$$F^{i\,j} = -\epsilon_{i\,j\,k}\, B^k = -\partial_i A^j + \partial_j A^i \qquad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \tag{3.2b}$$

4 Lorentz force

The Lorentz force law is logically independent from Maxwell's equations. The time component of the Lorentz force density four-vector f is the applied power density over c, and the spatial components are the three-vector Lorentz-force density, for the action of the electromagnetic field on charge and current distributions:

$$f^\mu = (f^0, \boldsymbol{f}) \qquad f^0 = \frac{1}{c}\boldsymbol{j}{\cdot}\boldsymbol{E} \qquad \boldsymbol{f} = \rho\,\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} \tag{4.1a}$$

$$f = \frac{1}{c} F{\cdot}j \tag{4.1b}$$

The Lorentz force three-vector is the spatial component of the volume integral of f. The contribution of the time and spatial components to the rate of change of the kinetic four-momentum P_q of the charge distribution is the following:

$$\frac{\mathrm{d}P_q}{\mathrm{d}t} = \int f(x)\,\mathrm{d}^3x \qquad q = \int \rho(x)\,\mathrm{d}^3x \tag{4.2}$$

5 Energy tensor

The dynamical quantities of the electromagnetic field are the following:

$$\text{energy density:} \qquad U = \frac{1}{4\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right) \tag{5.1a}$$

$$\text{momentum density:} \qquad \boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B} \tag{5.1b}$$

$$\text{energy flux:} \qquad \boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} \tag{5.1c}$$

$$\begin{aligned}\text{Maxwell stress tensor:} \quad T^{\mathrm{Max}}_{i\,j} &= \frac{1}{4\pi}\Big[\left(E^i E^J + B^i B^j\right) \\ &\qquad - \tfrac{1}{2}\delta_{i\,j}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\Big]\end{aligned} \tag{5.1d}$$

The energy flux $\boldsymbol{S}$ is the Poynting vector.

These quantities appear in the components of the standard energy-momentum tensor for the electromagnetic field, $\Theta^{\mu\nu}$, which is symmetric:

$$\Theta^{0\mu} = \Theta^{\mu 0} = (U, c\boldsymbol{g}) \qquad \Theta^{i\,j} = \Theta^{j\,i} = -T^{\mathrm{Max}}_{i\,j} \tag{5.2a}$$

$$P^{\mu}_{\text{field}} = \int \mathrm{d}^3x\,\Theta^{0\mu} \tag{5.2b}$$

The Maxwell equations (2.3a) and (2.3b) together with the defintion of the Lorentz force density in Eq. (4.1a) yield the conservation law:

$$\partial\cdot\Theta = -f \tag{5.3}$$

The covariant form of the energy-momentum tensor is the following:

$$\Theta = \frac{1}{4\pi}\left(F\cdot F + \tfrac{1}{4}\,g\,F : F\right) \qquad \Theta^{\mu}{}_{\mu} = 0 \tag{5.4a}$$

$$= \frac{1}{4\pi}\left(F_{\mathrm{D}}\cdot F_{\mathrm{D}} + \tfrac{1}{4}\,g\,F_{\mathrm{D}} : F_{\mathrm{D}}\right) \tag{5.4b}$$

In this form the conservation law can be derived from Eq. (2.5a) and the Bianchi identity in Eq. (2.5b), or from the vector potential by using Eqs. (3.1b) and (3.1c):

$$\partial\cdot\Theta = -\frac{1}{c}\,F\cdot j = -f \tag{5.5}$$

The standard energy-momentum tensor produces the covariant Lorentz force conservation law. Because it is symmetric, the corresponding covariant angular momemtum density produces the Lorentz torque conservation law:

$$\partial_\lambda \left(x^\mu\,\Theta^{\lambda\nu} - x^\nu\,\Theta^{\lambda\mu}\right) = -\left(x^\mu\,f^\nu - x^\nu\,f^\mu\right) \tag{5.6}$$

The equations of motion for a total system with a symmetric energy-momentum tensor, including moving matter, internal forces, the electromagnetic field, and external forces, imply conservation of four-momentum and covariant angular momentum:

$$\partial_\mu T^{\mu\nu}_{\text{total}} = 0 \qquad T^{\mu\nu}_{\text{total}} = T^{\mu\nu}_{\text{matter}} + T^{\mu\nu}_{\text{internal}} + T^{\mu\nu}_{\text{field}} + T^{\mu\nu}_{\text{external}} \tag{5.7}$$

6 Green functions

Lorentz gauge vector potentials satisfy the inhomogeneous wave equation (3.1c). The corresponding retarded and advanced Green functions are given by:

$$\begin{aligned} G_{\mathrm{R,A}}(x) &= -\lim_{\substack{\eta\to 0\\ \eta\in \mathrm{V}_\pm}} \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p\,\frac{e^{-ip\cdot x}}{(p+i\eta)^2} \\ &= \frac{1}{2\pi}\,\theta\left(\pm x^0\right)\,\delta(x\cdot x) \end{aligned} \tag{6.1a}$$

$$\Box\, G_{\mathrm{R,A}}(x) = \delta(x) \tag{6.1b}$$

Equation (6.1a) says that the retarded Green function G_R has its support on the future lightcone V_+, and the advanced Green function G_A has its support on the past lightcone V_-.

7 Point charge current

Let $y^\mu(\tau)$ be the (time-like) world line of a point charge q, where τ is the proper time. Then

$$y(\tau) = x(t) = [ct, \boldsymbol{x}(t)] \tag{7.1a}$$

$$u(\tau) \equiv \frac{\mathrm{d}y}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}x}{\mathrm{d}t} = (\gamma c, \gamma \boldsymbol{v}) \tag{7.1b}$$

The current density is

$$j(x) = qc \int \frac{\mathrm{d}y}{\mathrm{d}\tau}\, \delta[x - y(\tau)]\, \mathrm{d}\tau \tag{7.2}$$

$$= qc \int \frac{\mathrm{d}y}{\mathrm{d}\tau} \frac{\delta[(\tau - \tau(x^0)]}{|\mathrm{d}y^0/\mathrm{d}\tau|}\, \delta[\boldsymbol{x} - \boldsymbol{y}(\tau)]\, \mathrm{d}\tau \tag{7.3}$$

Here $\tau(x^0)$ is the solution of $t = y^0(\tau)/c$ which obeys $\mathrm{d}t/\mathrm{d}\tau = \gamma$; i.e., $\tau(x^0)$ is a proper time at which the time is t. Thus

$$j(x) = [qc,\ q\boldsymbol{v}(t)]\ \delta[\boldsymbol{x} - \boldsymbol{x}(t)] \tag{7.4}$$

8 Liénard-Wiechert potentials

Retarded and advanced solutions of the Lorentz-gauge Maxwell equations (3.1c) result from applying the Green's functions in Eq. (6.1a) to get the vector potentials:

$$A_{\mathrm{R,A}}(x) = \frac{4\pi}{c} \int G_{\mathrm{R,A}}(x - y)\, j(y)\, \mathrm{d}y \tag{8.1}$$

Note that if the integral is sufficiently well-defined in the distributional sense to allow integration by parts after taking its four-divergence, then $A_{\mathrm{R,A}}$ is in the Lorentz gauge class because the current j is conserved.

For a point charge this gives:

$$A_{\mathrm{R,A}}(x) = 2q \int \theta[\pm(x^0 - y^0)]\, \delta[(x-y)^2]\, \delta[x - y(\tau)]\, u(\tau)\, \mathrm{d}\tau\, \mathrm{d}y \tag{8.2a}$$

$$= 2q \int \theta\{\pm[x^0 - y^0(\tau)]\}\, \delta\{[x - y(\tau)]^2\}\, u(\tau)\, \mathrm{d}\tau \tag{8.2b}$$

Let $\tau_{\mathrm{R,A}}(x)$ be the retarded and advanced solutions of the light-cone condition:

$$r(x,\tau)\cdot r(x,\tau) = 0 \qquad r(x,\tau) \equiv x - y(\tau) \tag{8.3}$$

Then

$$\frac{\mathrm{d}r^2}{\mathrm{d}\tau} = -2r\cdot u \tag{8.4a}$$

$$A_{\mathrm{R,A}}(x) = 2q\int \frac{\delta[\tau - \tau_{\mathrm{R,A}}(x)]}{|\mathrm{d}r^2/\mathrm{d}\tau|}\, u(\tau)\,\mathrm{d}\tau \tag{8.4b}$$

$$= \left.\frac{qu}{|r\cdot u|}\right|_{\tau=\tau_{\mathrm{R,A}}(x)} \tag{8.4c}$$

These are the Liénard-Wiechert vector potentials for a point charge in arbitrary, sufficiently regular motion. In the next section, we shall see by direct calculation that they belong to the Lorentz gauge class.

9 Liénard-Wiechert fields

Notation for antisymmetric and symmetric combinations:

$$\left[V_1 V_2\right]_-^{\mu\nu} = V_1^\mu V_2^\nu - V_1^\nu V_2^\mu \qquad \left[V_1 V_2\right]_+^{\mu\nu} = V_1^\mu V_2^\nu + V_1^\nu V_2^\mu \tag{9.1}$$

In the following it is understood that τ is evaluated at $\tau_{\mathrm{R,A}}(x)$:

$$a(\tau) \equiv \frac{\mathrm{d}u}{\mathrm{d}\tau} \tag{9.2a}$$

$$0 = \partial\, r^2 = 2\, r\cdot\partial r = 2\,(r - r\cdot u\, \partial\tau) \quad \Rightarrow \quad \partial\tau = \frac{r}{r\cdot u} \tag{9.2b}$$

$$\partial\,(r\cdot u) = u + r\,\frac{r\cdot a - c^2}{r\cdot u} \tag{9.2c}$$

$$\partial\,|r\cdot u| = \mathrm{sgn}(r\cdot u)\,\partial\,(r\cdot u) \tag{9.2d}$$

$$\mathrm{sgn}(r\cdot u) = \mathrm{sgn}\, r^0 = \pm 1 \begin{cases} \text{ret} \\ \text{adv} \end{cases} \tag{9.2e}$$

$$\partial\, A_{\mathrm{R,A}}(x) = q\,\mathrm{sgn}(r\cdot u)\left[-uu\,\frac{1}{(r\cdot u)^2} + ru\,\frac{c^2 - r\cdot a}{(r\cdot u)^3} + ra\,\frac{1}{(r\cdot u)^2}\right] \tag{9.2f}$$

$$F_{\mathrm{R,A}}(x) = q\,\mathrm{sgn}(r\cdot u)\left[[ru]_-\,\frac{c^2 - r\cdot a}{(r\cdot u)^3} + [ra]_-\,\frac{1}{(r\cdot u)^2}\right] \tag{9.2g}$$

By contracting the two implicit four-vector indices in Eq. (9.2f), one readily verifies directly that $\partial\cdot A_{\mathrm{R,A}} = 0$, and hence that the point-charge vector potentials satisfy the Lorentz gauge condition.

The energy-momentum tensor (5.4a) for point-charge fields is given by:

$$\Theta_{\mathrm{R,A}} = \frac{1}{4\pi}\left(F_{\mathrm{R,A}}\cdot F_{\mathrm{R,A}} + \tfrac{1}{4} g\, F_{\mathrm{R,A}} : F_{\mathrm{R,A}}\right) \tag{9.3}$$

$$\begin{aligned} F_{\mathrm{R,A}}\cdot F_{\mathrm{R,A}} = \frac{q^2}{(r\cdot u)^4}\Bigg\{ -rr\left[a^2 + \frac{c^2(c^2 - r\cdot a)^2}{(r\cdot u)^2}\right] \\ + [ru]_+\,\frac{c^2(c^2 - r\cdot a)}{r\cdot u} + [ra]_+\, c^2 \Bigg\} \end{aligned} \tag{9.4}$$

$$\tfrac{1}{4} g\, F_{\mathrm{R,A}} : F_{\mathrm{R,A}} = -\tfrac{1}{2} g\,\frac{q^2 c^4}{(r\cdot u)^4} \tag{9.5}$$

It can be checked that $\partial\cdot\Theta_{\mathrm{R,A}}$ vanishes away from the charge, as expected. At the charge, however, both F terms have a leading singularity $1/(r\cdot u)^4$, which is not locally integrable in four dimensions. That prevents $\Theta_{\mathrm{R,A}}$, and hence the divergence, from existing as a tempered distribution. That can be remedied by rewriting the most singular terms as sums of weak derivatives of locally integrable functions, while preserving $\partial \cdot \Theta_{\mathrm{R,A}} = 0$ away from the particle, whereupon the four-divergence gives a mathematically well-defined contribution to the equation of motion. Fortunately or unfortunately, the only known way of doing that leads to the Abraham-Lorentz-Dirac equation of motion for a point charge.[1]

[1] We worked this out in October, 1974, but were so slow to write it up that we got scooped by the very nice paper of E. G. Peter Rowe, "Structure of the energy tensor in the classical electodynamics of point particles", Phys. Rev. D, **18** (1978), 3639–3654, and never submitted our work for publication.