Lorentz Group Formulas*

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Contents

 It is an expanded version of a composition book of handwritten notes, begun in late 1967. The date in the header will change as updates are made, even if they are only cosmetic.

These notes collect various formulas that we have found useful over the years for the homogeneous and inhomogeneous Lorentz groups and their unitary and spinorial representations. They are presented with a few comments, but without many details of the proofs. They are **PRELIMINARY**, not only because the content is under construction, but because the references are sketchy.

1 Conventions

Our matrix notation is M^{T} for transpose, M^* for hermitean conjugate, and $\overline{M}$ for complex conjugate.

When M is a second rank tensor or spinor, or a transformation on tensor or spinor indices, whose components we want to regard as matrix elements, we surround it with parentheses or brackets and write the indices as all lower, with no dots. For example:

$$(M)_{\mu\nu} = M^{\mu}{}_{\nu} \qquad (M)_{\alpha\beta} = M^{\dot{\alpha}\beta} \qquad \left[\mathcal{D}^j(U)\right]_{\alpha\beta} = \mathcal{D}^j(U)_{\alpha}{}^{\beta}$$

Our Lorentz signature is $(+---)$.

Unless otherwise stated, all homogeneous or inhomogeneous Lorentz transformations, along with any of their linear representations, are regarded as active. That is, they move vectors rather than changing their coordinates.

1.1 Lorentz metric

Four-vector indices are indicated by lowercase Greek letters and three-vector indices by lowercase Roman letters. Occasionaly we use x, y, z labels to disambiguate three-vector indices. Momentum and position are traditionally contravariant four vectors, with upper indices, and the corresponding gradients are covariant, with lower indices. We use notations like:

$$p^{\mu} = (p^0, \boldsymbol{p}) \qquad \partial_{\mu} = (\partial_0, \boldsymbol{\nabla}) \tag{1.1}$$

to indicate that the spatial components of p^{μ} are the components of the three-vector $\boldsymbol{p}$, and the spatial components of ∂_{μ} are those of the three-vector $\boldsymbol{\nabla}$.

The summation convention applies to repeated four-vector indices when one is upper and the other lower, and to repeated three-vector indices irrespective of upper or lower.

The homogeneuous Lorentz group is the set of real, 4×4 matrices Λ satisfying:

$$\Lambda^{\mathrm{T}} G \Lambda = G \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = G^{\mathrm{T}} = G^{-1} \tag{1.2a}$$

$$\det \Lambda = \pm 1 \tag{1.2b}$$

The *proper* Lorentz group L_+ is the subset of unimodular matrices; the *orthochronous* Lorentz group $L^\uparrow$ is the subset of matrices that preserve the sign of the time component of time-like vectors; and the *restricted* Lorentz group $L^\uparrow_+$, which is also the connected subgroup, is the subset of orthochronous, unimodular matrices.

The Lorentz metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ are both defined to be equal to the matrix elements $(G)_{\mu\nu}$. They can be used to raise or lower any lower or upper four-vector index, by contracting on either index of g, a rule that is consistent when applied to g itself. The matrix elements of the homogeneous Lorentz transformation Λ are written as $\Lambda^\mu{}_\nu$, corresponding to contravariant transformations of Minkowski space. As usual for a covariant coordinate transformation, lower indices transform according to the contragredient $\Lambda^{\mathrm{T}-1}$, whose matrix elements are naturally written as $\Lambda_\mu{}^\nu$. Equation (1.2a) says that the latter is related to the former by lowering and raising with the Lorentz metric tensor:

$$\Lambda_\mu{}^\nu = \Lambda^{-1\nu}{}_\mu\,. \tag{1.3}$$

It can be handy in some manipulations to define $\Lambda^{\mu\nu}$ or $\Lambda_{\mu\nu}$ by lowering or raising the upper or lower index of $\Lambda^\mu{}_\nu$, or by performing the opposite operations on $\Lambda_\mu{}^\nu$. It is easy to check from the definition of G that these operations are consistent.

The fundamental isotropic tensors of the Lorentz group are the metric tensor $g_{\mu\nu}$, and the pseudotensor alternating symbol $\epsilon_{\mu\nu\lambda\rho}$, which is odd under space inversion:

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \qquad g_{\mu\nu} = g^{\mu\nu} \tag{1.4a}$$

$$\epsilon_{0123} = -1 \qquad \epsilon_{123} = 1 \qquad \epsilon^{\mu\nu\lambda\rho} = -\epsilon_{\mu\nu\lambda\rho} \tag{1.4b}$$

$$\begin{aligned} \Lambda^\mu{}_\lambda\, g^\lambda{}_\nu &= g^\mu{}_\lambda\, \Lambda^\lambda{}_\nu \\ \Lambda^\mu{}_\lambda\, \Lambda^\nu{}_\rho\, g^{\lambda\rho} &= g^{\mu\nu} \end{aligned} \tag{1.5a}$$

$$\begin{aligned} \Lambda^\mu{}_\delta\, \epsilon^\delta{}_{\nu\lambda\rho} &= \epsilon^\mu{}_{\kappa\sigma\tau}\, \Lambda^\kappa{}_\nu\, \Lambda^\sigma{}_\lambda\, \Lambda^\tau{}_\rho\ \det\Lambda \\ \Lambda^\mu{}_\delta\, \Lambda^\nu{}_\kappa\, \epsilon^{\delta\kappa}{}_{\lambda\rho} &= \epsilon^{\mu\nu}{}_{\sigma\tau}\, \Lambda^\sigma{}_\lambda\, \Lambda^\tau{}_\rho\ \det\Lambda \\ \Lambda^\mu{}_\delta\, \Lambda^\nu{}_\kappa\, \Lambda^\lambda{}_\sigma\, \epsilon^{\delta\kappa\sigma}{}_\rho &= \epsilon^{\mu\nu\lambda}{}_\tau\, \Lambda^\tau{}_\rho\ \det\Lambda \\ \Lambda^\mu{}_\delta\, \Lambda^\nu{}_\kappa\, \Lambda^\lambda{}_\sigma\, \Lambda^\rho{}_\tau\, \epsilon^{\delta\kappa\sigma\tau} &= \epsilon^{\mu\nu\lambda\rho}\ \det\Lambda \end{aligned} \tag{1.5b}$$

It is a basic fact about determinants that (1.5b) holds when Λ is any complex 4×4 matrix. For Lorentz transformations, both sets of formulas remain consistent for any arrangement of up contracted with down indices. This consistency is maintained in the *pass-through rule*: any Λ acting on an invariant symbol to the right can be passed through the symbol, whereupon it acts on it to the left, or vice versa, passing through from right to left.

In the immediately following formulas, parenthesized combinations of tensor indices represent determinants of components of g, and $\mathcal{A}$ tensors are antisymmetrizing projections. For example, $\binom{\mu\nu}{\delta\kappa} = g^{\mu}{}_{\delta}\, g^{\nu}{}_{\kappa} - g^{\mu}{}_{\kappa}\, g^{\nu}{}_{\delta}$.

$$-\epsilon^{\mu\nu\lambda\rho}\,\epsilon_{\delta\kappa\sigma\tau} = 0!\,\binom{\mu\nu\lambda\rho}{\delta\kappa\sigma\tau} = 0!\cdot 4!\,\mathcal{A}^{\mu\nu\lambda\rho}_{\delta\kappa\sigma\tau} \tag{1.6a}$$

$$-\epsilon^{\mu\nu\lambda\tau}\,\epsilon_{\delta\kappa\sigma\tau} = 1!\,\binom{\mu\nu\lambda}{\delta\kappa\sigma} = 1!\cdot 3!\,\mathcal{A}^{\mu\nu\lambda}_{\delta\kappa\sigma} \tag{1.6b}$$

$$-\epsilon^{\mu\nu\lambda\rho}\,\epsilon_{\delta\kappa\lambda\rho} = 2!\,\binom{\mu\nu}{\delta\kappa} = 2!\cdot 2!\,\mathcal{A}^{\mu\nu}_{\kappa\delta} \tag{1.6c}$$

$$-\epsilon^{\mu\kappa\lambda\rho}\,\epsilon_{\delta\kappa\lambda\rho} = 3!\,g^{\mu}{}_{\delta} \tag{1.6d}$$

$$-\epsilon^{\mu\kappa\lambda\rho}\,\epsilon_{\mu\kappa\lambda\rho} = 4! \tag{1.6e}$$

1.2 Duals

Irreducible representations of the restricted Lorentz group for integer spin have self and antiself duality properties that make it useful to include a factor i in the definition of the dual. For antisymmetric, second-rank tensors:

$$T_{\mu\nu} = T^{\mathrm{A}}_{\mu\nu} \equiv \tfrac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right) \tag{1.7a}$$

$$T^{\mathrm{D}}_{\mu\nu} \equiv \tfrac{i}{2}\epsilon_{\mu\nu\lambda\rho}\,T^{\lambda\rho} \qquad (T^{\mathrm{D}})^{\mathrm{D}} = T \tag{1.7b}$$

$$T^{\mathrm{SD}}_{\mu\nu} \equiv \tfrac{1}{2}\left(T + T^{\mathrm{D}}\right)_{\mu\nu} \qquad (T^{\mathrm{SD}})^{\mathrm{D}} = T^{\mathrm{SD}} \tag{1.7c}$$

$$T^{\mathrm{ASD}}_{\mu\nu} \equiv \tfrac{1}{2}\left(T - T^{\mathrm{D}}\right)_{\mu\nu} \qquad (T^{\mathrm{ASD}})^{\mathrm{D}} = -T^{\mathrm{ASD}} \tag{1.7d}$$

$$(T^{\mathrm{SD}})^{\mathrm{SD}} = T^{\mathrm{SD}} \qquad (T^{\mathrm{ASD}})^{\mathrm{ASD}} = T^{\mathrm{ASD}} \tag{1.7e}$$

$$(T^{\mathrm{SD}})^{\mathrm{ASD}} = 0 \qquad (T^{\mathrm{ASD}})^{\mathrm{SD}} = 0 \tag{1.7f}$$

Projection operators for the self and antiself dual parts of any second rank tensor are defined by:

$$\mathcal{P}^{\pm}_{(\mu\nu)(\lambda\rho)} \equiv \tfrac{1}{4}\left(g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\lambda}\right) \pm \tfrac{i}{4}\epsilon_{\mu\nu\lambda\rho} \tag{1.8a}$$

$$= \mathcal{P}^{\pm}_{(\lambda\rho)(\mu\nu)} = \overline{\mathcal{P}^{\mp}_{(\lambda\rho)(\mu\nu)}} \tag{1.8b}$$

$$\mathcal{P}^{\pm}_{(\mu\nu)(\lambda\rho)}\, \mathcal{P}^{\pm(\lambda\rho)}{}_{(\sigma\tau)} = \mathcal{P}^{\pm}_{(\mu\nu)(\sigma\tau)} \tag{1.8c}$$

$$\mathcal{P}^{\pm}_{(\mu\nu)(\lambda\rho)}\, \mathcal{P}^{\mp(\lambda\rho)}{}_{(\sigma\tau)} = 0 \tag{1.8d}$$

Here self or antiself dual pairs of indices are surrounded by parentheses. The projection operators may of course be applied to any pair of indices of a tensor of higher rank.

1.3 Units

Before putting $\hbar = c = 1$, we have:

$$p^{\mu} = (E/c, \boldsymbol{p}) = \hbar k = \hbar(\omega/c, \boldsymbol{k}) \tag{1.9}$$

$$E = \sqrt{m^2c^4 + \boldsymbol{p}\cdot\boldsymbol{p}c^2} \qquad \omega/c = \sqrt{\mu^2 + \boldsymbol{k}\cdot\boldsymbol{k}} \qquad \mu = mc/\hbar \tag{1.10}$$

$$\begin{aligned} \text{energy} &= \text{MeV} \\ \text{momentum} &= \text{MeV}/c \qquad\qquad \dim k = L^{-1} = \dim \text{MeV}/\hbar c \\ \text{mass} &= \text{MeV}/c^2 \end{aligned}$$

From now on we put $\hbar = c = 1$, as in:

$$p^{\mu} = (\omega, \boldsymbol{p}) \qquad\qquad \omega = \sqrt{m^2 + \boldsymbol{p}\cdot\boldsymbol{p}} \tag{1.11}$$

2 Spinor calculus

2.1 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1a}$$

$$\left[\sigma_i, \sigma_j\right] = 2i\,\epsilon_{ijk}, \sigma_k \qquad \left\{\sigma_i, \sigma_j\right\} = 2\,\delta_{ij} \tag{2.1b}$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = \boldsymbol{\sigma}^* \qquad \sigma_\mu = (I, \boldsymbol{\sigma}) \qquad \widetilde{\sigma}_\mu = (I, -\boldsymbol{\sigma}) = \sigma^\mu \tag{2.1c}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\,\sigma_i\,\sigma_j = \delta_{ij} \qquad \tfrac{1}{2}\,\mathrm{Tr}\,\sigma_i\,\sigma_j\,\sigma_k = i\,\epsilon_{ijk} \qquad \tfrac{1}{2}\,\mathrm{Tr}\,\sigma_\mu \widetilde{\sigma}_\nu = g_{\mu\nu} \tag{2.1d}$$

Let $z^\mu = (z^0, \boldsymbol{z})$ be an arbitrary complex four vector. Then the mappings between $\mathbb{C}^4$ and the complex 2×2 matrices defined by (2.2a) below are holomorphic, 1-1, onto, inverses of each other:

$$\boldsymbol{M} \equiv z\cdot\sigma = z^0 I + \boldsymbol{z}\cdot\boldsymbol{\sigma} \qquad z^\mu = \tfrac{1}{2}\,\mathrm{Tr}\,\boldsymbol{M}\sigma^\mu \tag{2.2a}$$

$$\boldsymbol{M}^2 = [(z^0)^2 + \boldsymbol{z}\cdot\boldsymbol{z}]I + 2z^0\,\boldsymbol{z}\cdot\boldsymbol{\sigma} \tag{2.2b}$$

$$\mathrm{Tr}\,\boldsymbol{M} = 2z^0 \qquad \mathrm{Tr}\,\boldsymbol{M}^2 = 2[(z^0)^2 + \boldsymbol{z}\cdot\boldsymbol{z}] \tag{2.2c}$$

$$\det \boldsymbol{M} = z\cdot z = \tfrac{1}{2}\left[(\mathrm{Tr}\,\boldsymbol{M})^2 - \mathrm{Tr}\,\boldsymbol{M}^2\right] \tag{2.2d}$$

2.2 SL(2, C) transformation laws

Let Λ be the image of $A \in \mathrm{SL}(2,\mathrm{C})$, $\Lambda = \Lambda(\pm A)$, under the standard, two-to-one homomorphism of $\mathrm{SL}(2,\mathrm{C})$ onto $\mathrm{L}_+^\uparrow$. Then:

$$A\,\sigma_\mu A^* = \sigma_\nu\,\Lambda^\nu{}_\mu \qquad A^{*-1}\,\widetilde{\sigma}_\mu\,A^{-1} = \widetilde{\sigma}_\nu\,\Lambda^\nu{}_\mu \tag{2.3a}$$

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\,\mathrm{Tr}\left(\widetilde{\sigma}^\mu A\,\sigma_\nu A^*\right) \qquad A = A^* \;\Rightarrow\; \Lambda = \Lambda^{\mathrm{T}} \tag{2.3b}$$

2.3 Spinor labels

When considered as a generic matrix, the elements of a 2×2 matrix M are written as $(M)_{\alpha\beta}$. When considered as a second rank spinor, especially involving Pauli matrices or their products, the matrix elements of $\boldsymbol{M}$ are written as lower, upper, lower dotted, or upper dotted indices, as appropriate, corresponding respectively to the four faithful representations of SL(2, C) defined by A, $A^{\mathrm{T}-1}$, $\overline{A}$, and A^{*-1}.[1] The matrix elements are written according to their action towards the right: $A_\alpha{}^\beta$, $A^{\mathrm{T}-1\,\alpha}{}_\beta$, $\overline{A}_{\dot\alpha}{}^{\dot\beta}$, and $A^{*-1\,\dot\alpha}{}_{\dot\beta}$. All indices have the values $\pm 1/2$. Fortunately, one of the points of the spinor notation is that one rarely has to write down the matrix elements of A variants. The contragredients $A^{\mathrm{T}-1}$ and A^{*-1} can often be avoided by letting A and $\overline{A}$ act to the left, as in (2.8a) and (2.8b).

The spinor metric symbol ε raises and lowers spinor indices and relates contragradient spinor transformation matrices, although not in exactly the same way as g does for tensors. It implements unitary equivalence for lower and upper indices of the same type, because of (2.4b). Dotted indices are not equivalent to undotted indices.

$$\varepsilon \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon^{-1} \tag{2.4a}$$

$$\varepsilon\, M^{\mathrm{T}}\, \varepsilon^{-1} = M^{-1} \det M \tag{2.4b}$$

$$\varepsilon\, \sigma_\mu\, \varepsilon^{-1} = \widetilde{\sigma}_\mu{}^{\mathrm{T}} \qquad \varepsilon\, \widetilde{\sigma}_\mu\, \varepsilon^{-1} = \sigma_\mu{}^{\mathrm{T}} \tag{2.4c}$$

$$\sigma_{\mu\,\alpha\dot\beta} = (\sigma_\mu)_{\alpha\beta} \qquad \widetilde{\sigma}_\mu{}^{\dot\alpha\beta} = (\widetilde{\sigma}_\mu)_{\alpha\beta} = (\sigma^\mu)_{\alpha\beta} = \sigma^\mu{}_{\alpha\dot\beta} \tag{2.4d}$$

$$\varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = \varepsilon_{\dot\alpha\dot\beta} = (-1)^{\frac{1}{2}-\alpha}\, \delta_\alpha{}^{-\beta} = (\varepsilon)_{\alpha\beta} = -(\varepsilon^{-1})_{\alpha\beta} \tag{2.4e}$$

The raising and lowering operations are written:

$$\eta^\alpha = \varepsilon^{\alpha\beta}\,\eta_\beta \qquad \eta_\alpha = \varepsilon^{-1}_{\alpha\beta}\,\eta^\beta = -\varepsilon_{\alpha\beta}\,\eta^\beta = \eta^\beta\,\varepsilon_{\beta\alpha} \tag{2.5a}$$

$$\Rightarrow \qquad \zeta^\alpha\,\eta_\alpha = -\,\zeta_\alpha\,\eta^\alpha \tag{2.5b}$$

Thus a spinor index is raised by contracting on the right index of $\varepsilon^{\alpha\beta}$ or $\varepsilon^{\dot\alpha\dot\beta}$, and lowered by contracting on the left index of $\varepsilon_{\alpha\beta}$ or $\varepsilon_{\dot\alpha\dot\beta}$. The raising and lowering operations can be applied to σ, $\widetilde{\sigma}$, and ε itself; and complex conjugation converts between their undotted and dotted indices. The results turn out to be consistent

[1] We call these the four *natural automorphisms* of SL(2, C).

and natural:

$$\varepsilon^{\alpha\beta} = \varepsilon^{\alpha\gamma}\,\varepsilon^{\beta\delta}\,\varepsilon_{\gamma\delta} \qquad \varepsilon_{\alpha\beta} = \varepsilon^{\gamma\delta}\,\varepsilon_{\gamma\alpha}\,\varepsilon_{\delta\beta} \tag{2.6a}$$

$$\varepsilon^{\dot\alpha\dot\beta} = \varepsilon^{\dot\alpha\dot\gamma}\,\varepsilon^{\dot\beta\dot\delta}\,\varepsilon_{\dot\gamma\dot\delta} \qquad \varepsilon_{\dot\alpha\dot\beta} = \varepsilon^{\dot\gamma\dot\delta}\,\varepsilon_{\dot\gamma\dot\alpha}\,\varepsilon_{\dot\delta\dot\beta} \tag{2.6b}$$

$$\overline{\varepsilon_{\alpha\beta}} = \varepsilon_{\dot\alpha\dot\beta} \qquad \overline{\varepsilon^{\alpha\beta}} = \varepsilon^{\dot\alpha\dot\beta} \tag{2.6c}$$

$$\sigma_\mu{}^{\alpha\dot\beta} = (\varepsilon\,\sigma_\mu\,\varepsilon^{-1})_{\alpha\beta} = (\widetilde{\sigma}_\mu^{\mathrm{T}})_{\alpha\beta} = \widetilde{\sigma}_\mu{}^{\dot\beta\alpha} \tag{2.7a}$$

$$= (\overline{\widetilde{\sigma}_\mu})_{\alpha\beta} = \overline{\widetilde{\sigma}_\mu{}^{\dot\alpha\beta}} \tag{2.7b}$$

$$\widetilde{\sigma}_{\mu\,\dot\alpha\beta} = (\varepsilon^{-1}\,\widetilde{\sigma}_\mu\,\varepsilon)_{\alpha\beta} = (\sigma_\mu^{\mathrm{T}})_{\alpha\beta} = \sigma_{\mu\beta\dot\alpha} \tag{2.7c}$$

$$= (\overline{\sigma_\mu})_{\alpha\beta} = \overline{\sigma_{\mu\alpha\dot\beta}} \tag{2.7d}$$

The Pauli matrices and the spinor metric are isotropic spinors under SL(2, C) because of (2.3a) and (2.4b):

$$\Lambda^\mu{}_\nu\,A_\alpha{}^\gamma\,\overline{A}_{\dot\beta}{}^{\dot\delta}\,\sigma^\nu{}_{\gamma\dot\delta} = \sigma^\mu{}_{\alpha\dot\beta} \qquad \Lambda^\mu{}_\nu\,\widetilde{\sigma}^{\nu\dot\gamma\delta}\,\overline{A}_{\dot\gamma}{}^{\dot\alpha}\,A_\delta{}^\beta = \widetilde{\sigma}^{\mu\dot\alpha\beta} \tag{2.8a}$$

$$\begin{array}{ll} A_\alpha{}^\gamma\,A_\beta{}^\delta\,\varepsilon_{\gamma\delta} = \varepsilon_{\alpha\beta} & \varepsilon^{\gamma\delta}\,A_\gamma{}^\alpha\,A_\delta{}^\beta = \varepsilon^{\alpha\beta} \\ \overline{A}_{\dot\alpha}{}^{\dot\gamma}\,\overline{A}_{\dot\beta}{}^{\dot\delta}\,\varepsilon_{\dot\gamma\dot\delta} = \varepsilon_{\dot\alpha\dot\beta} & \varepsilon^{\dot\gamma\dot\delta}\,\overline{A}_{\dot\gamma}{}^{\dot\alpha}\,\overline{A}_{\dot\delta}{}^{\dot\beta} = \varepsilon^{\dot\alpha\dot\beta} \end{array} \tag{2.8b}$$

Note how we avoid having to write indices for $A^{\mathrm{T}-1}$ and A^{*-1} by acting to the left with A and $\overline{A}$ on upper indices. It is sometimes handy to do the analogous thing with Λ, as in (1.5b); but while allowing all positions for indices is a convenience for Λ, it is a notational burden for A and $\overline{A}$. In fact it is not true that $A^{\mathrm{T}-1}$ and A^{*-1} are obtained from A and $\overline{A}$ by formally raising and lowering. Instead:

$$\varepsilon^{\alpha\alpha'}A_{\alpha'}{}^{\beta'}\varepsilon_{\beta'\beta} = (\varepsilon\,A\,\varepsilon)_{\alpha\beta} = -\left(A^{\mathrm{T}-1}\right)_{\alpha\beta} = -A^{\mathrm{T}-1\,\alpha}{}_\beta \tag{2.9a}$$

$$\varepsilon^{\alpha\alpha'}\overline{A}_{\alpha'}{}^{\beta'}\varepsilon_{\beta'\beta} = (\varepsilon\,\overline{A}\,\varepsilon)_{\alpha\beta} = -\left(A^{*-1}\right)_{\alpha\beta} = -A^{*-1\,\alpha}{}_\beta \tag{2.9b}$$

As matrices, $\varepsilon^* = \varepsilon^{-1} = -\varepsilon$, so the above formulas are special cases of the contragredient formula (2.4b) that say $A^{\mathrm{T}-1}$ is unitary equivalent to A and $\overline{A}$ is unitary equivalent to A^{*-1}:

$$A^{\mathrm{T}-1} = \varepsilon A \varepsilon^* = \varepsilon A \varepsilon^{-1} \tag{2.10a}$$

$$A^{*-1} = \varepsilon \overline{A} \varepsilon^* = \varepsilon \overline{A} \varepsilon^{-1} \tag{2.10b}$$

It is a basic fact that the self representation of SL(2, C) by A is not unitary equivalent, nor even similar to, the A^{*-1} representation.

2.4 Orthogonality

The covariant Pauli matrices can be regarded as proportional to symbols for unitary transformation between vector and spinor forms of the irreducible $(\frac{1}{2}, \frac{1}{2})$ representation of SL(2, C). The Lorentz-invariant trace identity (2.1d) has several equivalent spinor forms, which can be regarded as orthogonality relations for transformation from a vector to an equivalent rank-two spinor of type $(\frac{1}{2}, \frac{1}{2})$ and back again.

There is an awkward point about the inverse orthgonality relation, from rank-two spinor to vector to rank-two spinor. Namely, the Kronecker symbol with one index down and one up changes sign when formally raised and lowered, because:

$$\varepsilon_{\alpha}{}^{\beta} = -\varepsilon^{\alpha}{}_{\beta} = (I)_{\alpha\beta} \tag{2.11}$$

To put it another way, there is no spinor δ such that $\delta_{\alpha}{}^{\beta}$ and $\delta^{\alpha}{}_{\beta}$ are both equal to the Kronecker delta.

In order to keep the convenient Kronecker notation, we define the spinor Kronecker delta as an isotropic convenience symbol with only one configuration of indices, *never to be raised or lowered*:

$$\delta_{\alpha}{}^{\beta} = \delta_{\dot{\alpha}}{}^{\dot{\beta}} \equiv (I)_{\alpha\beta} \tag{2.12}$$

The dotted and undotted versions are in fact isotropic SL(2, C) spinors, but we agree never to write down the other index configurations.

Some equivalent spinor forms of the trace orthogonality relation (2.1d) are given below in (2.13), and some for its inverse in (2.14a) and (2.14b):

$$\tfrac{1}{2}\,\sigma_{\mu\alpha\dot{\beta}}\,\widetilde{\sigma}_{\nu}{}^{\dot{\beta}\alpha} = \tfrac{1}{2}\,\widetilde{\sigma}_{\mu}{}^{\dot{\beta}\alpha}\,\sigma_{\nu\alpha\dot{\beta}} = \tfrac{1}{2}\,\sigma_{\mu\alpha\dot{\beta}}\,\sigma_{\nu}{}^{\alpha\dot{\beta}} = \tfrac{1}{2}\,\widetilde{\sigma}_{\mu}{}^{\dot{\beta}\alpha}\,\widetilde{\sigma}_{\nu\dot{\beta}\alpha} = g_{\mu\nu} \tag{2.13}$$

$$\tfrac{1}{2}\,\sigma_{\mu\alpha_1\dot{\beta}_1}\,\widetilde{\sigma}^{\mu\dot{\beta}_2\alpha_2} = \tfrac{1}{2}\,\widetilde{\sigma}_{\mu\dot{\beta}_1\alpha_1}\,\sigma^{\mu\alpha_2\dot{\beta}_2} = \delta_{\alpha_1}{}^{\alpha_2}\,\delta_{\dot{\beta}_1}{}^{\dot{\beta}_2} \tag{2.14a}$$

$$\tfrac{1}{2}\,\sigma_{\mu\alpha_1\dot{\beta}_1}\,\sigma^{\mu}{}_{\alpha_2\dot{\beta}_2} = \tfrac{1}{2}\,\widetilde{\sigma}_{\mu\dot{\beta}_1\alpha_1}\,\widetilde{\sigma}^{\mu}{}_{\dot{\beta}_2\alpha_2} = \varepsilon_{\alpha_1\alpha_2}\,\varepsilon_{\dot{\beta}_1\dot{\beta}_2} \tag{2.14b}$$

It is straightforward albeit tedious to compute the l.h.s. of (2.14b) by explicitly writing out the dot products of the four-vectors that result from fixing the values of the spinor index pairs in $\sigma_{\mu\alpha_1\dot{\beta}_1}$ and $\sigma^{\mu}{}_{\alpha_2\dot{\beta}_2}$, but the calculation becomes trivial after a Clebsch-Gordan analysis. The l.h.s. is an isotropic spinor with two lower undotted and two lower dotted indices. Being invariant, the only nonzero irreducible component has to have zero angular momentum content for each of the undotted and dotted pairs, and must be proportional to $\varepsilon_{\alpha_1\alpha_2}\,\varepsilon_{\dot\beta_1\dot\beta_2}$. The proportionality constant is fixed by choosing, for example, $(\alpha_1, \alpha_2, \dot\beta_1, \dot\beta_2) = (1/2, -1/2, 1/2, -1/2)$ and computing a single dot product. The rest of the inverses can be derived by raising and lowering.

2.5 Products

Let $a^\mu = (a^0, \boldsymbol{a})$, etc.

$$a\cdot\sigma\, b\cdot\widetilde{\sigma} + b\cdot\sigma\, a\cdot\widetilde{\sigma} = a\cdot\widetilde{\sigma}\, b\cdot\sigma + b\cdot\widetilde{\sigma}\, a\cdot\sigma = 2\, a\cdot b \tag{2.15a}$$

$$a\cdot\sigma\, b\cdot\widetilde{\sigma} = a\cdot b + \big(b^0\, \boldsymbol{a} - a^0\, \boldsymbol{b} - i\, \boldsymbol{a}\times\boldsymbol{b}\big)\cdot\boldsymbol{\sigma} \tag{2.15b}$$

$$b\cdot\sigma\, a\cdot\widetilde{\sigma} = a\cdot b + \big(a^0\, \boldsymbol{b} - b^0\, \boldsymbol{a} - i\, \boldsymbol{b}\times\boldsymbol{a}\big)\cdot\boldsymbol{\sigma} \tag{2.15c}$$

$$a\cdot\widetilde{\sigma}\, b\cdot\sigma = a\cdot b + \big(a^0\, \boldsymbol{b} - b^0\, \boldsymbol{a} - i\, \boldsymbol{a}\times\boldsymbol{b}\big)\cdot\boldsymbol{\sigma} \tag{2.15d}$$

$$b\cdot\widetilde{\sigma}\, a\cdot\sigma = a\cdot b + \big(b^0\, \boldsymbol{a} - a^0\, \boldsymbol{b} - i\, \boldsymbol{b}\times\boldsymbol{a}\big)\cdot\boldsymbol{\sigma} \tag{2.15e}$$

$$\sigma_\mu\, \widetilde{\sigma}_\nu = g_{\mu\nu} + \tfrac{i}{2}\epsilon_{\mu\nu\lambda\rho}\, \sigma^\lambda\, \widetilde{\sigma}^\rho \tag{2.15f}$$

$$\widetilde{\sigma}_\mu\, \sigma_\nu = g_{\mu\nu} - \tfrac{i}{2}\epsilon_{\mu\nu\lambda\rho}\, \widetilde{\sigma}^\lambda\, \sigma^\rho \tag{2.15g}$$

$$\tfrac{1}{2}\left(\sigma_\mu\, \widetilde{\sigma}_\nu - \sigma_\nu\, \widetilde{\sigma}_\mu\right) = \left(\sigma_\mu\, \widetilde{\sigma}_\nu\right)^{\mathrm{A}} = \tfrac{i}{2}\epsilon_{\mu\nu\lambda\rho}\, \sigma^\lambda \widetilde{\sigma}^\rho = \left(\sigma_\mu\, \widetilde{\sigma}_\nu\right)^{\mathrm{SD}} \tag{2.15h}$$

$$\tfrac{1}{2}\left(\widetilde{\sigma}_\mu\, \sigma_\nu - \widetilde{\sigma}_\nu\, \sigma_\mu\right) = \left(\widetilde{\sigma}_\mu\, \sigma_\nu\right)^{\mathrm{A}} = -\tfrac{i}{2}\epsilon_{\mu\nu\lambda\rho}\, \widetilde{\sigma}^\lambda \sigma^\rho = \left(\widetilde{\sigma}_\mu\, \sigma_\nu\right)^{\mathrm{ASD}} \tag{2.15i}$$

$$[a,b,c]^\mu \equiv \epsilon^{\mu\nu\lambda\rho}\, a_\nu b_\lambda c_\rho \tag{2.16a}$$

$$a\cdot\sigma\, b\cdot\widetilde{\sigma}\, c\cdot\sigma = a\cdot\sigma\, b\cdot c - b\cdot\sigma\, a\cdot c + c\cdot\sigma\, a\cdot b + i\,[a,b,c]\cdot\sigma \tag{2.16b}$$

$$a\cdot\widetilde{\sigma}\, b\cdot\sigma\, c\cdot\widetilde{\sigma} = a\cdot\widetilde{\sigma}\, b\cdot c - b\cdot\widetilde{\sigma}\, a\cdot c + c\cdot\widetilde{\sigma}\, a\cdot b - i\,[a,b,c]\cdot\widetilde{\sigma} \tag{2.16c}$$

$$\sigma_\mu\, \widetilde{\sigma}_\nu\, \sigma_\lambda = g_{\mu\nu}\, \sigma_\lambda - g_{\mu\lambda}\, \sigma_\nu + g_{\nu\lambda}\, \sigma_\mu - i\, \epsilon_{\mu\nu\lambda\rho}\, \sigma^\rho \tag{2.16d}$$

$$\widetilde{\sigma}_\mu\, \sigma_\nu\, \widetilde{\sigma}_\lambda = g_{\mu\nu}\, \widetilde{\sigma}_\lambda - g_{\mu\lambda}\, \widetilde{\sigma}_\nu + g_{\nu\lambda}\, \widetilde{\sigma}_\mu + i\, \epsilon_{\mu\nu\lambda\rho}\, \widetilde{\sigma}^\rho \tag{2.16e}$$

$$\tfrac{i}{2}\, \epsilon_{\mu\nu\sigma\tau}\, \sigma^\sigma\, \widetilde{\sigma}^\tau\, \sigma_\lambda = \quad \sigma_\mu\, g_{\nu\lambda} - \sigma_\nu\, g_{\mu\lambda} - i\, \epsilon_{\mu\nu\lambda\rho}\, \sigma^\rho \tag{2.16f}$$

$$\tfrac{i}{2}\, \epsilon_{\mu\nu\sigma\tau}\, \widetilde{\sigma}^\sigma\, \sigma^\tau\, \widetilde{\sigma}_\lambda = -\widetilde{\sigma}_\mu\, g_{\nu\lambda} + \widetilde{\sigma}_\nu\, g_{\mu\lambda} - i\, \epsilon_{\mu\nu\lambda\rho}\, \widetilde{\sigma}^\rho \tag{2.16g}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\left(\sigma_\mu\, \widetilde{\sigma}_\nu\, \sigma_\lambda\, \widetilde{\sigma}_\rho\right) = g_{\mu\nu}\, g_{\lambda\rho} - g_{\mu\lambda}\, g_{\nu\rho} + g_{\mu\rho}\, g_{\nu\lambda} - i\, \epsilon_{\mu\nu\lambda\rho} \tag{2.17a}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\left(\widetilde{\sigma}_\mu\, \sigma_\nu\, \widetilde{\sigma}_\lambda\, \sigma_\rho\right) = g_{\mu\nu}\, g_{\lambda\rho} - g_{\mu\lambda}\, g_{\nu\rho} + g_{\mu\rho}\, g_{\nu\lambda} + i\, \epsilon_{\mu\nu\lambda\rho} \tag{2.17b}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\left[\left(\sigma_\mu \widetilde{\sigma}_\nu\right)^{\mathrm{A}} \sigma_\lambda \widetilde{\sigma}_\rho\right] = -\left(g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\lambda} + i\, \epsilon_{\mu\nu\lambda\rho}\right) \tag{2.17c}$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\sigma_\mu \widetilde{\sigma}_\nu \left(\sigma_\lambda \widetilde{\sigma}_\rho\right)^{\mathrm{A}}\right] \tag{2.17d}$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\sigma_\mu \widetilde{\sigma}_\nu\right)^{\mathrm{SD}} \left(\sigma_\lambda \widetilde{\sigma}_\rho\right)^{\mathrm{SD}}\right] \tag{2.17e}$$

$$= -4\, \mathcal{P}^+_{(\mu\nu)(\lambda\rho)} \tag{2.17f}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\left[\left(\widetilde{\sigma}_\mu \sigma_\nu\right)^{\mathrm{A}} \widetilde{\sigma}_\lambda \sigma_\rho\right] = -\left(g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\lambda} - i\, \epsilon_{\mu\nu\lambda\rho}\right) \tag{2.17g}$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\widetilde{\sigma}_\mu \sigma_\nu \left(\widetilde{\sigma}_\lambda \sigma_\rho\right)^{\mathrm{A}}\right] \tag{2.17h}$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\widetilde{\sigma}_\mu \sigma_\nu\right)^{\mathrm{ASD}} \left(\widetilde{\sigma}_\lambda \sigma_\rho\right)^{\mathrm{ASD}}\right] \tag{2.17i}$$

$$= -4\, \mathcal{P}^-_{(\mu\nu)(\lambda\rho)} \tag{2.17j}$$

2.6 Specific transformations

2.6.1 Boosts

It is generally easier to compute the effects of $\Lambda \in \mathrm{L}_{+}^{\uparrow}$, as well as the matrix elements of Λ, from the $\mathrm{SL}(2, \mathrm{C})$ versions. Boosts correspond to changes of velocity, and are naturally parametrized in terms of rapidity and direction. The term is especially apt for describing the boost of a massive particle from rest to nonzero three-momentum, and we begin with a momentum parametrization for nonzero mass, keeping in mind that such an interpretation is not available for massless particles, which have no rest frame.

Boost corresponding to $(m, 0, 0, 0) \to p = (\omega, \boldsymbol{p})$:

$$A(p) \equiv \sqrt{\frac{p\cdot\sigma}{m}} = \frac{m + p\cdot\sigma}{\sqrt{2m(m+\omega)}} = A(p)^{*} \tag{2.18a}$$

$$p\cdot\sigma = A(p)\; m\sigma_0\; A(p)^{*} \tag{2.18b}$$

$$A(p)^{-1} = \sqrt{\frac{p\cdot\widetilde{\sigma}}{m}} = \frac{m + p\cdot\widetilde{\sigma}}{\sqrt{2m(m+\omega)}} \tag{2.18c}$$

The matrix square roots in Eqs. (2.18a) and (2.18c) are positive definite. That can be seen by rotating $\boldsymbol{p}$ into the three-direction with a similarity transformation in SU(2), which diagonalizes $A(p)$ and $A^{-1}(p)$ into linear combinations of σ_0 and σ_3:

$$U\;A(p)\,U^{*} = \begin{pmatrix} \dfrac{m+\omega+|\boldsymbol{p}|}{\sqrt{2m(m+\omega)}} & 0 \\ 0 & \dfrac{m+\omega-|\boldsymbol{p}|}{\sqrt{2m(m+\omega)}} \end{pmatrix} \tag{2.19a}$$

$$U\;A(p)^{-1}\,U^{*} = \begin{pmatrix} \dfrac{m+\omega-|\boldsymbol{p}|}{\sqrt{2m(m+\omega)}} & 0 \\ 0 & \dfrac{m+\omega+|\boldsymbol{p}|}{\sqrt{2m(m+\omega)}} \end{pmatrix} \tag{2.19b}$$

Boost in terms of rapidity λ and direction $\boldsymbol{n} = \boldsymbol{p}/|\boldsymbol{p}|$:

$$A(p) = \exp\left(\tfrac{1}{2}\,\lambda\,\boldsymbol{n}\cdot\boldsymbol{\sigma}\right) \tag{2.20a}$$

$$= \boldsymbol{I}\,\cosh\tfrac{1}{2}\,\lambda + \boldsymbol{n}\cdot\boldsymbol{\sigma}\,\sinh\tfrac{1}{2}\,\lambda \tag{2.20b}$$

$$\beta = \frac{|\boldsymbol{p}|}{\omega} \qquad\qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.20c}$$

$$\cosh\lambda = \gamma \qquad\qquad \sinh\lambda = \beta\gamma \tag{2.20d}$$

Equation (2.20b) follows from (2.20a) by substituting $(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = \boldsymbol{I}$ into the power series expansion for the exponential, which converges everywhere. It is straightforward to show that every hermitean, positive definite matrix in SL(2, C) has the form (2.20b).

Let $x\cdot\sigma = m + p\cdot\sigma$, and $\Lambda = \Lambda(p) = \Lambda[A(p)]$. Then:

$$\Lambda^{\mu}{}_{\nu} = \frac{x^{\mu}\,\widetilde{x}_{\nu}}{m(m+\omega)} - g_{\mu\nu} \qquad\qquad \Lambda^{-1\,\mu}{}_{\nu} = \frac{\widetilde{x}^{\mu}\,x_{\nu}}{m(m+\omega)} - g_{\mu\nu} \tag{2.21}$$

Let $u^{\mu} = p^{\mu}/m = (\gamma, \gamma\boldsymbol{v})$. Then:

$$\Lambda^{0}{}_{0} = \gamma \qquad \Lambda^{0}{}_{i} = \Lambda^{i}{}_{0} = \gamma v^{i} \qquad \Lambda^{i}{}_{j} = \delta_{ij} + \frac{\gamma^2}{1+\gamma}\,v^{i}\,v^{j} \tag{2.22a}$$

$$\Lambda = \begin{pmatrix} u^0 & \boldsymbol{u} \\ \boldsymbol{u} & \epsilon + \dfrac{\boldsymbol{u}\boldsymbol{u}}{1+u^0} \end{pmatrix} \tag{2.22b}$$

$$= \begin{pmatrix} \cosh\lambda & \boldsymbol{n}\sinh\lambda \\ \boldsymbol{n}\sinh\lambda & \epsilon + \boldsymbol{n}\boldsymbol{n}\,(\cosh\lambda - 1) \end{pmatrix} \qquad\qquad \boldsymbol{n} \equiv \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \tag{2.22c}$$

2.6.2 Rotations

The covering group of the rotation subgroup SO(3) of $\mathrm{L}_{+}^{\uparrow}$ is the SU(2) subgroup of SL(2, C).

Rotations leave the x^0 component of x^{μ} as well as the magnitude of $\boldsymbol{x}$ invariant:

$$U\,x^0\sigma^0\,U^* = x^0\sigma^0 \qquad \Rightarrow \qquad U^* = U^{-1} \tag{2.23a}$$

$$U\,\boldsymbol{x}\cdot\boldsymbol{\sigma}\,U^* = \boldsymbol{y}\cdot\boldsymbol{\sigma} \qquad \Rightarrow \qquad \boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{y}\cdot\boldsymbol{y} \tag{2.23b}$$

Rotation by θ about the $\boldsymbol{e} = (e^1, e^2, e^3)$ axis, $\boldsymbol{e}\cdot\boldsymbol{e} = 1$:

$$U(\theta\boldsymbol{e}) = \exp\left(-\tfrac{i}{2}\,\theta\,\boldsymbol{e}\cdot\boldsymbol{\sigma}\right) = U(-\theta\boldsymbol{e})^{-1} \tag{2.24a}$$

$$= \boldsymbol{I}\cos\tfrac{1}{2}\theta - i\,\boldsymbol{e}\cdot\boldsymbol{\sigma}\,\sin\tfrac{1}{2}\theta \tag{2.24b}$$

The proof that (2.24b) follows from (2.24a) imitates the power series argument for boosts. A straightforward calculation shows that all elements of SU(2) have the form (2.24b). Note that unitarity implies that:

$$U(\theta\boldsymbol{e})^{\mathrm{T}} = \overline{U(-\theta\boldsymbol{e})} \tag{2.25}$$

As an application of (2.24b), note that the spinor metric (2.4a) is a rotation by $-\pi$ about the y axis:$\boldsymbol{e}_2 \equiv (0,1,0)$:

$$\varepsilon = U(-\pi\boldsymbol{e}_2) \tag{2.26}$$

The three-dimensional rotation in $\mathrm{L}_+^\uparrow$ corresponding to θ and $\boldsymbol{e}$ is:

$$R(\theta\boldsymbol{e})^0{}_0 = 1 \qquad\qquad R(\theta\boldsymbol{e})^0{}_i = R(\theta\boldsymbol{e})^i{}_0 = 0 \tag{2.27a}$$

$$R(\theta\boldsymbol{e})^i{}_j = \tfrac{1}{2}\,\mathrm{Tr}\left[\sigma_i\,U(\theta\boldsymbol{e})\,\sigma_j\,U(\theta\boldsymbol{e})^*\right] \tag{2.27b}$$

$$= \delta_{ij}\,\cos\theta + e^i e^j\,(1-\cos\theta) - \epsilon_{ijk}\,e^k\sin\theta \tag{2.27c}$$

2.6.3 Homogeneous transformations

The fact that $A(p)$ and $U(\theta\boldsymbol{e})$ separately parameterize the hermitean positive definite and unitary matrices in SL(2, C) means, by polar decomposition, that every element A has the form:

$$A = A(p)\,U(\theta\boldsymbol{e}) \tag{2.28a}$$

Corresponding to that, every element Λ in $\mathrm{L}_+^\uparrow$ has the form:

$$\Lambda = \Lambda(p)\,R(\theta\boldsymbol{e}) \tag{2.28b}$$

2.6.4 Planar transformations

Let $y_1 \cdot y_1 = y_2 \cdot y_2 \equiv y \cdot y$:

$$M \equiv y \cdot y + y_2 \cdot \sigma \, y_1 \cdot \widetilde{\sigma} \tag{2.29a}$$

$$\det M = 2\, y \cdot y \left(y \cdot y + y_2 \cdot y_1 \right) \tag{2.29b}$$

Let $\det M \neq 0$, $\operatorname{sgn} {y_1}^0 = \operatorname{sgn} {y_2}^0$ when $y \cdot y > 0$, and $q \cdot y_1 = q \cdot y_2 = 0$:

$$B \equiv \frac{y \cdot y + y_2 \cdot \sigma \, y_1 \cdot \widetilde{\sigma}}{\sqrt{2\, y \cdot y \left(y \cdot y + y_2 \cdot y_1 \right)}} \in \mathrm{SL}(2, \mathrm{C}) \tag{2.30a}$$

$$B \, y_1 \cdot \sigma \, B^* = y_2 \cdot \sigma \qquad B \, q \cdot \sigma \, B^* = q \cdot \sigma \tag{2.30b}$$

$$A \, B(y_1, y_2) A^{-1} = B(\Lambda y_1, \Lambda y_2) \tag{2.30c}$$

$$\Lambda(B)^\mu{}_\nu = g^\mu{}_\nu - \frac{1}{y \cdot y + y_2 \cdot y_1} \Bigg[{y_1}^\mu \, y_{1\nu} + {y_1}^\mu \, y_{2\nu} + {y_2}^\mu \, y_{2\nu} - \frac{y \cdot y + 2\, y_2 \cdot y_1}{y \cdot y} \, {y_2}^\mu \, y_{1\nu} \Bigg] \tag{2.31a}$$

$$\Lambda(B) \, y_1 = y_2 \qquad \Lambda(B) \, q = q \tag{2.31b}$$

Note that the boost $A(p)$ in (2.18a), as well as its Λ counterpart in (2.21), is a planar transformation with $y_1 = (m, 0, 0, 0)$ and $y_2 = (\omega, \boldsymbol{p})$.

2.6.5 Discrete transformations

Our notation for 4×4 inversions is inspired by that of Wightman [1, p. 171]:

$$I_P = G \,, \qquad I_T = -G \,, \qquad I_Y \equiv I_{PT} = -I \tag{2.32}$$

Together with the identity, these three matrices form the discrete symmetry subgroup of the homogeneous Lorentz group. Wightman gives a construction of covering groups by extending SL(2, C) to include them along with complex conjugation, which results in eight, nonisomorphic solutions [1, p. 172]. Only two of the covering group solutions turn out to be of interest for the representations of the Poincaré group considered in the following sections, and those can be covered by reformulating the problem in terms of automorphisms of SL(2, C) that correspond to I_P, I_T, and I_Y similarity transformations of $\mathrm{L}_+^\uparrow$, to be realized eventually in representations of the Poincaré group by a unitary operator $U(I_P)$ and antiunitary operators $U(I_T)$ and $U(I_Y)$.

The subgroup of automorphisms of $\mathrm{L}_+^\uparrow$ induced by the discrete symmetry subgroup of all Lorentz transformations consists of:

$$I\,\Lambda\,I = \Lambda \tag{2.33a}$$

$$I_P\,\Lambda\,I_P^{-1} = \Lambda^{\mathrm{T}-1} \equiv \widetilde{\Lambda} \qquad\qquad \widetilde{\Lambda}(A) = \Lambda(A^{*-1}) \tag{2.33b}$$

$$I_T\,\Lambda\,I_T^{-1} = \widetilde{\Lambda} \tag{2.33c}$$

$$I_Y\,\Lambda\,I_Y^{-1} = \Lambda \tag{2.33d}$$

Thus the discrete symmetries generate only two distinct $\mathrm{L}_+^\uparrow$ automorphisms, which correspond to two of the four natural SL(2, C) automorphisms, $A \to A$ and $A \to A^{*-1}$. We saw in (2.10a) and (2.10b) that the other two natural automorphisms are unitary equivalent to these. To label the automorphisms, let I_σ, $\sigma = I, P, T, Y$, stand for any of the elements of the 4×4 discrete symmetry group, and let $\mathcal{I}_\sigma$ be the corresponding automorphism of SL(2, C). Then we have:

$$I_\sigma\,\Lambda(A)\,I_\sigma^{-1} = \Lambda\left[\mathcal{I}_\sigma(A)\right] \tag{2.34a}$$

$$\mathcal{I}_P(A) = \mathcal{I}_T(A) = A^{*-1} \tag{2.34b}$$

$$\mathcal{I}_I(A) = \mathcal{I}_Y(A) = A \tag{2.34c}$$

Note that there is no SL(2, C) sign ambiguity in the definitions on the r.h.s. because $\mathcal{I}_\sigma$ preserves the SL(2, C) group law.

3 Clebsch-Gordan coefficients

The usual Clebsch-Gordan (CG) coefficients for the addition of angular momenta also appear in the reduction of tensor products of finite-dimensional representations of SL(2, C) and $L_+^\uparrow$ into irreducible components. Indeed they serve that function for the proper, homogeneous complex Lorentz group, for which the covering group is SL(2, C)⊗SL(2, C). This section reviews our notational conventions for spinors of higher spin and discusses the invariance of the CG coefficients under SL(2, C). In particular, CG coefficients are isotropic spinors, just as $g_{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\rho}$ are isotropic tensors.

3.1 Definition

There are various notations for the coefficients, but one dominant convention for their values, that of Wigner [2, 3] and Condon and Shortley [4], which we call the *standard phase convention*. The standard convention has two parts, the first, for angular momentum in general:

(i) The choice of real, positive coefficients in the ladder recursion formulas to define the relative phases of normalized eigenstates of $\boldsymbol{J}_z$ in an irreducible subspace.

(ii) The choice of a real, positive amplitude for the transition between total angular momentum eigenstates with maximum magnetic quantum number and product eigenstates of two angular momenta with maximum magnetic quantum number in the first factor.

Rose [5] follows the standard convention,[2] and his notation $C(j_1 j_2 j; m_1 m_2 m)$ for the coefficient is common. The Particle Data Group adopts the standard convention, and has an online link for CG coefficients, spherical harmonics and d-functions.

Expressed in terms of normalized angular momentum eigenvectors with conventional phases, the CG coefficient is the transition amplitude:[3]

$$C(j_1 j_2 j; m_1 m_2 m) = \langle j_1 m_1 j_2 m_2 \big| j_1 j_2 j m\rangle \tag{3.1a}$$

[2]Other authors who follow it include Blatt and Weisskopf [6], Edmonds [7], Fano and Racah [8], Messiah [9, 10], Merzbacher [11], Cohen-Tannoudji et al. [12, 13].

[3]We do not follow the common labeling $\langle j_1 j_2 m_1 m_2 \big| j_1 j_2 j m\rangle$ for the bra in the amplitude. The notation in (3.1a) emphasizes the tensor product nature of the bra.

The normalized bra is a simultaneous eigenvector of the commuting obsevables,

$$\boldsymbol{J_1}\cdot\boldsymbol{J_1},\ \boldsymbol{J}_{1z},\ \boldsymbol{J_2}\cdot\boldsymbol{J_2},\ \boldsymbol{J}_{2z},$$

with eigenvalues,

$$j_1(j_1+1),\ m_1,\ j_2(j_2+1),\ m_2;$$

and the normalized ket is a simultaneous eigenvector of the commuting observables,

$$\boldsymbol{J_1}\cdot\boldsymbol{J_1},\ \boldsymbol{J_2}\cdot\boldsymbol{J_2},\ \boldsymbol{J}\cdot\boldsymbol{J},\ \boldsymbol{J}_{z},$$

with eigenvalues,

$$j_1(j_1+1),\ \rho_2(j_2+1),\ j(j+1),\ m,$$

where $\boldsymbol{J}$ is the total angular momentum,

$$\boldsymbol{J} = \boldsymbol{J}_1+\boldsymbol{J}_2.$$

The bras and kets are orthogonal unless

$$m = m_1 + m_2\,, \tag{3.1b}$$

because both are eigenvectors of $\boldsymbol{J}_z = \boldsymbol{J}_{1z}+\boldsymbol{J}_{2z}$. It follows from the angular momentum algebra that the j's are nonnegative integers or half-integers that satisfy the *triangle condition*,

$$|j_1-j_2| \le j \le j_1+j_2\,. \tag{3.1c}$$

The standard phase convention guarantees that the transition amplitudes, and hence the CG coefficients, are real; so it is also true that

$$C(j_1j_2j;m_1m_2m) = \langle j_1j_2jm \big| j_1m_1j_2,m_2\rangle\ . \tag{3.1d}$$

Spinor notation for the CG coefficients replaces the magnetic quantum numbers m_1, m_2, and m by appropriately lower or upper, undotted or dotted spinor indices for the corresponding spins, taking the same values as the magnetic quantum numbers [16], e.g., $j, j-1, \cdots - j$. What is appropriate is determined by the transformation law of angular momentum eigenstates under the representation of SU(2) defined by the angular momentum Lie algebra.

For spin-1/2, the infinitesimal generators of the self-representation of SU(2), $\boldsymbol{J} = \boldsymbol{\sigma}/2$, obey the standard convention, so

$$(U)_{mm'} = \left\langle \tfrac{1}{2}m \right| \exp(-i\theta\boldsymbol{e}\cdot\boldsymbol{\sigma}/2) \left| \tfrac{1}{2}m' \right\rangle . \tag{3.2a}$$

In spite of the fact that we have written the spinor indices of the Pauli matrices as $\boldsymbol{\sigma}_{\alpha\dot{\beta}}$, the appropriate index type for U is:

$$U_{\alpha}{}^{\beta} = (U)_{\alpha\beta} \tag{3.2b}$$

This clash of conventions is unavoidable. First of all, the assignment of index types for the SL(2, C) spinor calculus is, as far we know, historically near universal. Furthermore, having one undotted and one dotted index for the Pauli matrix three-vector is consonant with the complex conjugate relationship between $\left\langle \tfrac{1}{2}m \right|$ and $\left| \tfrac{1}{2}m \right\rangle$. But if we think of U as a special $A \in \mathrm{SL}(2, \mathrm{C})$, it makes no sense to write $U_{\alpha\dot{\beta}}$, because dotted and undotted spinor types are inequivalent under SL(2, C).

Although there is a clash, there is no paradox. Under SU(2), the formerly inequivalent lower dotted and upper undotted indices are not only unitary equivalent; the matrix representations $\overline{U}$ and $U^{\mathrm{T}-1}$ are identical as unitary matrices. If we restrict to SU(2), it makes perfect sense to write the spinor indices of the Pauli matrices as $\sigma_{\mu\alpha}{}^{\beta}$, including the $\sigma_0 = I$ component.

3.2 Restriction to SU(2)

The basic properties of CG coefficients are determined by the angular momentum algebra. That includes the construction of the unitary, irreducible representations of SU(2), the reduction of tensor products of those representations by CG coefficients, and the SU(2) invariance of the coefficients. We describe the facts here, and refer to any of several texts for the proofs.[4]

The standard unitary irreducible representations of SU(2) in the active view[5]

[4]Cf. Rose [5], Edmonds [7], Merzbacher [11], which all use the standard phase convention. Merzbacher is especially clear.

[5]The minus sign in the exponential in (3.3a) corresponds to the active view; the passive view would have a plus sign. Wigner [2] and [3], Rose [5, pp. 16,17,48], Merzbacher [11, p. 413], and Cohen-Tannoudji et al. [13] take the active view. Edmonds [7], Fano and Racah [8] take the passive view.

are given by:

$$\left[\mathcal{D}^j(U)\right]_{mm'} \equiv \left\langle jm\right| \exp(-i\theta \boldsymbol{e}\cdot\boldsymbol{J}) \left|jm'\right\rangle = \left[\mathcal{D}^j(U)^{*-1}\right]_{mm'} \tag{3.3a}$$

$$\overline{\mathcal{D}^j(U)} = \mathcal{D}^j(\overline{U}) \tag{3.3b}$$

$$\mathcal{D}^j(U)^{\mathrm{T}} = \mathcal{D}^j(U^{\mathrm{T}}) \tag{3.3c}$$

$$\mathcal{D}^j(U)^{*} = \mathcal{D}^j(U^{*}) \tag{3.3d}$$

$$\mathcal{D}^j(U)^{-1} = \mathcal{D}^j(U^{-1}) \tag{3.3e}$$

$$\mathcal{D}^j(U_1)\,\mathcal{D}^j(U_2) = \mathcal{D}^j(U_1U_2) \tag{3.3f}$$

$$\mathcal{D}^j(U)_\alpha{}^\beta \equiv \left[\mathcal{D}^j(U)\right]_{\alpha\beta} \tag{3.3g}$$

The unitarity expressed in (3.3a) follows from the hermiticity of $\boldsymbol{J}$. Persistence of complex conjugation in (3.3b) follows by conjugating the transition amplitude in (3.3a) to change the sign of θ and transpose the magnetic quantum numbers, then applying (2.25). Persistence of transposition (3.3c) follows from that of complex conjugation, plus unitarity. Persistence of hermitian conjugation (3.3d) follows from that of complex conjugation and transposition, as does persistence of inversion (3.3e). The group composition law (3.3f) is less obvious; it follows as a special case from (**??**) in Section **??**, which expresses irreducible representations of SL(2, C) in terms of a tensor product of spin-1/2's.

The raising and lowering symbol $[\,j\,]$ is the natural generalization of ε. The evaluation in (3.4a) follows from (2.26) and explicit calculation of $\mathcal{D}^j[U(-\pi\boldsymbol{e}_2)]$:[6]

$$[\,j\,]^{\alpha\beta} = [\,j\,]_{\alpha\beta} \equiv \left[\mathcal{D}^j(\varepsilon)\right]_{\alpha\beta} = (-1)^{j-\alpha}\,\delta_\alpha{}^{-\beta} \tag{3.4a}$$

$$[\,j\,]^{\dot\alpha\dot\beta} = [\,j\,]_{\dot\alpha\dot\beta} \equiv \overline{[\mathcal{D}^j(\varepsilon)]_{\alpha\beta}} = \left[\mathcal{D}^j(\overline{\varepsilon})\right]_{\alpha\beta} = \left[\mathcal{D}^j(\varepsilon)\right]_{\alpha\beta} = [\,j\,]^{\alpha\beta} \tag{3.4b}$$

Thus the matrix $[\,j\,]$ is real and unitary, with the action:

$$[\,j\,]\,\mathcal{D}^j(U)\,[\,j\,]^{-1} = \mathcal{D}^j\left(\varepsilon U\varepsilon^{-1}\right) = \mathcal{D}^j\left(U^{\mathrm{T}-1}\right) = \mathcal{D}^j(U)^{\mathrm{T}-1} \tag{3.5a}$$

As with spin-1/2, we raise by contracting on the right index of $[\,j\,]^{\alpha\beta}$ or $[\,j\,]^{\dot\alpha\dot\beta}$ and lower by contracting on the left index of $[\,j\,]_{\alpha\beta}$ or $[\,j\,]_{\dot\alpha\dot\beta}$.

[6] Cf. Edmonds [7, p. 59].

The transformation law for angular momentum eigenstates is immediate from (3.3a):

$$\exp(-i\theta\boldsymbol{e}\cdot\boldsymbol{J})\,|jm\rangle = \sum_{m'} \left|jm'\right\rangle \mathcal{D}^j(U)_{m'}{}^m \tag{3.6a}$$

$$\langle jm|\exp(-i\theta\boldsymbol{e}\cdot\boldsymbol{J}) = \sum_{m'} \mathcal{D}^j(U)_m{}^{m'} \left\langle jm'\right| \tag{3.6b}$$

This gives two transformation laws for CG coefficients:

$$\begin{aligned}
&\langle jm_1 jm_2 \big| \exp(-i\theta\boldsymbol{e}\cdot\boldsymbol{J})\,\big|j_1 j_2 jm\rangle = \\
&\sum_{m_1' m_2' m'} \mathcal{D}^{j_1}(U)_{m_1}{}^{m_1'}\,\mathcal{D}^{j_2}(U)_{m_2}{}^{m_2'} \left\langle j_1 m_1' j_2 m_2' \middle| j_1 j_2 jm\right\rangle \\
&\qquad = \sum_{m'} \left\langle j_1 m_1 j_2 m_2 \middle| j_1 j_2 jm'\right\rangle \mathcal{D}^j(U)_{m'}{}^m
\end{aligned} \tag{3.7a}$$

$$\begin{aligned}
&\langle j_1 j_2 jm \big| \exp(-i\theta\boldsymbol{e}\cdot\boldsymbol{J})\,\big|jm_1 jm_2\rangle = \\
&\sum_{m'} \mathcal{D}^j(U)_m{}^{m'} \left\langle j_1 j_2 jm' \middle| j_1 m_1 j_2 m_2\right\rangle \\
&\qquad = \sum_{m_1' m_2' m'} \left\langle j_1 j_2 jm \middle| j_1 m_1' j_2 m_2'\right\rangle \mathcal{D}^{j_1}(U)_{m_1'}{}^{m_1}\,\mathcal{D}^{j_2}(U)_{m_2'}{}^{m_2}
\end{aligned} \tag{3.7b}$$

Among the two numerically equal ways of writing the CG coefficient, $\langle j_1 m_1 j_2 m_2 \, \big| j_1 j_2 jm\rangle$ transforms as an isotropic spinor with two lower indices and one upper index,[7] while $\langle j_1 j_2 jm \big| j_1 m_1 j_2 m_2\rangle$ is isotropic with one lower and two upper indices. As isotropic spinors they are not identical; raising and lowering on one produces the other multiplied by a phase factor.

We define "the" CG spinor as the one that reduces two lower indices, corresponding to a tensor product of representations of SU(2), to a single lower index, corresponding to an irreducible component of the equivalent direct sum:

$$\begin{aligned}
[\,j j_1 j_2\,]_\alpha{}^{\alpha_1\alpha_2} &\equiv C(j_1 j_2 j;\alpha_1\alpha_2\alpha) = \langle j_1 j_2 j\alpha \big| j_1\alpha_1 j_2\alpha_2\rangle \\
&= \overline{[\,j j_1 j_2\,]_\alpha{}^{\alpha_1\alpha_2}}
\end{aligned} \tag{3.8a}$$

$$[\,j j_1 j_2\,]_{\dot\alpha}{}^{\dot\alpha_1\dot\alpha_2} \equiv \overline{[\,j j_1 j_2\,]_\alpha{}^{\alpha_1\alpha_2}} \tag{3.8b}$$

[7] This property is discussed by Wigner [3, pp. 292–296] and Edmonds [7, p. 46].

Here are some standard properties of CG coefficients in spinor notation. They all have obvious dotted versions, because everything is real.

Reduction to zero angular momentum:

$$[\,0jj\,]_0{}^{\alpha\beta} = (-1)^{2j}[\,0jj\,]_0{}^{\beta\alpha} = \frac{[\,j\,]^{\alpha\beta}}{\sqrt{2j+1}} \tag{3.9}$$

Index inversion:

$$[\,jj_1j_2\,]^{\alpha}{}_{\alpha_1\alpha_2} = (-1)^{2j}\,[\,jj_1j_2\,]_{\alpha}{}^{\alpha_1\alpha_2} \qquad (-1)^{2j} = (-1)^{2j_1+2j_2} \tag{3.10}$$

Symmetry:

$$[\,jj_2j_1\,]_{\alpha}{}^{\alpha_2\alpha_1} = (-1)^{j-j_1-j_2}\,[\,jj_1j_2\,]_{\alpha}{}^{\alpha_1\alpha_2} \tag{3.11}$$

Orthogonality:

$$\delta_{jj'}\,\delta_{\alpha}{}^{\alpha'} = (-1)^{2j}\,[\,jj_1j_2\,]_{\alpha}{}^{\alpha_1\alpha_2}\,[\,j'j_1j_2\,]^{\alpha'}{}_{\alpha_1\alpha_2} \tag{3.12a}$$

$$\delta_{\alpha_1}{}^{\alpha_1'}\,\delta_{\alpha_2}{}^{\alpha_2'} = \sum_j (-1)^{2j}\,[\,jj_1j_2\,]_{\alpha}{}^{\alpha_1\alpha_2}\,[\,jj_1j_2\,]^{\alpha}{}_{\alpha_1'\alpha_2'} \tag{3.12b}$$

Equation (3.9), including the phase change for exchange of indices, follows by comparing (3.4a) with a calculation of the CG coefficient.[8] The CG index inversion formula (3.10) is the spinor form of magnetic quantum number reflection symmetry:[9]

$$C(j_1j_2j;-m_1,-m_2,-m) = C(j_1j_2j;m_1m_2m) \tag{3.13}$$

It follows from (3.4a).

We defer the spinor version of the SU(2) invariance laws (3.7a) and (3.7b) to Section 3.4, as a special case of SL(2, C) invariance.

3.3 Orthogonality

3.4 Extension to SL(2, C)

[8]Cf. [13, p. 1041]. Proportionality follows from the CG positive sign convention and ladder recursion, and the normalization follows by taking traces on the l.h.s. of the CG orthogonality condition (3.12b).

[9]Cf. [13, pp. 1039,1041].

4 Semi-bivectors

Semi-bispinors, semi-bivectors, semi-trivectors, etc., are discussed for the real and complex orthogonal groups with definite and indefinite metric by Élie Cartan [14]. The treatment here is based on the physics literature before around 1965, especially that of Maxwell "solid spherical harmonics" by Hans Joos [18, p. 72], adapted to our spinor notation.

For us, the two SL(2, C)-inequivalent types of semi-bivector are equivalent to any of the following three inequivalent pairs of types:

(i) second-rank, selfdual or antiselfdual $\mathrm{L}_{+}^{\uparrow}$ tensors;

(ii) $\mathrm{O}_{+}(3, \mathrm{C})$ three-vectors or their complex conjugates;

(iii) second-rank, symmetric SL(2, C) spinors with spin-$1/2$ lower undotted or upper dotted indices.

5 Spherical harmonics

6 Lightlike projections

Unitary, irreducible representations of the Poincaré group for massless particles obey a special calculus, which stems from properties of lightlike projection operators elevated from spin $1/2$ to spin s with D^s matrices. Throughout this section, p is on the positive momentum light cone:

$$p = (\omega, \boldsymbol{p}) \qquad \omega = |\boldsymbol{p}| > 0 \qquad p \cdot p = 0$$

7 Poincaré group

The inhomogeneous Lorentz group, or Poincaré group, has ten parameters, four for spacetime translations, and six for homogeneous Lorentz transformations. The group elements may be written as (a, Λ), $a \in \mathbb{R}^4$, $\Lambda \in \mathrm{L}(4, \mathrm{R})$, $a^\mu = (a^0, \boldsymbol{a})$. The covering group of the connected part of the Poincaré group $\mathrm{P}_+^\uparrow$ is the inhomogeneous SL(2, C) group, or iSL(2, C), with elements (a, A), $A \in \mathrm{SL}(2, \mathrm{C})$. Group laws:

$$(a, \Lambda)\,(a', \Lambda') = (a + \Lambda\, a', \Lambda\, \Lambda') \tag{7.1a}$$

$$(a, A)\,(a', A') = \big(a + \Lambda(A)\, a', A\, A'\big) \tag{7.1b}$$

7.1 Poincaré generators

For iSL(2, C), (or $\mathrm{P}_+^\uparrow$):

$$\big[\, M_{\mu\nu},\, M_{\lambda\rho} \,\big] = -i \left(g_{\mu\lambda}\, M_{\nu\rho} - g_{\mu\rho}\, M_{\nu\lambda} + g_{\nu\rho}\, M_{\mu\lambda} - g_{\nu\lambda}\, M_{\mu\rho} \right) \tag{7.2a}$$

$$\big[\, P_\mu,\, P_\nu \,\big] = 0 \tag{7.2b}$$

$$\big[\, M_{\mu\nu},\, P_\lambda \,\big] = -i \left(g_{\mu\lambda}\, P_\nu - g_{\nu\lambda}\, P_\mu \right) \tag{7.2c}$$

Scalar, vector, and second rank tensor operators relative to $M_{\mu\nu}$:

$$\big[\, M_{\mu\nu},\, S \,\big] = 0 \tag{7.3a}$$

$$\big[\, M_{\mu\nu},\, V_\lambda \,\big] = -i \left(g_{\mu\lambda}\, V_\nu - g_{\nu\lambda}\, V_\mu \right) \tag{7.3b}$$

$$\big[\, M_{\mu\nu},\, T_{\lambda\rho} \,\big] = -i \left(g_{\mu\lambda}\, T_{\nu\rho} - g_{\mu\rho}\, T_{\nu\lambda} + g_{\nu\rho}\, T_{\mu\lambda} - g_{\nu\lambda}\, T_{\mu\rho} \right) \tag{7.3c}$$

Here S is a generic notation for any scalar. Later it will be used for spin.

Three-vector forms:

$$\boldsymbol{J} = \big(M_{23}, M_{31}, M_{12}\big) \qquad J_i = \tfrac{1}{2}\,\epsilon_{ijk}\,M_{jk} \tag{7.4a}$$

$$\boldsymbol{K} = \big(M_{01}, M_{02}, M_{03}\big) \qquad K_i = M_{0i} \tag{7.4b}$$

$$\big[\,J_i,\,J_j\,\big] = i\,\epsilon_{ijk}\,J_k \tag{7.4c}$$

$$\big[\,J_i,\,K_j\,\big] = i\,\epsilon_{ijk}\,K_k \tag{7.4d}$$

$$\big[\,K_i,\,K_j\,\big] = -i\,\epsilon_{ijk}\,J_k \tag{7.4e}$$

$$\big[\,J_i,\,P^j\,\big] = i\,\epsilon_{ijk}\,P^k \qquad \big[\,J_i,\,P^0\,\big] = 0 \tag{7.4f}$$

$$\big[\,K_i,\,P^j\,\big] = i\,\delta_{ij}\,P^0 \qquad \big[\,K_i,\,P^0\,\big] = i\,P^i \tag{7.4g}$$

$$\tfrac{1}{2}\,M_{\mu\nu}\,M^{\mu\nu} = \boldsymbol{J}\cdot\boldsymbol{J} - \boldsymbol{K}\cdot\boldsymbol{K} \tag{7.4h}$$

$$\tfrac{1}{8}\,\epsilon_{\mu\nu\lambda\rho}\,M^{\mu\nu}\,M^{\lambda\rho} = \boldsymbol{J}\cdot\boldsymbol{K} = \boldsymbol{K}\cdot\boldsymbol{J} \tag{7.4i}$$

For any four-vector operator V^μ:

$$\big[\,J_i,\,V_j\,\big] = i\,\epsilon_{ijk}\,V_k \qquad \big[\,J_i,\,V_0\,\big] = 0 \tag{7.5a}$$

$$\big[\,K_i,\,V_j\,\big] = -i\,\delta_{ij}\,V_0 \qquad \big[\,K_i,\,V_0\,\big] = -i\,V_i \tag{7.5b}$$

The sign of the rotation generator $\boldsymbol{J}$ is unambiguous in these formulas. When it has an oribtal part, the sign of the boost generator $\boldsymbol{K}$ is fixed by its physical interpretation as the Energieschwerpunkt operator, Eq. (7.33b). The sign of any finite-dimensional part[10] generally results from the choice of representation.

7.2 Finite Lorentz transformations

The exponential parameterization of the active representation of the element (a, A) of iSL(2, C) corresponding to P^μ, $M_{\mu\nu}$ has the form:

$$U(a, A) = T(a)\,U(A) \tag{7.6a}$$

$$T(a) = \exp\big(i P\cdot a\big) \tag{7.6b}$$

$$U(A) = \exp\big(i\lambda\,\boldsymbol{n}\cdot\boldsymbol{K}\big)\;\exp\big(-i\theta\,\boldsymbol{e}\cdot\boldsymbol{J}\big) \tag{7.6c}$$

[10] Or signs, for reducible representations.

Here the representations are linear and generic, obeying the group law (7.1b) but not required to be unitary:

$$U(a,A)\,U(a',A') = U\left[a + \Lambda(A)a',\, AA'\right] \tag{7.7}$$

The group law gives the action of Lorentz transformations on the translation operator, which yields their action on general functions of momentum:

$$U(A)\,\exp\!\big(iP\cdot a\big)\,U(A)^{-1} = \exp\!\big[i\Lambda(A)^{-1}P\cdot a\big] \tag{7.8a}$$

$$\Rightarrow \qquad U(A) f(P)\, U(A)^{-1} = f\left[\Lambda(A)^{-1}P\right] \tag{7.8b}$$

The action of finite Lorentz transformations on scalar and vector operators corresponding to that of the infinitesimal generators in (7.3a) and (7.3b) is the following:

$$U(A)\,S\,U(A)^{-1} = S \tag{7.9a}$$

$$U(A) V_\mu\, U(A)^{-1} = V_\nu\, \Lambda(A)^\nu{}_\mu \tag{7.9b}$$

The extension to tensor operators is analogous.

7.3 Casimir operators

The Casimir invariants of the Poincaré group are the mass and the magnitude of spin. The mass operator

$$M^2 = P\cdot P \tag{7.10}$$

is a scalar function of P^μ, and hence commutes will all generators of the Poincaré group. We only consider $M^2 \geq 0$ in these notes.

The spin is described in Section 8.1 with the help of the Pauli-Lubanski vector:

$$w_\mu \equiv \tfrac{1}{2}\,\epsilon_{\mu\nu\lambda\rho}\, P^\nu\, M^{\lambda\rho} \tag{7.11a}$$

$$\left[\,P_\mu,\, w_\nu\,\right] = 0 \tag{7.11b}$$

$$\left[\,M_{\mu\nu},\, w_\lambda\,\right] = -i\left(g_{\mu\lambda}\, w_\nu - g_{\nu\lambda}\, w_\mu\right) \tag{7.11c}$$

$$\left[\,w_\mu,\, w_\nu\,\right] = i\,\epsilon_{\mu\nu\lambda\rho}\, P^\lambda\, w^\rho \tag{7.11d}$$

$$= i\left(P_\mu\, P^\lambda\, M_{\nu\lambda} - P_\nu\, P^\lambda\, M_{\mu\lambda} + M^2\, M_{\mu\nu}\right) \tag{7.11e}$$

7.4 Finite-dimensional SL(2, C) generators

$$\boldsymbol{M} \equiv \tfrac{1}{2}\left(\boldsymbol{J} + i\,\boldsymbol{K}\right) \qquad \boldsymbol{N} \equiv \tfrac{1}{2}\left(\boldsymbol{J} - i\,\boldsymbol{K}\right) \tag{7.12a}$$

$$\left[M_i,\, M_j\right] = i\,\epsilon_{ijk}\,M_k \quad \left[N_i,\, N_j\right] = i\,\epsilon_{ijk}\,N_k \quad \left[M_i,\, N_j\right] = 0 \tag{7.12b}$$

The finite-dimensional, irreducible representations are labeled by $(m.n)$, where m and n half integers:

$$\boldsymbol{M}\cdot\boldsymbol{M} = m(m+1) \qquad \boldsymbol{N}\cdot\boldsymbol{N} = n(n+1) \tag{7.13a}$$

$$(s,0):\ \ \boldsymbol{K} = -i\,\boldsymbol{J} \qquad \boldsymbol{M} = \boldsymbol{J} \qquad \boldsymbol{N} = 0 \tag{7.13b}$$

$$(0,s):\ \ \boldsymbol{K} = i\,\boldsymbol{J} \qquad \boldsymbol{M} = 0 \qquad \boldsymbol{N} = \boldsymbol{J} \tag{7.13c}$$

For any second rank, antisymetric tensor:

$$T^{\mathrm{D}}_{\mu\nu} \equiv \tfrac{i}{2}\,\epsilon_{\mu\nu\lambda\rho}\,T^{\lambda\rho} \qquad T_{\mu\nu} = T^{\mathrm{D}D}_{\mu\nu} \tag{7.14a}$$

$$T^{\mathrm{S}D}_{\mu\nu} = \tfrac{1}{2}\left(T_{\mu\nu} + \tfrac{i}{2}\,\epsilon_{\mu\nu\lambda\rho}\,T^{\lambda\rho}\right) \tag{7.14b}$$

$$T^{\mathrm{AS}D}_{\mu\nu} = \tfrac{1}{2}\left(T_{\mu\nu} - \tfrac{i}{2}\,\epsilon_{\mu\nu\lambda\rho}\,T^{\lambda\rho}\right) \tag{7.14c}$$

$$T_{\mu\nu} = T^{\mathrm{S}D}_{\mu\nu} + T^{\mathrm{AS}D}_{\mu\nu} \tag{7.14d}$$

$$T^{\mathrm{D}}_{\mu\nu} = T^{\mathrm{S}D}_{\mu\nu} - T^{\mathrm{AS}D}_{\mu\nu} \tag{7.14e}$$

The representations $(s,0)$ and $(0,s)$ are respectively selfdual and antiselfdual:

$$(s,0):\ \ M_{\mu\nu} = M^{\mathrm{S}D}_{\mu\nu} \qquad M^{\mathrm{AS}D}_{\mu\nu} = 0 \tag{7.15a}$$

$$(0,s):\ \ M_{\mu\nu} = M^{\mathrm{AS}D}_{\mu\nu} \qquad M^{\mathrm{S}D}_{\mu\nu} = 0 \tag{7.15b}$$

$$\left(\tfrac{1}{2},0\right):\ \ M_{\mu\nu} = \tfrac{i}{4}\left(\sigma_\mu\,\widetilde{\sigma}_\nu - \sigma_\nu\,\widetilde{\sigma}_\mu\right) \qquad \boldsymbol{J} = \tfrac{1}{2}\,\boldsymbol{\sigma} \qquad \boldsymbol{K} = -\tfrac{i}{2}\,\boldsymbol{\sigma} \tag{7.15c}$$

$$\left(0,\tfrac{1}{2}\right):\ \ M_{\mu\nu} = \tfrac{i}{4}\left(\widetilde{\sigma}_\mu\,\sigma_\nu - \widetilde{\sigma}_\nu\,\widetilde{\sigma}_\mu\right) \qquad \boldsymbol{J} = \tfrac{1}{2}\,\boldsymbol{\sigma} \qquad \boldsymbol{K} = \tfrac{i}{2}\,\boldsymbol{\sigma} \tag{7.15d}$$

The contravariant and covariant four-vector representations are equivalent to $(\frac{1}{2},\frac{1}{2})$. Generators for the contravariant vector representation:

$$\left[M^{\mathrm{V}}_{\mu\nu}\right]^{\lambda}{}_{\rho} = i\,(g^{\lambda}{}_{\mu}\,g_{\rho\nu} - g^{\lambda}{}_{\nu}\,g_{\rho\mu}) \tag{7.16a}$$

$$\left[J^{\mathrm{V}}_{i}\right]^{j}{}_{k} = -i\,\epsilon_{ijk} \qquad \left[J^{\mathrm{V}}_{i}\right]^{0}{}_{0} = \left[J^{\mathrm{V}}_{i}\right]^{0}{}_{j} = \left[J^{\mathrm{V}}_{i}\right]^{j}{}_{0} = 0 \tag{7.16b}$$

$$\left[K^{\mathrm{V}}_{i}\right]^{0}{}_{j} = \left[K^{\mathrm{V}}_{i}\right]^{j}{}_{0} = -i\,\delta_{ij} \qquad \left[K^{\mathrm{V}}_{i}\right]^{0}{}_{0} = \left[K^{\mathrm{V}}_{i}\right]^{j}{}_{k} = 0 \tag{7.16c}$$

To calculate (7.16a), use the fact that near the identity the matrices $\Lambda(A)$, A, and A^{*} can be written as exponentials,[11] which can be approximated by

$$\Lambda^{\lambda}{}_{\rho} \approx g^{\lambda}{}_{\rho} + i\,a^{\mu\nu}\left[M^{\mathrm{V}}_{\mu\nu}\right]^{\lambda}{}_{\rho} \qquad a^{\mu\nu} = -a^{\nu\mu} = \overline{a}^{\mu\nu} \tag{7.17a}$$

$$A \approx \boldsymbol{I} - \; a^{\mu\nu}\,\tfrac{1}{4}\left(\sigma_{\mu}\,\widetilde{\sigma}_{\nu} - \sigma_{\nu}\,\widetilde{\sigma}_{\mu}\right) \tag{7.17b}$$

$$A^{*} \approx \boldsymbol{I} - \; a^{\mu\nu}\,\tfrac{1}{4}\left(\widetilde{\sigma}_{\nu}\,\sigma_{\mu} - \widetilde{\sigma}_{\mu}\,\sigma_{\nu}\right) \tag{7.17c}$$

Then use Eqs. (2.3b) and (2.17b).

The exponential forms of the vector representations for rotations and boosts are:

$$R(\theta\boldsymbol{e}) = \exp\left(-i\,\theta\,\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right) \qquad \boldsymbol{e} = (e^{1}, e^{2}, e^{3}) \qquad \boldsymbol{e}\cdot\boldsymbol{e} = 1 \tag{7.18a}$$

$$= \boldsymbol{I} - i\,\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\sin\theta - \left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^{2}(1-\cos\theta) \tag{7.18b}$$

$$L(\lambda\boldsymbol{n}) = \exp\left(i\,\boldsymbol{n}\cdot\boldsymbol{K}^{\mathrm{V}}\right) \qquad \boldsymbol{n} = (n^{1}, n^{2}, n^{3}) \qquad \boldsymbol{n}\cdot\boldsymbol{n} = 1 \tag{7.19a}$$

$$= \boldsymbol{I} + i\,\boldsymbol{n}\cdot\boldsymbol{K}^{\mathrm{V}}\sinh\lambda - \left(\boldsymbol{n}\cdot\boldsymbol{K}^{\mathrm{V}}\right)^{2}(\cosh\lambda - 1) \tag{7.19b}$$

These formulas can be computed by power series manipulation with the help of

$$\left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^{3} = -\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}} \tag{7.20a}$$

$$\left(\boldsymbol{n}\cdot\boldsymbol{K}^{\mathrm{V}}\right)^{3} = \quad \boldsymbol{n}\cdot\boldsymbol{K}^{\mathrm{V}} \tag{7.20b}$$

[11] It is well-known that every $\Lambda \in \mathrm{L}^{\uparrow}_{+}$ can be written as the exponential of an element of its Lie algebra, while not every $A \in \mathrm{SL}(2,\mathbb{C})$ can. However, at least one of $\pm A$ is an exponential. A good exposition of this nontrivial matter can be found in [].

They can be connected to Eqs. (2.27a), (2.27c), and (2.22c) with the help of

$$\left[\left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^n\right]^0{}_0 = 1 \qquad \left[\left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^n\right]^0{}_i = \left[\left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^n\right]^i{}_0 = 0 \tag{7.21a}$$

$$\left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^i{}_j = -i\,\epsilon_{ijk}\,e^k \tag{7.21b}$$

$$\left[\left(\boldsymbol{e}\cdot\boldsymbol{J}^{\mathrm{V}}\right)^2\right]^i{}_j = \delta_{ij} - e^i e^j \tag{7.21c}$$

7.5 Orbital generators

Let $\mathcal{F}$ be a linear space of sufficiently differentiable complex functions $f(p)$, $p \in \mathbb{R}^4$. The following action defines a representation of $\mathrm{L}_+^\uparrow$ by linear operators $\mathrm{U}(\Lambda)$ on $\mathcal{F}$:

$$\mathrm{U}(\Lambda)f(p) = f(\Lambda^{-1}p) \tag{7.22}$$

We use the same symbol U in later sections to indicate a unitary representation; but here we are interested in infinitesimal generators for $\mathrm{L}_+^\uparrow$ on $\mathcal{F}$ only as differential operators, leaving aside any Hilbert space structure. Define the orbital generators $L_{\mu\nu}$ by the approximate action near the identity:

$$\mathrm{U}(\Lambda)f(p) \approx f(p) + i\,a^{\mu\nu}\,L_{\mu\nu}f(p) \approx f(p - i\,a^{\mu\nu}\,\boldsymbol{M}^{\mathrm{V}}_{\mu\nu}p) \tag{7.23}$$

where $\boldsymbol{M}^{\mathrm{V}}_{\mu\nu}$ is the generator for the four-vector representation in Eq. (7.16a). That equation together with Taylor expansion of the r.h.s. gives the result:

$$L_{\mu\nu} = i\left(p_\mu\,\frac{\partial}{\partial p^\nu} - p_\nu\,\frac{\partial}{\partial p^\mu}\right) \tag{7.24}$$

It is easy to check that these linear operators on $\mathcal{F}$, together with P^μ defined as multiplication by p^μ, obey the Poincaré Lie algebra.

Now consider the restriction of $\mathcal{F}$ to the mass shell. Let $\widehat{\mathcal{F}}$ be a linear space of sufficiently differentiable functions $\widehat{f}(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{R}^3$. Then a direct calculation shows that the following linear operators on $\widehat{\mathcal{F}}$ also obey the Poincaré Lie algebra:

$$\widehat{L}_{0j} = i\omega\,\frac{\partial}{\partial p^j} \tag{7.25a}$$

$$\widehat{L}_{ij} = i\left(p_i\,\frac{\partial}{\partial p^j} - p_j\,\frac{\partial}{\partial p^i}\right) = -i\left(p^i\,\frac{\partial}{\partial p^j} - p^j\,\frac{\partial}{\partial p^i}\right) \tag{7.25b}$$

$$\widehat{P}^\mu = \widehat{p}^\mu = (\omega, \boldsymbol{p}) \qquad \omega = \sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + m^2} \qquad m \geq 0 \tag{7.25c}$$

Definition. Consider the mass shell restriction map from $\mathcal{F}$ onto $\widehat{\mathcal{F}}$, $\widehat{f}(\boldsymbol{p}) = f(\widehat{p})$. A linear operator $\mathcal{O}$ on $\mathcal{F}$ is said to be restrictable to the mass shell if there exists a linear operator $\widehat{\mathcal{O}}$ on $\widehat{\mathcal{F}}$ such that:

$$\widehat{\mathcal{O}}\widehat{f} = \widehat{\mathcal{O}f} \tag{7.26}$$

Note that $\widehat{\mathcal{O}}$ is unique, and that the mapping $\mathcal{O} \to \widehat{\mathcal{O}}$ is an operator algebra homomorphism.

Lemma 1. $\widehat{L}_{\mu\nu}$ defined by Eqs. (7.25a) and (7.25b) is the mass shell restriction of $L_{\mu\nu}$.

Proof.

$$\begin{aligned}
\frac{\partial \widehat{f}}{\partial p^j}(\boldsymbol{p}) &= \frac{\partial f}{\partial p^0}(\omega, \boldsymbol{p})\,\frac{\partial \omega}{\partial p^j} + \frac{\partial f}{\partial p^j}(\omega, \boldsymbol{p}) \\
&= \frac{\partial f}{\partial p^0}(\omega, \boldsymbol{p})\,\frac{p^j}{\omega} + \frac{\partial f}{\partial p^j}(\omega, \boldsymbol{p}) \\
&= -\frac{p_j}{\omega}\,\frac{\partial f}{\partial p^0}(\omega, \boldsymbol{p}) + \frac{\partial f}{\partial p^j}(\omega, \boldsymbol{p})
\end{aligned}$$

The proof for $\mu\nu = 0j$ or $j0$ then follows directly from the definition of $\widehat{L}_{0j}$, Eq. (7.25a). For $\mu\nu = ij$, it follows from the definition of $\widehat{L}_{ij}$, Eq. (7.25b), especially its antisymmetry.

This lemma justifies an informal notation[12] where $L_{\mu\nu}$ is intended to be applied on the mass shell, but is written as in Eq. (7.24). Because of the algebra homomorphism, the lemma makes it unnecessary to do a calculation to show that $\widehat{L}_{\mu\nu}$ and $\widehat{P}^{\mu}$ obey the Poincaré algebra.

The following formula holds whether K_i is taken to be $\widehat{L}_{0i}$, with $P^0 = \sqrt{\boldsymbol{P}\cdot\boldsymbol{P} + M^2}$ and M fixed, or is taken to be L_{0i}, with P^0 independent of $\boldsymbol{P}$:

$$\left[\, K_i,\, f(P^0) \right] = i\, P^i\, f'(P^0) \tag{7.27}$$

[12] Sometimes seen, not recommended.

7.6 Formal position operators

The "formal" in "formal position operator" refers to the neglect of any Hilbert space structure, hermiticity in particular, for operators on $\mathcal{F}$ or $\widehat{\mathcal{F}}$. The basic requirements are that position operators be vectors under $\mathrm{L}_+^\uparrow$ or $\mathrm{O}_+(3)$ and satisfy canonical commutation relations (CCR). Satisfying CCR amounts to four- or three-dimensional translation covariance.

Note that on $\mathcal{F}$ the operators

$$x_\mu = -i\,\frac{\partial}{\partial p^\mu} \tag{7.28}$$

satisfy the $\mathrm{L}_+^\uparrow$ vector law (7.3b), and CCR:

$$\left[\,x^\mu,\,p^\nu\,\right] = -i\,g^{\mu\nu} \qquad\qquad \left[\,x^\mu,\,x^\nu\,\right] = \left[\,p^\mu,\,p^\nu\,\right] = 0 \tag{7.29}$$

We call such operators *Lorentz-covariant position operators*. We call the corresponding three-vector operators on $\mathcal{F}$ or $\widehat{\mathcal{F}}$ *rotatation-covariant position operators*.

Lemma 2. All Lorentz-covariant position operators on $\mathcal{F}$, respectively, rotation-covariant position operators on $\mathcal{F}$ or $\widehat{\mathcal{F}}$, have the form:

$$x^\mu = -i\,\frac{\partial}{\partial p_\mu} + p^\mu\,h(p\cdot p) \tag{7.30a}$$

$$x^i = i\,\frac{\partial}{\partial p^i} + p^i\,h(\boldsymbol{p}\cdot\boldsymbol{p}) \tag{7.30b}$$

It is a well-known fact that the CCR cannot be satisfied by physical observables for time and total energy in quantum mechanics, because that forces the spectra of both to be unbounded above and below, which violates the stability of total energy.[13] But as we said, formal position operators disgregard Hilbert space structure.

Note that $L_{\mu\nu}$ on $\mathcal{F}$, as defined in Eq. (7.24), has the same expression for all Lorentz-covariant position operators, and is independent of h:

$$L_{\mu\nu} = -p_\mu\,x_\nu + p_\nu\,x_\mu = x_\mu\,p_\nu - x_\nu\,p_\mu \tag{7.31}$$

The same applies to L_{ij} on $\mathcal{F}$ and $\widehat{L}_{ij}$ on $\widehat{\mathcal{F}}$ in terms of rotation-covariant position operators on the respective spaces.

[13] Often called *Pauli's Theorem* [15, p. 63, fn. 2].

Lemma 3. There is no Lorentz-covariant position operator on $\widehat{\mathcal{F}}$.

Proof. Suppose that $\boldsymbol{x}$ is a rotation-covariant position operator on $\widehat{\mathcal{F}}$. Then from the vector commutation law (7.3b), any x^0 that extends x^i to a fourvector must obey:

$$\left[\,\widehat{L}_{0i},\, x^j\,\right] = -i\left(g_0{}^j\, x_i - g_i{}^j x_0\right) = i\,\delta_i{}^j\, x^0 \tag{7.32}$$

But after substituting Eq. (7.25a) for $\widehat{L}_{0i}$ and Eq. (7.30b) for x^i into the commutator, it is straightforward to check that there is no h for which it is proportional to $\delta_i{}^j$.

For Lorentz-covariant position operators, the following lemma is a simple corollary of the preceding lemma. However, we give an independent proof.

Lemma 4. Neither the time nor the spatial components of x^μ on $\mathcal{F}$ can be restricted to the mass shell.

Proof. The point is that neither $\partial/\partial p^0$ nor $\partial/\partial p^i$ is restrictable. Let $f_1 = p^0 f(p)$ and $f_2 = \omega f(p)$. Then $\widehat{f}_1 - \widehat{f}_2 = 0$. But consider

$$\frac{\partial f_1}{\partial p^0} = f + p^0\,\frac{\partial f}{\partial p^0} \qquad \frac{\partial f_2}{\partial p^0} = \omega\,\frac{\partial f}{\partial p^0}$$

$$\frac{\partial f_1}{\partial p^i} = p^0\,\frac{\partial f}{\partial p^i} \qquad \frac{\partial f_2}{\partial p^i} = \frac{p^i}{\omega}\, f + \omega\,\frac{\partial f}{\partial p^i}$$

whence

$$\widehat{\frac{\partial f_1}{\partial p^\mu}} - \widehat{\frac{\partial f_2}{\partial p^\mu}} = \frac{\widehat{p}_\mu}{\omega}\, f$$

for which neither temporal nor spatial components vanish.

The upshot of the lemmas is that rotation-covariant position operators can be defined on $\widehat{\mathcal{F}}$ by Eq. (7.30b), but can be neither part of a four-vector nor the mass shell restriction of a rotation-covariant position operator on $\mathcal{F}$. The next lemma states the role of a special choice of rotation-covariant position.

Lemma 5. There is a unique rotation-covariant position operator on $\widehat{\mathcal{F}}$ for which $\widehat{L}_{0i}$ is the *Energieschwerpunkt* operator

$$\widehat{L}_{0i} = \tfrac{1}{2}\left(x^i\,\omega + \omega\, x^i\right)$$

namely

$$x^i = i\,\frac{\partial}{\partial p^i} - i\,\frac{p^i}{2\omega^2}$$

Proof. It is straightforward to check that putting the above x^i into the Energieschwerpunkt operator gives back Eq. (7.25a). Next, assume that an x^i exists for which $\widehat{L}_{0i}$ has the above form. Then

$$i\,\omega\,\frac{\partial}{\partial p^i} = \tfrac{1}{2}\left[\,x^i,\,\omega\,\right] + \omega x^i = i\,\frac{p^i}{2\omega} + \omega\,x^i$$

gives the result for x^i.

Of course this same position operator can be used to define $\widehat{L}_{ij}$. For convenience we collect the formulas:

$$x^i = i\,\frac{\partial}{\partial p^i} - i\,\frac{p^i}{2\omega^2} \tag{7.33a}$$

$$\widehat{L}_{0i} = \tfrac{1}{2}\left(x^i\,\omega + \omega\,x^i\right) \tag{7.33b}$$

$$\widehat{L}_{ij} = x^i\,p^j - x^j\,p^i \tag{7.33c}$$

If we replace $\widehat{\mathcal{F}}$ by $\mathrm{L}^2\,(\mathrm{d}^3 p/2\omega)$, a straightforward calculation shows that x^i in Eq. (7.33a) is the only hermitean, rotation-covariant position operator of the form in Eq. (7.30b). A similar calculation starting from Eqs. (7.25a) and (7.25b) shows that $\widehat{L}_{\mu\nu}$ is also hermitean. Instead we can just note that the hermiticity of x^i makes the right-hand sides of Eqs. (7.33b) and (7.33c) manifestly hermitian.

The operator in Eq. (7.33a) is the Newton-Wigner position operator, which we denote by x^i_{nw} when we want to be explicit.

Finally, let us mention that similarity transformation by a nonzero operator function of $\widehat{P}$ preserves the Poincaré commutator algebra, as does any similarity transformation, but changes the action of $\widehat{L}_{0i}$ and x^i on $\widehat{\mathcal{F}}$. We can use that to transform the generators and Newton-Wigner position from operators hermitean on $\mathrm{L}^2(\mathrm{d}^3p/2\omega)$ into operators hermitean on $\mathrm{L}^2(\mathrm{d}^3p)$:

$$\omega^{-\frac{1}{2}}\,\widehat{L}_{ij}\,\omega^{\frac{1}{2}} = -i\left(p^i\,\frac{\partial}{\partial p^j} - p^j\,\frac{\partial}{\partial p^i}\right) = \widehat{L}_{ij} \tag{7.34a}$$

$$\omega^{-\frac{1}{2}}\,\widehat{L}_{0i}\,\omega^{\frac{1}{2}} = i\,\omega\,\frac{\partial}{\partial p^i} + i\,\frac{p^i}{2\omega} = \widehat{L}_{0i} + i\,\frac{p^i}{2\omega} \tag{7.34b}$$

$$\omega^{-\frac{1}{2}}\,\widehat{P}^{\mu}\,\omega^{\frac{1}{2}} = \widehat{p}^{\mu} \tag{7.34c}$$

$$\omega^{-\frac{1}{2}}\,x^i_{\mathrm{nw}}\,\omega^{\frac{1}{2}} = i\,\frac{\partial}{\partial p^i} \tag{7.34d}$$

8 One massive particle

This section describes two equivalent, unitary, irreducible representations of the covering group $\mathrm{iSL}(2,\mathrm{C})$ of the Poincaré group, for discrete mass $m > 0$ and spin s, the Wigner and spinor representations.

8.1 Wigner representation

8.1.1 Hilbert space

The Hilbert space for the Wigner representation is

$$\psi(p)_\lambda = \langle p, \lambda | \psi \rangle \in \mathrm{L}^2(\mathrm{d}^3 p/2\omega) \otimes \mathbb{C}^{2s+1} \qquad p^\mu = (\omega, \boldsymbol{p}) \tag{8.1a}$$

$$\langle \psi, \psi \rangle = \int \frac{\mathrm{d}^3 p}{2\omega}\, \Sigma_\lambda \left|\psi(p)_\lambda\right|^2 \tag{8.1b}$$

For inner products, we tend to use Dirac bra-ket notation like that on the r.h.s. of (8.1a) when focusing on eigenstates of momentum, and otherwise comma notation like that on the l.h.s. of (8.1b).

8.1.2 Transformation law

In the following, note the distinction between $\mathrm{U}(A)$, which is a unitary operator on $\mathrm{L}^2(\mathrm{d}^3 p/2\omega) \otimes \mathbb{C}^{2s+1}$, and $U(p)$, which is a 2×2 unitary matrix in SU(2). For $A \in \mathrm{SL}(2,\mathrm{C})$, $\mathrm{D}^s(A)$ is the $(2s{+}1)$-dimensional, irreducible representation (s,0), corresponding to a lower, undotted spinor index. When restricted to the SU(2) subgroup, it coincides with the standard unitary, irreducible representation.

$$\mathrm{U}(a, A) = \exp(i P \cdot a)\, \mathrm{U}(A) \tag{8.2a}$$

$$\langle p, \lambda |\, \exp(i P \cdot a)\, \psi \rangle = e^{ip\cdot a} \langle p, \lambda | \psi \rangle \tag{8.2b}$$

$$\langle p, \lambda | \mathrm{U}(A)\, \psi \rangle = \mathrm{D}^s\!\left[B(p)^{-1} A\, B(\Lambda^{-1} p)\right]_\lambda{}^{\lambda'} \left\langle \Lambda^{-1} p, \lambda' \middle| \psi \right\rangle \tag{8.2c}$$

$$B(p) = \sqrt{\frac{p\cdot\sigma}{m}}\, U(p) \qquad U(p)^* = U(p)^{-1} \tag{8.2d}$$

$$\langle \mathrm{U}(A)\, \psi | \mathrm{U}(A)\, \psi \rangle = \langle \psi | \psi \rangle \qquad \mathrm{U}(A)^* = \mathrm{U}(A)^{-1} \tag{8.2e}$$

The unitarity of $\mathrm{U}(A)$ expressed by Eq. (8.2e) follows from the unitarity of the argument of D^s in the transformation law (8.2c). That in turn follows from the definition (8.2d) of $B(p)$, and a threefold application of the transformation law (2.3a) for covariant Pauli matrices:

$$\left[B(p)^{-1} A\, B(\Lambda^{-1} p)\right] m\, \sigma_0 \left[B(\Lambda^{-1} p)^{*}\, A^{*}\, B(p)^{-1*}\right] = m\, \sigma_0 \tag{8.3}$$

The combination $B(p)^{-1} A\, B(\Lambda^{-1} p) \in \mathrm{SU}(2)$ corresponds to a member of the Wigner little group of the rest frame value of the four-vector p.

8.1.3 Infinitesimal generators

We rename the infinitesimal generators (7.15a) and (7.15c) of the selfdual, (s,0) representation as

$$\sigma_{\mu\nu}(s,0) = \tfrac{i}{2}\, \epsilon_{\mu\nu\lambda\rho}\, \sigma^{\lambda\rho}(s,0) \tag{8.4a}$$

$$\sigma_{\mu\nu}(1/2,0) = \tfrac{i}{4}\, \left(\sigma_\mu\, \widetilde{\sigma}_\nu - \sigma_\mu\, \widetilde{\sigma}_\mu\right) \tag{8.4b}$$

$$\mathrm{D}^s(A)\ \sigma_{\mu\nu}(s,0)\, \mathrm{D}^s\left(A^{-1}\right) = \sigma_{\lambda\rho}(s,0)\, \Lambda^\lambda{}_\mu\, \Lambda^\rho{}_\nu \tag{8.4c}$$

We use the names $\widehat{\boldsymbol{L}} = (\widehat{L}_1, \widehat{L}_2, \widehat{L}_3)$ and $\widehat{\boldsymbol{K}} = (\widehat{K}_1, \widehat{K}_2, \widehat{K}_3)$ for the on-shell orbital generators of rotations and boosts:

$$\widehat{L}_i = \tfrac{1}{2}\, \epsilon_{ijk}\, \widehat{L}_{jk} \qquad\qquad \widehat{K}_i = \widehat{L}_{0i} \tag{8.5}$$

Recall the definitions (7.25a) and (7.25b) for $\widehat{L}_{\mu\nu}$.

To calculate the action of the generators $\boldsymbol{M}_{\mu\nu}$ of $\mathrm{U}(A)$, let the group parameters $a^{\mu\nu} = -a^{\nu\mu} = \overline{a}^{\,\mu\nu}$ be sufficiently small to write:

$$A = \exp[i\, a^{\mu\nu} \sigma_{\mu\nu}(1/2, 0)]$$

Then

$$\mathrm{D}^s(A) \approx I + i\, a^{\mu\nu} \sigma_{\mu\nu}(s,0) \tag{8.6a}$$

$$f(\Lambda^{-1} p) \approx f(p) + i\, a^{\mu\nu} \widehat{L}_{\mu\nu}\, f(p) \tag{8.6b}$$

$$\mathrm{U}(A) \approx I + i\, a^{\mu\nu} \boldsymbol{M}_{\mu\nu} \tag{8.6c}$$

Using function instead of braket notation, we can read off the first-order terms in the expansion of the r.h.s. of (8.2c), taking the D-matrix factors in left to right order:

$$
\begin{aligned}
&\mathrm{D}^s\big[B(p)^{-1}A\,B(\Lambda^{-1}p)\big]_\lambda{}^{\lambda'}\,\psi(\Lambda^{-1}p)_{\lambda'} - \psi(p)_\lambda \\
&\qquad \approx i\,a^{\mu\nu}\Big\{\Big(\mathrm{D}^s\big[B(p)^{-1}\big]\sigma_{\mu\nu}(s,0)\mathrm{D}^s\big[B(p)\big]\Big)_\lambda{}^{\lambda'} \\
&\qquad\qquad + \Big(\mathrm{D}^s\big[B(p)^{-1}\big]\Big[\,\widehat{L}_{\mu\nu},\,\mathrm{D}^s\big[B(p)\big]\,\Big]\Big)_\lambda{}^{\lambda'} + \delta_\lambda{}^{\lambda'}\,\widehat{L}_{\mu\nu}\Big\}\,\psi(p)_{\lambda'}
\end{aligned}
\tag{8.7}
$$

Reordering the terms to put the orbital generator first, we find:

$$M_{\mu\nu} = \widehat{L}_{\mu\nu} + \mathrm{D}^s\big[B(p)^{-1}\big]\,\big[\,\widehat{L}_{\mu\nu},\,\mathrm{D}^s\big[B(p)\big]\,\big] \tag{8.8a}$$

$$\qquad + \mathrm{D}^s\big[B(p)^{-1}\big]\sigma_{\mu\nu}(s,0)\,\mathrm{D}^s\big[B(p)\big] \tag{8.8b}$$

$$
\begin{aligned}
\widehat{S}_{\mu\nu} \equiv\ &\mathrm{D}^s\big[B(p)^{-1}\big]\,\big[\,\widehat{L}_{\mu\nu},\,\mathrm{D}^s\big[B(p)\big]\,\big] \\
&+ \mathrm{D}^s\big[B(p)^{-1}\big]\sigma_{\mu\nu}(s,0)\,\mathrm{D}^s\big[B(p)\big]
\end{aligned}
\tag{8.8c}
$$

$$= \mathrm{D}^s\big[B(p)^{-1}\big]\,\Big[\,\widehat{L}_{\mu\nu} + \sigma_{\mu\nu}(s,0),\,\mathrm{D}^s\big[B(p)\big]\,\Big] + \sigma_{\mu\nu}(s,0) \tag{8.8d}$$

$$M_{\mu\nu} = \widehat{L}_{\mu\nu} + \widehat{S}_{\mu\nu} \tag{8.8e}$$

$$= \mathrm{D}^s\big[B(p)^{-1}\big]\,\big[\widehat{L}_{\mu\nu} + \sigma_{\mu\nu}(s,0)\big]\,\mathrm{D}^s\big[B(p)\big] \tag{8.8f}$$

Here Eq. (8.8f) follows directly from the (8.8c) form of $\widehat{S}_{\mu\nu}$. Note that in (8.8b–8.8f), $B(p)$ and $\mathrm{D}^s\big[B(p)\big]$ may be replaced by the corresponding operator functions of P, namely, $B(P)$ and $\mathrm{D}^s\big[B(P)\big]$. We intend to do that wherever appropriate in the rest of the discussion.

Add hermitian conjugation. Hermitiicty of orbital and spin parts.

8.1.4 Spin

Let $\Lambda(\boldsymbol{P}) = \Lambda[B(\boldsymbol{P})]$. The Pauli-Lubanski vector (7.11a) obeys:

$$w_\mu = \tfrac{1}{2}\,\epsilon_{\mu\nu\lambda\rho}\,\boldsymbol{P}^\nu \boldsymbol{M}^{\lambda\rho} = \tfrac{1}{2}\,\epsilon_{\mu\nu\lambda\rho}\,\boldsymbol{P}^\nu \widehat{S}^{\lambda\rho} \tag{8.9a}$$

$$= \tfrac{1}{2}\,\mathrm{D}^s\big[B(\boldsymbol{P})^{-1}\big]\,\epsilon_{\mu\nu\lambda\rho}\,\boldsymbol{P}^\nu \sigma^{\lambda\rho}(s,0)\,\mathrm{D}^s\big[B(\boldsymbol{P})\big] \tag{8.9b}$$

$$= \tfrac{1}{2}\,\epsilon_{\mu\nu}{}^{\lambda\rho}\,\boldsymbol{P}^\nu \sigma_{\lambda'\rho'}(s,0)\,\Lambda(\boldsymbol{P})^{-1\lambda'}{}_{\lambda}\,\Lambda(\boldsymbol{P})^{-1\rho'}{}_{\rho} \tag{8.9c}$$

$$= \epsilon_{\nu 0}{}^{ij}\,\tfrac{1}{2}\,m\,\sigma_{ij}(s,0)\,\Lambda(\boldsymbol{P})^{-1\nu}{}_{\mu} \tag{8.9d}$$

The spin operator $\boldsymbol{S} = (S_1, S_2, S_3)$ is defined as the spatial part of the Pauli-Lubanski vector in the rest frame of $\boldsymbol{P}$ corresponding to $\Lambda(\boldsymbol{P})^{-1}$, divided by the mass m:

$$s_\mu \equiv \Lambda(\boldsymbol{P})^{-1}{}_{\mu}{}^{\nu}\,\frac{w_\nu}{m} = (0, \boldsymbol{S}) \tag{8.10a}$$

$$S_i = \tfrac{1}{2}\,\epsilon_{ijk}\,\sigma_{jk}(s,0) \tag{8.10b}$$

$$\big[\,S_i,\ S_j\,\big] = i\epsilon_{ijk}\,S_k \qquad \big[\,S_i,\ \boldsymbol{S}\cdot\boldsymbol{S}\,\big] = 0 \tag{8.10c}$$

$$\boldsymbol{S}\cdot\boldsymbol{S} = -\frac{w\cdot w}{m} \tag{8.10d}$$

8.1.5 Commuting observables

A complete commuting set of observables for the massive irreducible representation can now be identified as $\boldsymbol{P}$ and S_3, where the two Casimir invariants $\boldsymbol{P}\cdot\boldsymbol{P} = m^2$ and $\boldsymbol{S}\cdot\boldsymbol{S} = s(s+1)$ are represented as multiples of the identity. The commutativity of $\boldsymbol{P}$ and w from (7.11b), together with the definition (8.10a) of the rest-frame Pauli-Lubansky vector s in terms of w and a function of $\boldsymbol{P}$, imply the commutativity of momentum and spin.

8.1.6 Helicity

The helicity operator is defined as the projection of the total angular momentum operator $\boldsymbol{J} = \widehat{\boldsymbol{L}} + \widehat{\boldsymbol{S}}$ onto the direction of the three-momentum, $\boldsymbol{P}\cdot\boldsymbol{J}/|\boldsymbol{P}|$. From definition (7.11a), the time component of the Pauli-Lubanski vector is related to the helicity:

$$w_0 = -\tfrac{1}{2}\,\epsilon_{ijk}\,P^j M_{jk} = -\boldsymbol{P}\cdot\boldsymbol{J} \tag{8.11}$$

Since $\boldsymbol{P}\cdot\widehat{\boldsymbol{L}} = 0$, Eq. (8.8f) gives an expression that relates the helicity operator to the spin operator (8.10b):

$$\boldsymbol{P}\cdot\boldsymbol{J} = \mathrm{D}^s\big[B(p)^{-1}\big]\,\boldsymbol{p}\cdot\boldsymbol{S}\;\mathrm{D}^s\big[B(p)\big] = \mathrm{D}^s\big[U(p)^*\big]\,\boldsymbol{p}\cdot\boldsymbol{S}\;\mathrm{D}^s\big[U(p)\big] \tag{8.12}$$

The last equality follows because, from Eq. (7.13b), the exponential form of the $(s,0)$ representation of the boost $A(p)$ in $B(p)$ is

$$\mathrm{D}^s\big[A(p)\big] = \exp\big(\lambda\,\boldsymbol{p}\cdot\boldsymbol{S}/|\boldsymbol{p}|\big) \tag{8.13}$$

which commutes with $\boldsymbol{p}\cdot\boldsymbol{S}$.

8.1.7 Canonical convention

There are two common conventions for the rotation $U(p)$ in the definition of $B(p)$ in (8.2d). This section covers the canonical convention, corresponding to a pure boost from $(m,0,0,0)$ to p:

$$U(p) = I \qquad B(p) = A(p) \equiv \sqrt{p\cdot\sigma/m} \tag{8.14}$$

In this case the rotation part of $\widehat{S}_{\mu\nu}$ simplifies:

$$\widehat{S}_i = \tfrac{1}{2}\,\epsilon_{ijk}\,\sigma_{jk}(s,0) = S_i \tag{8.15}$$

Proof. The most direct argument[14] is to restrict the transformation law (8.2c) to SU(2), and to use the rotation covariance of SL(2, C) boosts:

$$\begin{aligned}
&\mathrm{D}^s\Big(\sqrt{p\cdot\widetilde{\sigma}/m}\;U\sqrt{R^{-1}p\cdot\sigma/m}\Big)\\
&\quad= \mathrm{D}^s\Big(\sqrt{p\cdot\widetilde{\sigma}/m}\;U\sqrt{R^{-1}p\cdot\sigma/m}\;U^*U\Big)\\
&\quad= \mathrm{D}^s\Big(\sqrt{p\cdot\widetilde{\sigma}/m}\;\sqrt{p\cdot\sigma/m}\;U\Big)\\
&\quad= \mathrm{D}^s(U)
\end{aligned} \tag{8.16}$$

Hence

$$\langle p,\lambda|\mathrm{U}(U)\,\psi\rangle = \mathrm{D}^s(U)_\lambda{}^{\lambda'}\,\Big\langle R^{-1}p,\lambda'\Big|\psi\Big\rangle \tag{8.17}$$

from which the spin part of the rotation generator is clearly the rotation generator of the $(s,0)$ representation.

[14] Which I learned from Greg Weeks, private commnunication, 2012. I am astonished not to have known this fact before.

We sketch a more involved approach. Think of the commutator in Eq. (8.8d) as the action of the total generator on the rotation-covariant, second rank spinor function of $\boldsymbol{p}$:

$$\mathrm{D}^s\left(\sqrt{p\cdot\sigma/m}\right)_\lambda{}^\rho \in \widehat{\mathcal{F}} \otimes \mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$$

The above spinor function is *invariant* if $\boldsymbol{R}^{-1}$ is applied to its momentum argument at the same time that $\mathrm{D}^s(\boldsymbol{U}) \otimes \mathrm{D}^s(U)^{-1\,\mathrm{T}}$ is applied to its spinor indices. In this language, the total rotation generator must be zero, which means the commutator in (8.8d) must vanish for the canonical convention when restricted to rotations. This argument can be extended to covariant spinor functions, relative to either SU(2) or SL(2, C), of any rank with any number of vector arguments.

8.1.8 Helicity convention

The helicity convention for the rotation in $\boldsymbol{B}(p)$ obeys:

$$U(p)\,\sigma_3\,U(p)^* = \boldsymbol{p}\cdot\boldsymbol{\sigma}/|\boldsymbol{p}| \tag{8.18}$$

This of course determines $U(p)$ only up to a rotation about the 3-axis.

From Eq. (8.12) it follows that in the helicity convention the helicity operator is the third component of the spin operator:

$$\boldsymbol{P}\cdot\boldsymbol{J} = S_3 \tag{8.19}$$

8.1.9 Change of spin basis

Any sufficiently regular choice of the rotation $U(p)$ in the definition of $B(p)$ gives rise to a unitary, irreducible representation of the Poincaré group for nonzero mass m and spin s. Any two such choices must therefore produce unitary equivalent representations, and it is straightforward to find the unitary transformation between them. Let $B_1(p) = \sqrt{p\cdot\sigma/m}\,U_1(p)$ and $B_2(p) = \sqrt{p\cdot\sigma/m}\,U_2(p)$ correspond to $\mathrm{U}_1(a, A)$ and $\mathrm{U}_2(a, A)$ for the same mass and spin, with actions as defined in Sec. 8.1.2. Then:

$$\mathrm{U}_2(a, A) = \mathrm{D}^s\left[B_2(P)^* B_1(P)\right] \mathrm{U}_1(a, A)\,\mathrm{D}^s\left[B_1(P)^* B_2(P)\right] \tag{8.20a}$$

$$= \mathrm{D}^s\left[U_2(P)^* U_1(P)\right] \mathrm{U}_1(a, A)\,\mathrm{D}^s\left[U_1(P)^* U_2(P)\right] \tag{8.20b}$$

8.2 Spinor representation

The spinor representation factors the transformations to and from the particle rest frame out of the spin little group to produce a simple, $(2s+1)$-component spinor action, and adjusts the Hilbert-space metric to maintain the unitarity of the Poincaré group representation. As far as we know, this elegant idea originated independently with Henry Stapp [17] and Hans Joos [18].[15]

8.2.1 Metric induced by similarity

There is already an example of inducing a different Hilbert space metric by a similarity transformation in Sec. 7.6 on "Formal position operators". In that case, the similarity transformation is a function of momentum, and that will also be true for the spinor representation. But first we sketch the basic facts for the general case.[16]

Let S be a densely defined, invertible operator on a Hilbert space $\mathcal{H}$ with dense range such that $S^{-1*}S^{-1}$ is essentially self adjoint (and positive definite). We adopt the usual abuse of notation

$$S : \mathcal{H} \to \mathcal{H}_S \quad \text{and} \quad S^{-1} : \mathcal{H}_S \to \mathcal{H}$$

where S or S^{-1} may be only densely defined.

Let $\mathcal{H}_S$ be the Hilbert space induced from $\mathcal{H}$ by S; that is, for ψ in the domain of S and $\varphi = S\psi$ in the range of S:

$$\langle \varphi, \varphi \rangle_S \equiv \langle \psi, \psi \rangle = \left\langle \varphi, S^{-1*}S^{-1}\varphi \right\rangle \equiv \langle \varphi, \mathcal{M}\varphi \rangle \tag{8.21}$$

Note that when S is unitary, the induced metric operator $\mathcal{M} \equiv S^{-1*}S^{-1}$ is the identity. Any linear operator X on $\mathcal{H}$ compatible with S maps onto $Y \equiv SXS^{-1}$ compatible with S^{-1} on $\mathcal{H}_S$:

$$\langle \varphi, Y\varphi \rangle = \langle \psi, X\psi \rangle \tag{8.22}$$

[15]It was a key idea for Stapp's M-function formulation of analytic S-matrix theory, in particular for his proof of the CPT theorem, and the connection between spin and statistics. Joos presented it as a logical development in his definitive treatment of the representation theory of the Poincaré group. Neither author was aware of the other's work at the time of writing. Both works quickly became widely recognized.

[16]A summary of some of the technical issues, and further references, can be found in [19].

The $\mathcal{H}_S$ adjoint Y^A of a linear operator Y with domain dense in $\mathcal{H}_S$ is defined by:

$$\left\langle \varphi, Y^A \varphi \right\rangle_S \equiv \langle Y\varphi, \varphi\rangle_S \tag{8.23a}$$

$$\Rightarrow \quad Y^A = \mathcal{M}^{-1} Y \mathcal{M} = \mathcal{M}^* Y \mathcal{M} \tag{8.23b}$$

From (8.23b) it follows that the image of the $\mathcal{H}$ adjoint of X is the $\mathcal{H}_S$ adjoint of the image of X:

$$S X^* S^{-1} = (S X S^{-1})^A \tag{8.24}$$

Thus the image Y of an operator X Hermitean on $\mathcal{H}$ is Hermitian on $\mathcal{H}_S$:

$$X = X^* \quad \Rightarrow \quad Y = Y^A \tag{8.25}$$

8.2.2 Hilbert space

The lower undotted spinor realization of a unitary, irreducible representaion for a particle with nonzero mass m and spin s is the spinor Hilbert space $\mathcal{H}_S$ obtained from the Wigner space $\mathcal{H}$ by the following similarity transformation:

$$B : \mathcal{H} \to \mathcal{H}_S \qquad \varphi = B\psi \tag{8.26a}$$

$$B = \mathrm{D}^s\big[B(P)\big] \tag{8.26b}$$

$$B^{-1*} B^{-1} = \mathrm{D}^s\big(P\cdot\widetilde{\sigma}/m\big) \tag{8.26c}$$

$$\langle\varphi, \varphi\rangle_S = \left\langle \varphi, \mathrm{D}^s\big(P\cdot\widetilde{\sigma}/m\big)\ \varphi \right\rangle = \langle\psi, \psi\rangle \tag{8.26d}$$

$$= \int \frac{\mathrm{d}^3 p}{2\omega}\, \overline{\varphi_\alpha(p)}\ \mathrm{D}^s\big(p\cdot\widetilde{\sigma}/m\big)^{\dot{\alpha}\beta}\ \varphi_\beta(p) \tag{8.26e}$$

As in Eqs. (8.1a, 8.1b) for the Wigner representation, p is on the mass shell. Because the spin metric matrix $\mathrm{D}^s(p\cdot\widetilde{\sigma}/m)$ is not diagonal,[17] the components of the spinor wave function are not orthogonal in $\mathcal{H}_S$.

[17] Except at zero three-momentum, a set of zero measure.

The upper dotted realization is the spinor Hilbert space $\mathcal{H}^{\dot{S}}$:

$$B^{-1*} : \mathcal{H} \to \mathcal{H}^{\dot{S}} \qquad \varphi = B^{-1*}\psi \tag{8.27a}$$

$$B^{-1*} = \mathrm{D}^s\big[B(P)^{-1*}\big] \tag{8.27b}$$

$$BB^* = \mathrm{D}^s(P\cdot\sigma/m) \tag{8.27c}$$

$$\langle\varphi,\varphi\rangle^{\dot{S}} = \langle\varphi, \mathrm{D}^s(P\cdot\sigma/m)\ \varphi\rangle = \langle\psi,\psi\rangle \tag{8.27d}$$

$$= \int \frac{\mathrm{d}^3 p}{2\omega}\, \overline{\varphi^{\dot{\alpha}}(p)}\ \mathrm{D}^s(p\cdot\sigma/m)_{\alpha\dot{\beta}}\, \varphi^{\dot{\beta}}(p) \tag{8.27e}$$

The mappings between the two spaces are realized by raising and lowering with the spinor metric matrices:

$$\mathrm{D}^s\left(p\cdot\widetilde{\sigma}/m\right)^{\dot{\alpha}\beta}\varphi_\beta(p) = \varphi^{\dot{\alpha}}(p) \in \mathcal{H}^{\dot{S}} \tag{8.28a}$$

$$\mathrm{D}^s(p\cdot\sigma/m)_{\alpha\dot{\beta}}\,\varphi^{\dot{\beta}}(p) = \varphi_\alpha(p) \in \mathcal{H}_S \tag{8.28b}$$

8.2.3 Transformation law

A short calculation gives the simplified action and unitarity of the lower undotted spinor form of the irrecudible representation of the Poincaré group:

$$\big[\mathrm{U}_S(a,A)\varphi\big]\,(p)_\alpha = \exp(ip\cdot a)\,\mathrm{D}^s(A)_\alpha{}^\beta\varphi_\beta(\Lambda^{-1}p) \tag{8.29a}$$

$$\langle\mathrm{U}_S(a,A)\,\varphi, \mathrm{U}_S(a,A)\,\varphi\rangle_S = \langle\varphi,\varphi\rangle_S \tag{8.29b}$$

$$\mathrm{U}_S(a,A)^{-1} = \mathrm{U}_S(a,A)^A \tag{8.29c}$$

Note that the unitary equivalence expressed in (8.20a) between Wigner representations with same mass and spin based on B_1 and B_2 amounts to transforming from the Wigner representation $\mathrm{U}_1(a,A)$ to the spinor representation $\mathrm{U}_S(a,A)$, then back again to the Wigner representation $\mathrm{U}_2(a,A)$. The resultant induced Hilbert space metric operator is the identity, because the resultant similarity transformation is unitary.

8.2.4 Infinitesimal generators

Since the similarity transformation B is a function of momemtum, it has no effect on the four-momentum operator. The calculation for the Lorentz generators is trivial, given Eq. (8.8f). The results are:

$$P_S^\mu = B P^\mu B^{-1} = P^\mu = P_S^{\mu A} \tag{8.30a}$$

$$M_{S\mu\nu} = \widehat{L}_{\mu\nu} + \sigma_{\mu\nu}(s,0) = M_{S\mu\nu}{}^A \tag{8.30b}$$

Non-Hermiticity of orbital and spin parts.

8.2.5 Spin

8.2.6 Commuting observables

8.2.7 Helicity

8.3 Discrete symmetries

In this section we represent the discrete symmetries P, T, and Y by operators V_σ on the Hilbert space of an irreducible representation of iSL(2, C) for mass $m > 0$ and spin s. The general procedure will be the same for mass zero. By the standard argument, positive energy requires that V_P be unitary, and that V_T and V_Y be antiunitary.

The procedure is to solve the requirement that V_σ induce the following automorphism of the representation of iSL(2, C), where I_σ is given by (2.34a), (2.34b), and (2.34c):

$$V_\sigma \,\mathrm{U}(a, A)\, V_\sigma^* = \mathrm{U}\left[I_\sigma a, \mathcal{I}_\sigma(A)\right] \tag{8.31a}$$

It is easily seen that any solution for V_σ is unique up to a phase factor that depends only on σ, because if V_σ and V'_σ are both solutions, then $V_\sigma V'^{-1}_\sigma$ commutes with the irreducible representation $\mathrm{U}(a, A)$, and hence is a multiple of the identity. The phase factors are then constrained by the discrete group law. We simply list the results.

8.3.1 Wigner representation

In the following, $\widetilde{p} \equiv I_P\, p = (\omega, -\boldsymbol{p})$, and $U(p)$ is the SU(2) spin-axis rotation factor in $B(p)$ in (8.2d):

$$[V_P\, \psi](p)_\lambda = e^{i\alpha_P}\, V_P(p)_\lambda{}^{\lambda'}\, \psi(\widetilde{p})_{\lambda'} \qquad V_P(p) = U(p)^* U(\widetilde{p}) \tag{8.32a}$$

$$[V_T\, \psi](p)_\lambda = e^{i\alpha_T}\, \overline{V_T(p)_\lambda{}^{\lambda'}}\, \overline{\psi(\widetilde{p})_{\lambda'}} \qquad V_T(p) = \varepsilon\, U(p)^* U(\widetilde{p}) \tag{8.32b}$$

$$[V_Y\, \psi](p)_\lambda = e^{i\alpha_Y}\, \overline{V_Y(p)_\lambda{}^{\lambda'}}\, \overline{\psi(p)_{\lambda'}} \tag{8.32c}$$

9 One massless particle

9.1 Wigner representation

9.2 Zwanziger representation

10 Two massive particles

WIP: section title and organization

The technology for the decomposition of tensor products of irreducible representations of the Poincaré group into direct integrals of irreducible representations was fully developed in the late 1950's and early 1960's. We mention especially the work of Joos [], Wightman [1], Michel [], Luzzatto, Epstein, and Wightman [], Jacob and Wick [], Moussa and Stora [], MacFarlane [], Lomont [].

Formulas for generalized Clebsch-Gordan coefficients for the reduction of two-particle states were worked out in the references by Moussa and Stora, and by MacFarlane. We organize those results in the sections that follow.

10.1 Gårding-Wightman variables

First, we address the Lorentz-covariant kinematics for two particles with four-momenta p_1 and p_2 in the formulation of Gårding and Wightman,[18] along with the corresponding orbital infinitesimal generators.[19]

$$P = p_1 + p_2 \qquad q = \frac{\varphi_- \, p_1 - \varphi_+ \, p_2}{M\sqrt{\lambda}} \tag{10.1a}$$

$$p_1 = \frac{\varphi_+}{2M^2} P + \frac{\sqrt{\lambda}}{2M} q \qquad p_2 = \frac{\varphi_-}{2M^2} P - \frac{\sqrt{\lambda}}{2M} q \tag{10.1b}$$

$$p_1 \cdot p_1 = m_1^2 \qquad p_2 \cdot p_2 = m_2^2 \qquad P \cdot P = M^2 \tag{10.1c}$$

$$\varphi_\pm \equiv M^2 \pm \left({m_1}^2 - {m_2}^2 \right) \tag{10.1d}$$

$$\lambda \equiv M^4 + {m_1}^4 + {m_2}^4 - 2 \left(M^2 \, {m_1}^2 + M^2 {m_2}^2 + {m_1}^2 {m_2}^2 \right) \tag{10.1e}$$

Given that P is the total momentum, and allowing an arbitary sign for $\sqrt{\lambda}$, the above formula for q expresses the exactly the two linear combinations that satisfy the following two constraints:

$$P \cdot q = 0 \qquad q \cdot q = -1 \tag{10.2}$$

[18] See [1, pp. 198, 199].

[19] As before, the term *orbital* refers to the spinless part of the unitary representation of $\mathrm{L}_+^\uparrow$.

10.1.1 Threshhold

10.1.2 Equal mass

10.1.3 One zero mass

10.1.4 Two zero masses

10.2 Jacobians

The corresponding measures on the Hilbert space of two-particle functions are the following:

$$ \tag{10.3} $$

10.3 Wigner representation

10.3.1 Infinitesimal Generators

10.3.2 LS coupling

10.3.3 Helicity coupling

10.4 Spinor representation

11 One massive and one massless particle

11.1 Gårding-Wightman variables

11.2 Wigner representation

11.2.1 Poincaré Generators

11.2.2 LS coupling

11.2.3 Helicity coupling

11.3 Spinor representation

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