Lorentz Force for a Continuous Charge*

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February 10. 2013

Contents

This note describes our notation and conventions for the Lorentz force exerted by a continuous charge distribution on itself.

1 Classical electrodynamics

We use Jackson's special relativity and Gaussian unit conventions:

$$\mathrm{sig} = (+---) \qquad \mu = 0, 1, 2, 3 \tag{1.1}$$

$$x^\mu = (ct, \boldsymbol{x}) \qquad \partial_\mu = \left(\frac{\partial}{\partial ct}, \boldsymbol{\nabla}\right) \tag{1.2}$$

$$j^\mu = (c\rho, \boldsymbol{j}) \qquad A^\mu = (\phi, \boldsymbol{A}) \tag{1.3}$$

$$\partial\cdot F = \frac{4\pi}{c} j \qquad \partial\cdot j = 0 \qquad \partial\cdot F^{\mathrm{D}} = 0 \tag{1.4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu \equiv [\partial A]_{\mu\nu} \qquad \partial\cdot A = 0 \ \text{(Lorentz gauge)} \tag{1.5}$$

*This document was started on August 6, 2012.

$$\Box A \equiv \partial\cdot\partial A = \frac{4\pi}{c} j \tag{1.6}$$

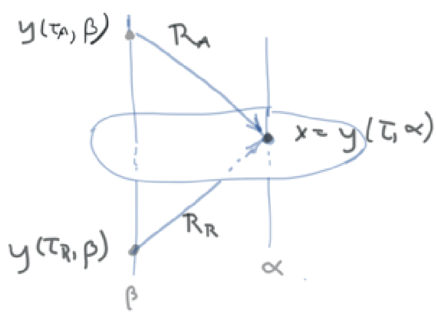
$$\Theta = \frac{1}{4\pi}\left(F\cdot F + \frac{1}{4} g\, F : F\right) \qquad \partial\cdot\Theta = -\frac{1}{c} F\cdot j \tag{1.7}$$

$$f = \frac{1}{c} F\cdot j \quad \text{(Lorentz force density)} \tag{1.8}$$

$$\frac{\mathrm{d}P_Q}{\mathrm{d}t} = \int f(x)\,\mathrm{d}^3x \quad \text{(Lorentz force)} \qquad Q = \int \rho(x)\,\mathrm{d}^3x \tag{1.9}$$

In Eq. (1.9), P is the total four-momentum of the charge distribution.

2 Retarded and advanced kinematics



Retarded and advanced sources for field point x.

Four-vector spacetime points on the world lines of charge elements are parametrized by proper time τ plus three, invariant spatial coordinates $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, corresponding to three-positions in a reference body. We also call $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ *body coordinates*. We use $\boldsymbol{\alpha}$ for spacetime points $x = y(\tau, \boldsymbol{\alpha})$ which experience the electromagnetic field, also called field points, and $\boldsymbol{\beta}$ for retarded or advanced spacetime source points $y(\tau_{\mathrm{R,A}}, \boldsymbol{\beta})$. We use R for the displacement from a source to a field point.

$$x = [ct, \boldsymbol{x}(t, \boldsymbol{\alpha})] = y(\tau, \boldsymbol{\alpha}) \tag{2.1}$$

$$R = x - y(\tau', \boldsymbol{\beta}) = y(\tau, \boldsymbol{\alpha}) - y(\tau', \boldsymbol{\beta}) \tag{2.2}$$

Here are the two solutions of the lightcone condition $R \cdot R = 0$ for τ' and $\boldsymbol{R}$, along with the corresponding retarded and advanced four-velocity u, four-acceleration a, and four-lurch l:

$$\tau_{\text{R,A}} = \tau_{\text{R,A}}(x, \boldsymbol{\beta}) = \tau_{\text{R,A}}[y(\tau, \boldsymbol{\alpha}), \boldsymbol{\beta}] \tag{2.3}$$

$$R_{\text{R,A}} = R_{\text{R,A}}(x, \boldsymbol{\beta}) = y(\tau, \boldsymbol{\alpha}) - y(\tau_{\text{R,A}}, \boldsymbol{\beta}) \tag{2.4}$$

$$u_{\text{R,A}} = u_{\text{R,A}}(x, \boldsymbol{\beta}) = \frac{\partial y}{\partial \tau}(\tau_{\text{R,A}}, \boldsymbol{\beta}) \tag{2.5}$$

$$a_{\text{R,A}} = a_{\text{R,A}}(x, \boldsymbol{\beta}) = \frac{\partial u}{\partial \tau}(\tau_{\text{R,A}}, \boldsymbol{\beta}) \tag{2.6}$$

$$l_{\text{R,A}} = l_{\text{R,A}}(x, \boldsymbol{\beta}) = \frac{\partial a}{\partial \tau}(\tau_{\text{R,A}}, \boldsymbol{\beta}) \tag{2.7}$$

We use the term *lurch* instead of *jerk* for $\dot{a}$ because we want to use j for current.

Let $\boldsymbol{J}$ be the Jacobian of the transformation $\boldsymbol{\alpha} \to \boldsymbol{x}(t, \boldsymbol{\alpha})$ at time t. Then

$$\mathrm{d}^3 x = \boldsymbol{J}\, \mathrm{d}^3 \alpha \tag{2.8}$$

$$\mathrm{d}q = \rho(x)\, \mathrm{d}^3 x = \rho_0(\boldsymbol{\alpha})\, \mathrm{d}^3 \alpha \qquad \rho(x) = \frac{\rho_0}{\boldsymbol{J}} \qquad Q = \int \mathrm{d}q \tag{2.9}$$

$$j(x) = \frac{\rho_0}{\boldsymbol{J}\gamma}\, u(\tau, \boldsymbol{\alpha}) \qquad \gamma \equiv u^0(\tau, \boldsymbol{\alpha}) = \gamma(x) \tag{2.10}$$

Note that point charge formulas for derivatives with respect to x of $\tau_{\text{R,A}}$, $\boldsymbol{R}_{\text{R,A}}$, $u_{\text{R,A}}$, etc., also hold for continuous charge, with the understanding that $\partial/\partial x$ is taken at constant source reference point $\boldsymbol{\beta}$.

3 Liénard-Wiechert potentials

In the following, blackboard bold symbols are used for reference body volume densities.

$$A_{\text{R,A}} = A_{\text{R,A}}(x) = A_{\text{R,A}}[y(\tau, \boldsymbol{\alpha})] \tag{3.1}$$

$$= \int \mathrm{d}^3\beta\, \rho_0(\boldsymbol{\beta})\, \frac{u_{\text{R,A}}}{|R_{\text{R,A}} \cdot u_{\text{R,A}}|} = \int \mathrm{d}^3\beta\, \rho_0(\boldsymbol{\beta})\, \mathbb{A}_{\text{R,A}} \tag{3.2}$$

$$\mathbb{A}_{\text{R,A}} = \mathbb{A}_{\text{R,A}}(x, \boldsymbol{\beta}) \quad \text{(vector potential body density)} \tag{3.3}$$

$$F_{\mathrm{R,A}} = F_{\mathrm{R,A}}(x) = \int \mathrm{d}^3\beta\, \rho_0(\boldsymbol{\beta})\, \mathbb{F}_{\mathrm{R,A}} \tag{3.4}$$

$$\mathbb{F}_{\mathrm{R,A}} = \mathbb{F}_{\mathrm{R,A}}(x, \boldsymbol{\beta}) = \left[\partial \mathbb{A}_{\mathrm{R,A}}\right](x, \boldsymbol{\beta}) \tag{3.5}$$

4 Lorentz force body density

Now we call $f(x)$ the *Lorentz force field*, and write it in terms of a Lorentz force body density, $\mathbb{f}(x, \boldsymbol{\beta})$.

$$f_{\mathrm{R,A}}(x) = \int \mathrm{d}^3\beta\, \rho_0(\boldsymbol{\beta})\, \mathbb{f}_{\mathrm{R,A}} \tag{4.1}$$

$$\mathbb{f}_{\mathrm{R,A}} = \mathbb{f}_{\mathrm{R,A}}(x, \boldsymbol{\beta}) = \frac{\mathbb{F}_{\mathrm{R,A}}(x, \boldsymbol{\beta}) \cdot j(x)}{c} = \frac{\mathbb{F}_{\mathrm{R,A}}(x, \boldsymbol{\beta}) \cdot \rho_0(\boldsymbol{\alpha})\, u(\tau, \boldsymbol{\alpha})}{J \gamma c} \tag{4.2}$$

$$\boxed{f_{\mathrm{R,A}}(x) = \frac{\rho_0(\boldsymbol{\alpha})}{J \gamma c} \int \mathrm{d}^3\beta\, \rho_0(\boldsymbol{\beta})\, \mathbb{F}_{\mathrm{R,A}}(x, \boldsymbol{\beta}) \cdot u(\tau, \boldsymbol{\alpha})} \tag{4.3}$$