Review Notes on Waves

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The notes contain an independent treatment of much of the material in Serway, *Physics for Scientists and Engineers*, third edition updated version, Chapter 16, and Chapter 18, §§1-3. They are intended as a review of wave concepts relevant for sophomore level special relativity and quantum mechanics, not as a systematic development. They were used in Physics 242, Introduction to Modern Physics, at the University of Michigan in the Fall '95 term.

Contents

1   Characteristics of Waves

- Typically, a wave is a disturbance that propagates in a medium, which provides whatever it is that "waves". It is best to think in terms of a medium for an intuitive grounding. But the two important kinds of waves in this course are light waves (special relativity) and probability waves (quantum mechanics), neither of which propagates in a conventional medium.

- Waves undergo interference and diffraction (e.g., bending around corners). The engine for these phenomena is *superposition*.

- Waves tend to be spread out in space (or in the medium).

- The energy/intensity carried by waves is proportional to the square of whatever waves (at least for linear waves).

- Waves obey a wave equation. Although a somewhat nonintuitive mathematical abstraction (a partial differential equation),[1] it really packages many properties of waves in a very convenient form; and many kinds of linear waves obey the same wave equation, called "the" wave equation—so if you learn about one of those kinds you know about them all.

Actually a very large fraction of the waves of interest in physics are *linear waves*, the meaning of which we review later.

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[1] Don't worry—this isn't a course in partial differential equations!

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2   Characteristics of Particles

Now let's contrast the properties of waves with those of particles. Here we have in mind classical, not quantum particles. This partly points up the fact that particle notions were historically more intuitive than wave notions, and partly anticipates the "paradox" of wave-particle duality in quantum mechanics.

- Typically particles don't need a medium, and travel on well-defined orbits.

- Particles don't interfere or bend around corners.

- Particles are confined in space. Indeed, the natural abstraction in classical physics is to build matter out of point particles.

- The energy carried by particles is typically kinetic energy plus energy (often potential energy) for the interaction of some particle property, such as mass or charge, with a field, such as the gravitational field or an electric field.

- The motion of a particle is determined by equations of motion (ordinary differential equations) founded on Newton's law $f = ma$.

3   Propagation of Shapes

For concreteness, think of a moving pulse on a string stretched along the horizontal, x-direction, with displacements along the vertical, y-direction. The physical particles in the string actually move only in the vertical direction, but the shape of the pulse appears to move horizontally. To be concrete, we choose the pulse to have the particular shape plotted in the figure, called a *Gaussian*:

$$y = f(x) = C \exp\left(-\frac{x^2}{a^2}\right), \qquad C, a = \text{const.}$$
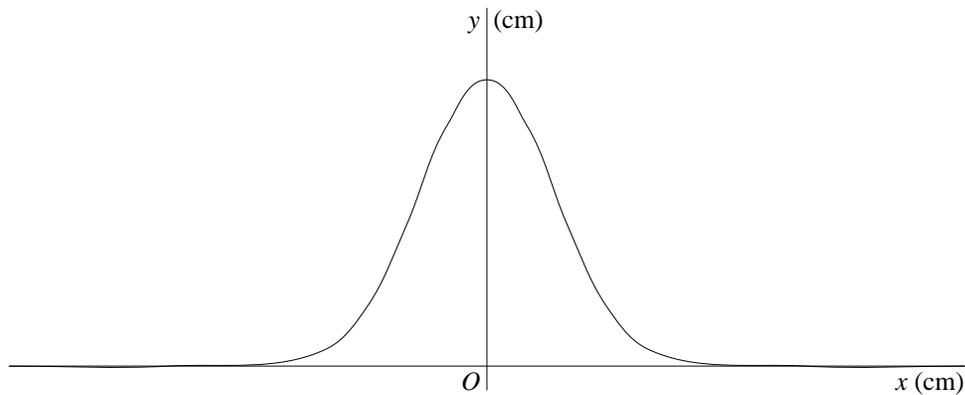
Here, x and y have the dimensions of length (cm).

Question: What is the dimension of the constant a?

Answer: *Length*, because the arguments of the standard mathematical functions are *dimensionless*.

Question: What feature of the curve is associated with the length a?

Answer: Its *width*.

Question: What is the dimension of the constant $C > 0$?

Answer: *Length*, the maximum height, at $x = 0$.

Now translate the shape a distance $b > 0$ to the right:

$$y = f(x - b) = C \exp\left(-\frac{(x - b)^2}{a^2}\right)$$
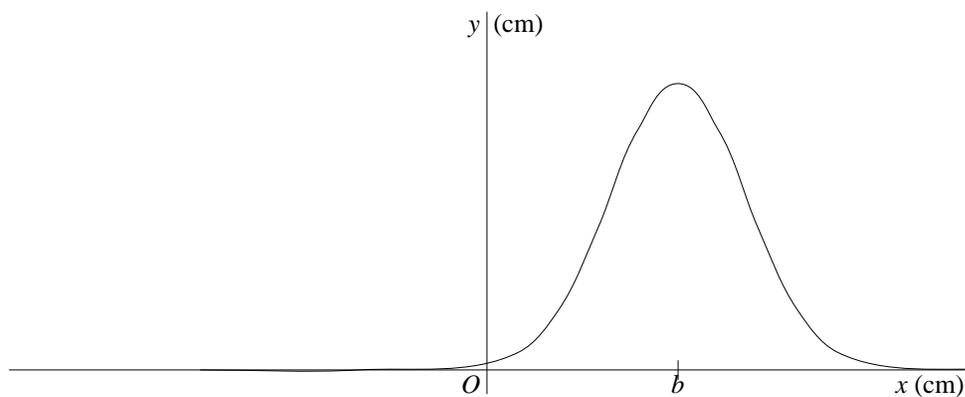
To get the shape moving to the right, we write the translation as proportional to time t (for positive time):

$$b = vt, \qquad \text{choose } v > 0, \qquad \dim t = \sec$$
$$\Rightarrow \dim v = \text{cm/sec (speed)}$$
$$y = f(x - vt) = C \exp\left(-\frac{(x - vt)^2}{a^2}\right)$$

To keep track of where the shape is, follow some feature, such as the maximum:

$$y = f(0) = C \qquad \Rightarrow \qquad x - vt = 0$$
$$x = vt \quad \text{position of max moving to the right}$$

To move the shape to the left, put

$$y = f(x + vt) = C \exp\left(-\frac{(x + vt)^2}{a^2}\right)$$

Now we can follow $y = C = f(0)$ by putting

$$x + vt = 0 \qquad \Rightarrow \qquad x = -vt \quad \text{position of max moving to the left}$$

We could follow any other point on the curve. For example, there are two points $\pm x_1$ where $f(\pm x_1) = C/3$. They all move to the right (or to the left) with the same speed v:

$$x_1 = x - vt \qquad \Rightarrow \qquad x = vt + x_1 \quad x \text{ still moves to the right}$$

Elaborating the notation: The waves moving to the right or left above are special cases of the more general situation that the spatial shapes of waves change with time. We indicate that mathematically by writing the displacement as a function of two variables, x and t:

$$y = y(x, t) \qquad \text{y is whatever waves}$$
$$\text{plot function of } x \text{ at fixed } t: \text{ "snapshot"}$$
$$\text{plot function of } t \text{ at fixed } x: \text{ waving at } x \text{ as time passes}$$

4 Superposition and Interference

For *linear* waves (the only kind we deal with in this course), if $y_1(x,t)$ and $y_2(x,t)$ are waves, then so is

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

which is called the *superposition* of y_1 and y_2. Linear waves add algebraically. This simple law is what gives rise to the fact that waves pass through each other without affecting each other.

Suppose y_1 and y_2 are waves with shapes like the Gaussian we used before, with y_1 traveling to the right, but y_2 traveling to the left and upside down. Let's say the snapshot of the superposition

$$y(x,t) = y_1(x - vt) + y_2(x + vt)$$

at time t looks like the plot below:
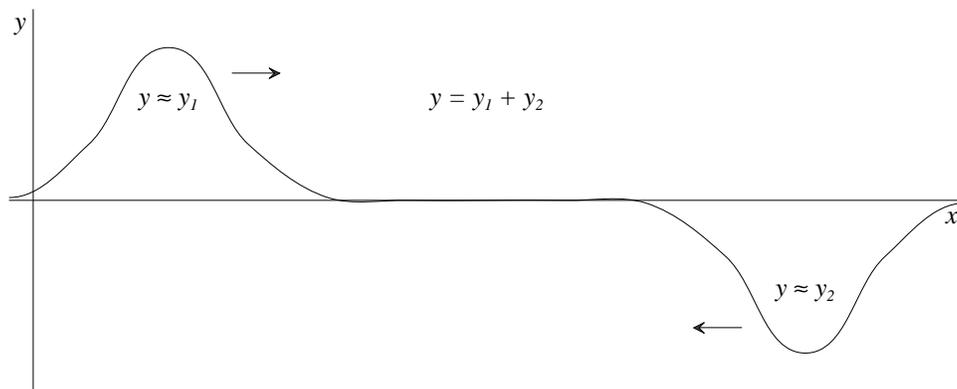
At some time t_0 later, the two shapes will be exactly opposite:

$$y_1(x,t_0) = -y_2(x,t_0) \qquad \Rightarrow \qquad y(x,t_0) = 0$$

At $t = t_0$, the snapshot is

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y | time $t=t_0$

$$y = y_1 + y_2 \equiv 0$$

x
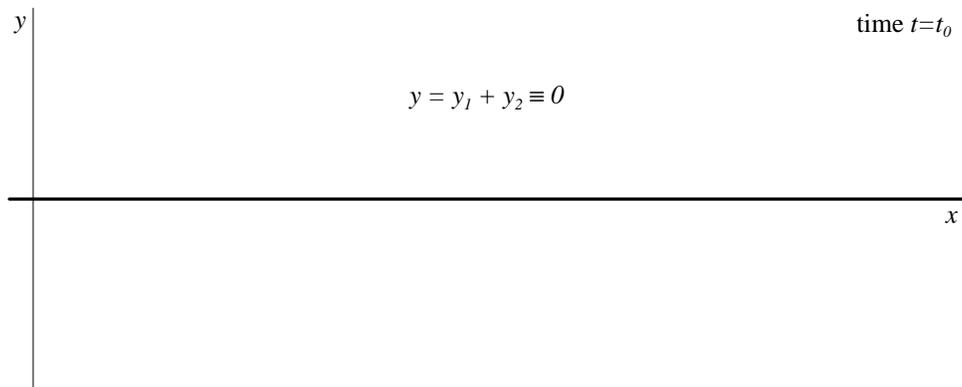
Somewhat later still, say at $t = t_1$, the waves have moved through each other, and the snapshot is

y | time $t=t_1$

$$y = y_1 + y_2$$

$y \approx y_1$

x

$y \approx y_2$
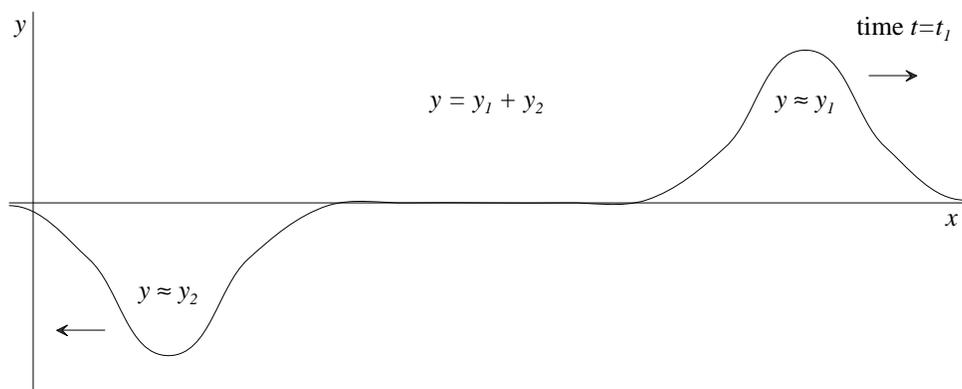
If the waves y_1 and y_2 have shapes that are only approximately upside down from each other, their superposition will still interfere destructively when the shapes are at the same position, but won't exactly cancel.

5 Wave Speed in Strings

A medium often supports waves with only a single wave speed, depending on the properties of the medium. That is true for strings stretched with the same tension T everywhere (true for massless strings), and the same mass per unit length μ.[2]

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[2]This is true for light in a vacuum, but not for light in glass, where the index of refraction and the wave speed vary with the frequency of the light.

We reproduce here the standard simple argument to find the wave speed in strings where the transverse displacements are small.

We assume the wave speed v for a small amplitude pulse moving to the right on the string is the same throughout the string. Suppose a snapshot of the moving shape looks like this:
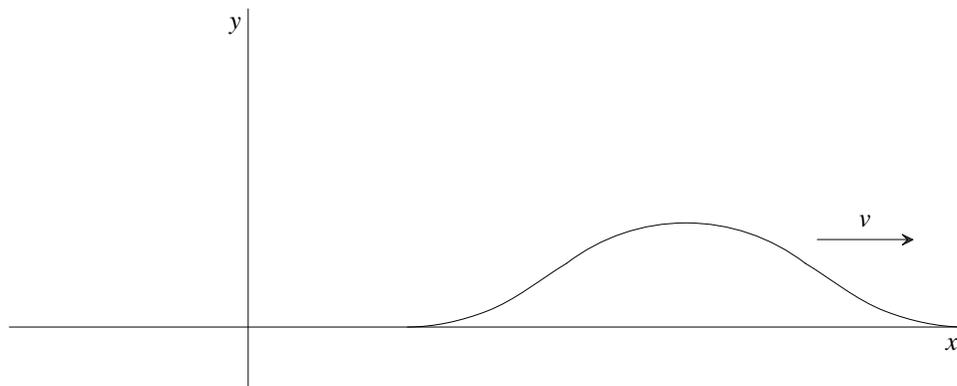
Recall that although the shape moves to the right, the particles of the string do not—they only move up or down. For convenience, we look at a small length of the string at the top of the pulse, and we put ourselves in the inertial frame in which the shape is at rest. In this frame the element of the string at the top of the pulse has not only a vertical motion, but is also moving to the left with speed v, as indicated below:
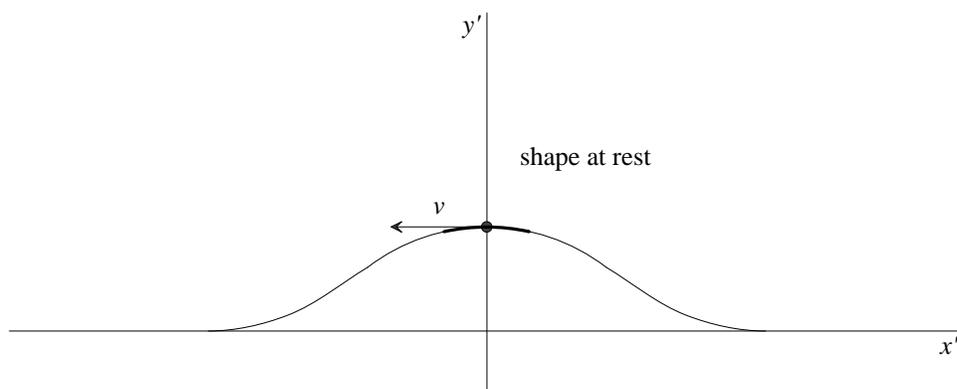
According to the principle of Gallilean relativity, the laws of physics are the same in all inertial (unaccelerated) frames, so we can apply the usual force laws to figure things out. The small segment of string we are looking at is very nearly the

arc of a circle, with a radius we'll call R. The geometry is shown in the magnified sketch of the string segment below:
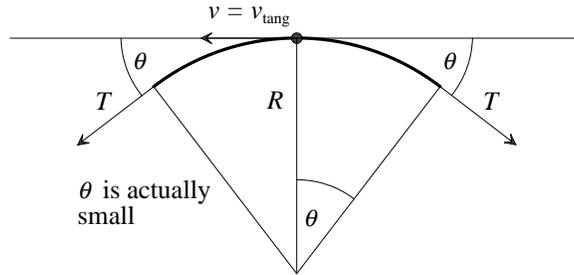
Because the segment is chosen to be small, and the vertical displacement of the string is assumed to be small, the angle θ (in radians) between the ends of the string segment and the horizontal is small. Since the tangential component of the velocity of the segment in our frame is the same as minus the wave speed v, the centripetal acceleration and radial component F_r of the force on it are related by

$$\frac{mv^2}{R} = F_r = 2T \sin \theta$$

where m is the mass of the small string segment. In terms of the mass per unit length μ, the mass is $m = \mu \times 2\theta R$, so

$$\frac{\mu 2\theta R v^2}{R} = 2T \sin \theta \approx 2T\theta$$

$$\Rightarrow \quad \boxed{v = \sqrt{\frac{T}{\mu}}}$$

So for a given tension T, a less dense string has faster wave speed, and for a given density μ, a tauter string has faster wave speed.

6 Reflection and Transmission

Typically when a wave travels through a medium and meets the boundary of another medium, it is partially transmitted and partially reflected. The reflected wave

does or does not undergo a phase change of 180° (π radians) depending on the media on either side of the boundary. To describe this situation for strings, we first consider two cases where there is no transmission, only reflection.

In the first case, we suppose that one end of the string is tied to a wall, or just something heavy, as shown below. When the shape incident from the left reaches the wall, the tension in the string tends to pull upward on the wall. From Newton's third law, the reaction force exerted by the wall on the string is then downward, making the reflected shape inverted.

end tied

incident
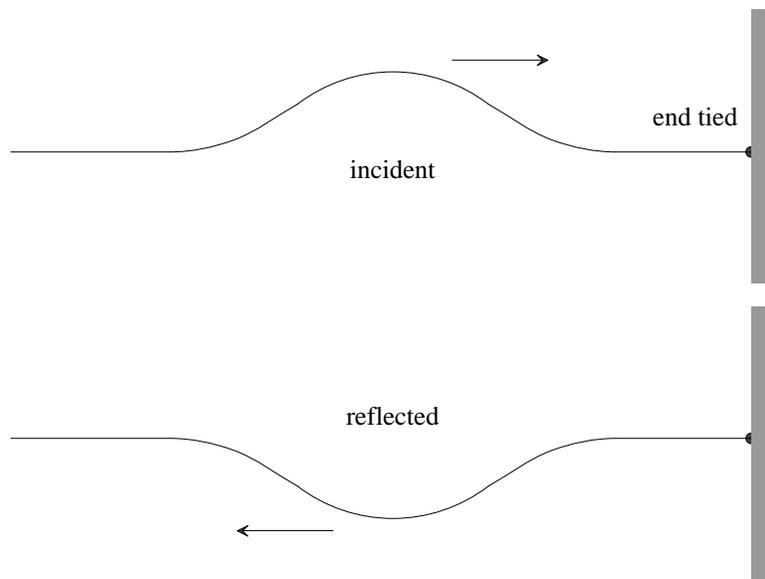
reflected

In the second case, the end of the string is tied to a frictionless ring, free to move up and down on a rod, as shown below. This time a pulse incident from the left just moves the ring upward when it reaches the wall, making the reflected shape also upward, not inverted.

incident

end slides
up and down

reflected
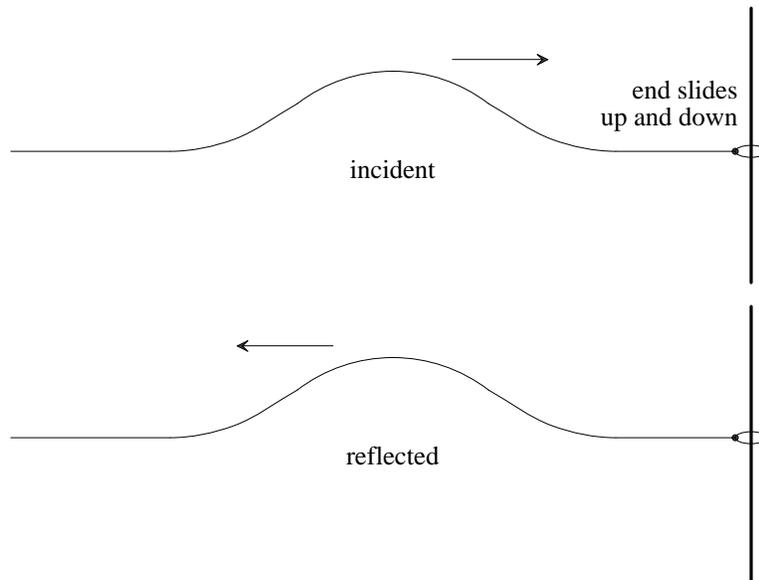
Next, we replace the wall by another piece of string, with the same tension in both strings. In the first case, shown below, the new string on the right has bigger mass per unit length compared to the string on the left. When a shape incident from the left reaches the boundary between the two strings, part of the energy goes into a transmitted wave, and part into a reflected wave, which is inverted, as with the string tied to a wall or something heavy.

incident

reflected

transmitted
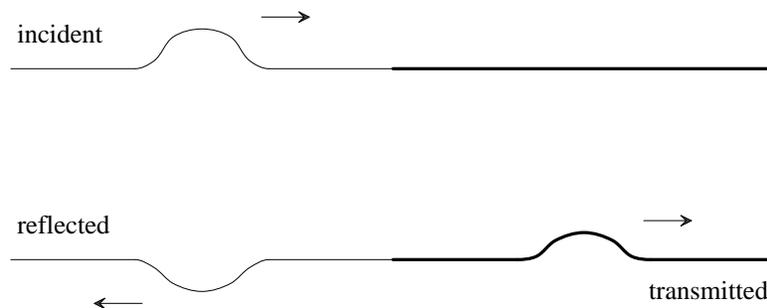
In the second case, as shown below, the string with the incident wave on the left is heavy and the string on the right is light. A shape hitting the boundary between the strings from the left finds it easy to lift the string on the right, like the

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frictionless ring above, so the reflected wave is not inverted, and it is easy to make a transmitted wave in the light string on the right.

incident

reflected   transmitted
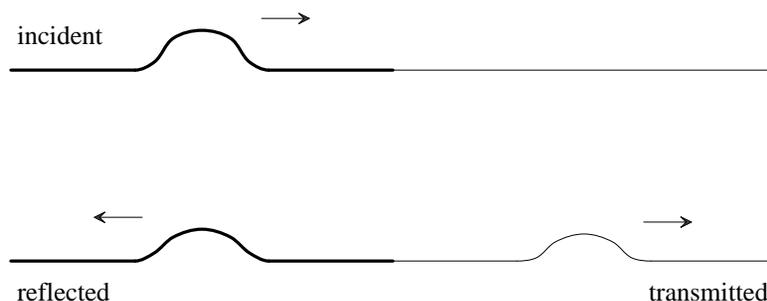
We shall see later that probability waves in quantum mechanics also reflect and transmit when they are incident on a "potential barrier".

7   Harmonic (Periodic) Waves

We have indicated that waves can have any shape, but there is a special set of shapes that are basic for our intuition, namely, the sinusoidal shapes, with definite frequency and wavelength. You will learn in more advanced courses that all linear waves can be written as superpositions of sinusoidal waves (Fourier series and integrals), and their properties are the foundation for the way we think about waves.

Here is a snapshot of a sinusoidal wave with wavelength λ at time $t = 0$:

$$y(x, 0) = A \sin \frac{2\pi x}{\lambda}$$
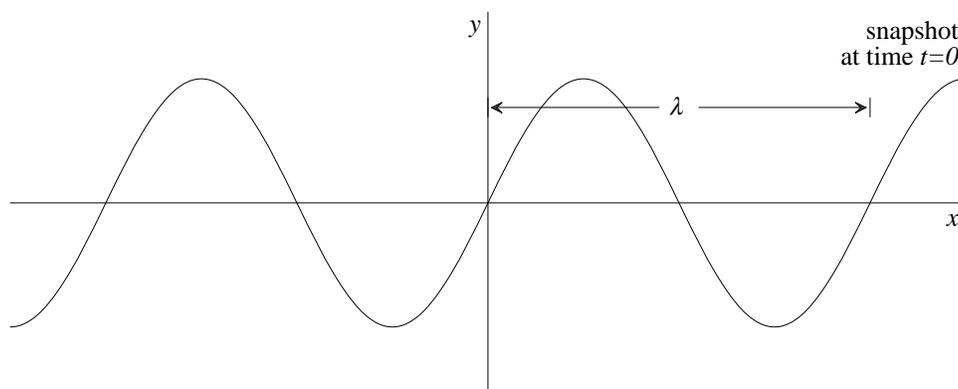
Such waves are *periodic*, that is, they repeat when their phase changes by $\pm 2\pi$ radians, in this case, when x changes by $\pm\lambda$. We take this wave to be moving to the right with wave speed v, so at time t a little later its snapshot is:

$$y(x,t) = A \sin \frac{2\pi}{\lambda}(x - vt)$$
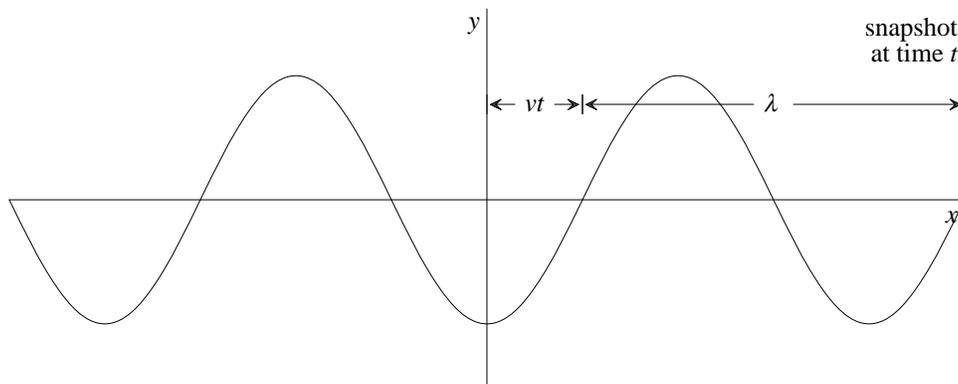
7.1   Wavelength, Period, Frequency

The sinusoidal waves above have been written in terms of wavelength λ and wave speed v. The period T (not to be confused with string tension), and frequency f, are also convenient paramaters. We summarize them here:

wavelength λ: The distance in which the phase changes by $\pm 2\pi$ at fixed time t.

period T: The time in which the phase changes by $\pm 2\pi$ at fixed position x.

$$\frac{2\pi}{\lambda}vT = 2\pi\,, \qquad \boxed{T = \frac{\lambda}{v}}$$

frequency f: The number of complete vibrations (cycles) per unit time at fixed position x (e.g., the number of times a crest passes by x in one second).

$$\boxed{f = \frac{1}{T} = \frac{v}{\lambda}}\,, \qquad \dim f = \sec^{-1} \equiv \text{Hz}\,, \qquad \boxed{v = f\lambda}$$

Putting in the period T gives

$$y(x,t) = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right)\,,$$

which makes it clear that the wave is *periodic* when $x \to x \pm \lambda$ or $t \to t \pm T$.

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7.2   Wavenumber, Angular Frequency

It is traditional and convenient to introduce another notation:

wavenumber k:

$$k = \frac{2\pi}{\lambda}, \qquad \dim k = \frac{1}{\text{length}} = \text{m}^{-1}$$

angular frequency ω:

$$\omega = 2\pi f = \frac{2\pi}{T}, \qquad \dim \omega = \frac{1}{\text{time}} = \text{s}^{-1}$$

In this notation

$$y = A\sin(kx - \omega t).$$

The phase in radians is now expressed as $kx - \omega t$. In this language the wave speed is

$$v = \lambda f = \frac{\omega}{k}$$

7.3   Phase Constant

To compare and superpose sinusoidal waves of the same wavelength and frequency, we need one more parameter, the *phase constant ϕ*, which adjusts the phase origin:

$$y(x, t) = A\sin(kx - \omega t - \phi)$$

For example, if we choose the phase constant to correspond to 90°, $\phi = \pi/2$:

$$y = A\left[\sin(kx - \omega t)\cos\frac{\pi}{2} - \cos(kx - \omega t)\sin\frac{\pi}{2}\right]$$
$$= -A\cos(kx - \omega t)$$

So sinusoidal waves are also "cosinusoidal", when the appropriate phase constant is put in. The plots below show that this calculation agrees with our discussion of the translation of shapes.

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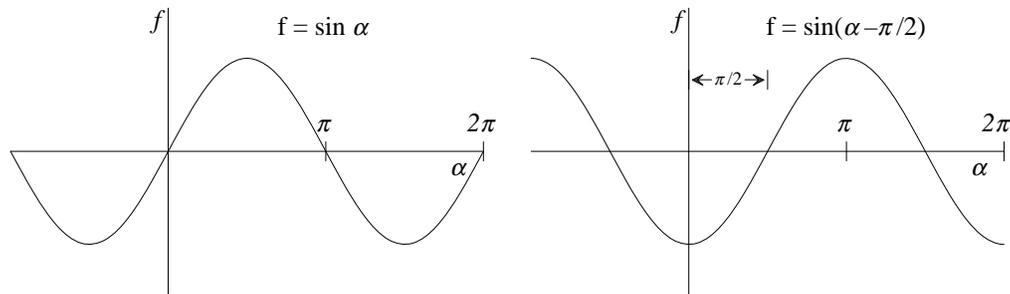
7.4 Velocity and Acceleration of Displacement

Consider a sinusoidal string wave with transverse displacement[3]

$$y = A \sin(kx - \omega t).$$

According to our previous discussion, this is a wave traveling to the right, because it is a function of $x - vt$; we just have to factor out the k to see what v is. It is a basic fact that the wave speed $v = \omega/k$ is different from the speed with which the particles of the string move. In particular, the wave speed is constant, while the transverse velocity is not.

The calculation is not difficult:

$$v_y = \left.\frac{dy}{dt}\right|_{\text{const } x} = \frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t)$$

$$a_y = \left.\frac{d^2 y}{dt^2}\right|_{\text{const } x} = \left.\frac{dv_y}{dt}\right|_{\text{const } x} = \frac{\partial^2 y}{\partial t^2} = \frac{\partial v_y}{\partial t}$$

$$= -\omega^2 A \sin(kx - \omega t)$$

Note that the displacement y and transverse velocity v_y are 90° out of phase. Note also that at each fixed x, the displacement obeys the differential equation for simple harmonic motion with angular frequency ω.

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[3]We leave it as an exercise for the student to check that the discussion in this section is just as easy with a nonzero phase constant ϕ.

8 Energy Transmitted by Harmonic Waves

For a particle in simple harmonic motion under a restoring force $F = -ky$, where k is the "spring constant", the energy is

$$E = \frac{1}{2}mv_y^2 + \frac{1}{2}ky^2, \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \quad E = \frac{1}{2}mv_y^2 + \frac{1}{2}m\omega^2 y^2 = \text{const.}$$

Since the velocity is $v_y = 0$ at maximum displacement $y = A$, and the energy is constant, we find

$$E = \frac{1}{2}m\omega^2 A^2$$

Although the particles on the string are not connected to springs, their transverse displacements y do undergo simple harmonic motion; and we take over this expression for the energy of each particle.

We also assume that the displacements are *small*, so that the length of string in a single wavelength λ is approximately the same as the wavelength (in fact the piece of string is a little longer). Then the energy in one wavelength of string is the sum of energies of each of its particles, with total mass

$$m = \mu\lambda, \qquad \mu = \frac{\text{mass}}{\text{length}}$$

Thus we find the energy in one wavelength of string

$$E = \frac{1}{2}\mu\lambda\omega^2 A^2.$$
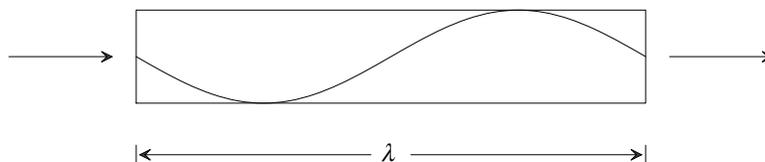
For a wave moving to the right, all of this energy leaves the box of length λ indicated above in one period T, so the *rate* at which energy is transported to the right

is

$$P = \frac{\frac{1}{2}\mu\lambda\omega^2 A^2}{T} = \frac{1}{2}\mu\lambda f\omega^2 A^2$$

$$\boxed{P = \frac{1}{2}\mu v\omega^2 A^2}, \qquad \text{power} = \frac{\text{energy}}{\text{time}}, \qquad \text{dim } P = \text{watts}$$

The fact that the transmitted power is proportional to the square of the frequency and to the square of the amplitude of whatever waves is generic for linear waves, and not limited to transverse waves on strings.

9   The Wave Equation

At the very beginning of the course, we indicated that "the" wave equation packages many properties of waves in a very convenient form. We certainly are not going to derive these properties in any systematic way from the wave equation, but there are several things that are both easy and useful that we can say about it.

First, let's derive it in the case of waves on strings. We assume as usual that the string is stretched along the x-axis, with the same tension T everywhere. We also assume that the transverse displacements are small—strictly speaking, the wave equation is only true for strings in this approximation. So we focus on the small segment of string shown in the snapshot at a fixed time t below.
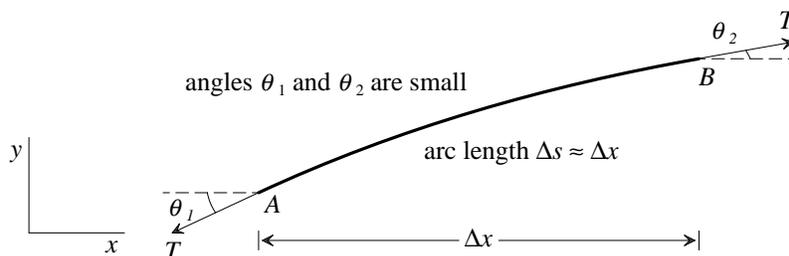
Since the displacement y is small, the shape of the wave is nearly flat, and the angles θ_1 and θ_2 between the tangents to the string at the ends and the horizontal are small; so we can approximate $\sin\theta \approx \tan\theta$ to get the vertical component of

the force on the segment, which is entirely due to the tension if the string is light:

$$F_y = T \sin\theta_2 - T \sin\theta_1$$
$$\approx T \tan\theta_2 - T \tan\theta_1$$
$$= T \left.\frac{\partial y}{\partial x}\right|_B - T \left.\frac{\partial y}{\partial x}\right|_A$$

In the last step, we used the fact that $\tan\theta$ is the slope of the curve, which is also $\partial y/\partial x$, at fixed time.

Next we use Newton's force law, and the fact that for small displacements the length of the segment is nearly the same as the horizontal component of the displacement between points A and B, $\Delta s \approx \Delta x$:

$$F_y = ma_y = \mu\Delta x\frac{\partial^2 y}{\partial t^2}$$
$$= T\left(\left.\frac{\partial y}{\partial x}\right|_B - \left.\frac{\partial y}{\partial x}\right|_A\right)$$

$$\Rightarrow \quad \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\left.\frac{\partial y}{\partial x}\right|_B - \left.\frac{\partial y}{\partial x}\right|_A}{\Delta x} \gg \Delta x \to 0 > \frac{\partial^2 y}{\partial x^2}$$

We put in $v = \sqrt{T/\mu}$,

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0,$$

and finally we replace y by ψ to represent a generic waving quantity to get "the" wave equation in one dimension

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0}$$

9.1 Linearity

If ψ_1 and ψ_2 are solutions of the wave equation, then

$$\frac{\partial^2 \psi_1}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2}$$
$$+ \frac{\partial^2 \psi_2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi_2}{\partial t^2} = 0 = \frac{\partial^2(\psi_1 + \psi_2)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2(\psi_1 + \psi_2)}{\partial t^2}$$
$$\Rightarrow \quad \psi_1 + \psi_2 \quad \text{is a solution.}$$

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Clearly we could have taken differences here, and also general linear combinations with constants c_1 and c_2 in front of ψ_1 and ψ_2. This is the mathematical statement that the wave equation is linear, and is the root of interference and diffraction phenomena for linear waves.

9.2   General Solution

Note that any function of the form $\psi(x,t) = f(x - vt)$ is a solution of the wave equation:

$$\frac{\partial f}{\partial x} = f', \qquad\qquad \frac{\partial^2 f}{\partial x^2} = f'',$$

$$\frac{\partial f}{\partial t} = -vf', \qquad\qquad \frac{\partial^2 f}{\partial t^2} = (-v)^2 f'' = v^2 f''$$

$$\Rightarrow \qquad \frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = f'' - \frac{1}{v^2}v^2 f'' = 0$$

Exercise: Verify that any function of the form $\psi(x,t) = f(x + vt)$ is a solution of the wave equation.

Given the result of the exercise just above, and the linearity of the wave equation, it follows that any function of the form

$$\psi(x,t) = f_1(x - vt) + f_2(x + vt)$$

is a solution of the wave equation.

Fact: The general solution of the wave equation in one dimension has the above form, the sum of an arbitrary wave traveling to the right and an arbitrary wave traveling to the left.

The mathematical proof is a bit technical, but not hard as those things go. Ignoring the technicalities, we sketch the ideas:

- For fixed wavenumber k, $\sin(kx \pm \omega t) = \sin k(x \pm vt)$ and $\cos(kx \pm \omega t) = \cos k(x \pm vt)$ are four "independent" solutions of the wave equation.

- Sums and/or integrals of these functions are also solutions. (Since $\omega = kv$ is fixed when k is given, these may be regarded as sums or integrals over values of k.)

- According to the theory of Fourier series and integrals, these independent solutions are "complete", that is, any solution of the wave equation can be respresented by their sums and/or integrals. By separately collecting together the terms with $x - vt$ and $x + vt$ in these sums/integrals, we see that they have the form

$$f_1(x - vt) + f_2(x + vt)$$

10 Superposition of Harmonic Waves

This section is based on Serway, *Physics for Scientists and Engineers*, Chapter 18, §§1-3. We now investigate some basic interference phenomena arising from the superposition of harmonic waves.

10.1 Interference

First, let's superpose two waves with the same frequency f, wavelength λ, and amplitude A, but different phase. Note that the wave speed is also the same.

$$y_1 = A \sin(kx - \omega t), \qquad y_2 = A \sin(kx - \omega t - \phi)$$
$$y = y_1 + y_2 = A \sin(kx - \omega t) + A \sin(kx - \omega t - \phi)$$

As we recall from alternating current theory[4] or elsewhere, even if the amplitudes were different, say A_1 and A_2, we could write the superposition in the form

$$y = A' \sin(kx - \omega t - \phi')$$

where the resultant amplitude A' and phase ϕ' can be determined from the vector diagram:
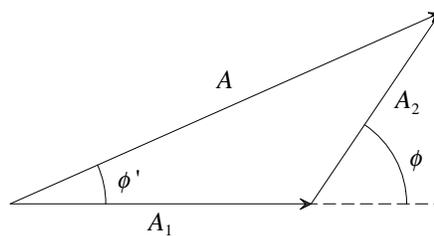
[4]Remember phasors?

The math is based on the following trigonometric identity:

$$\sin a + \sin b = 2 \cos \frac{a - b}{2} \sin \frac{a + b}{2}$$

(check it out). Thus

$$y = 2A \cos \frac{\phi}{2} \sin \left(kx - \omega t - \frac{\phi}{2} \right).$$

Now look what happens when $\phi = \pm\pi$ (or $\phi/2 = \pm\pi/2$). We get $y = 0$ everywhere (all x) and any time (all t). When a cancellation like this occurs at a spatial point x, we call it *destructive* interference. In this case we have destructive intererence at all x. The snapshots at $t = 0$ below show that it makes sense:

result is total destructive interference
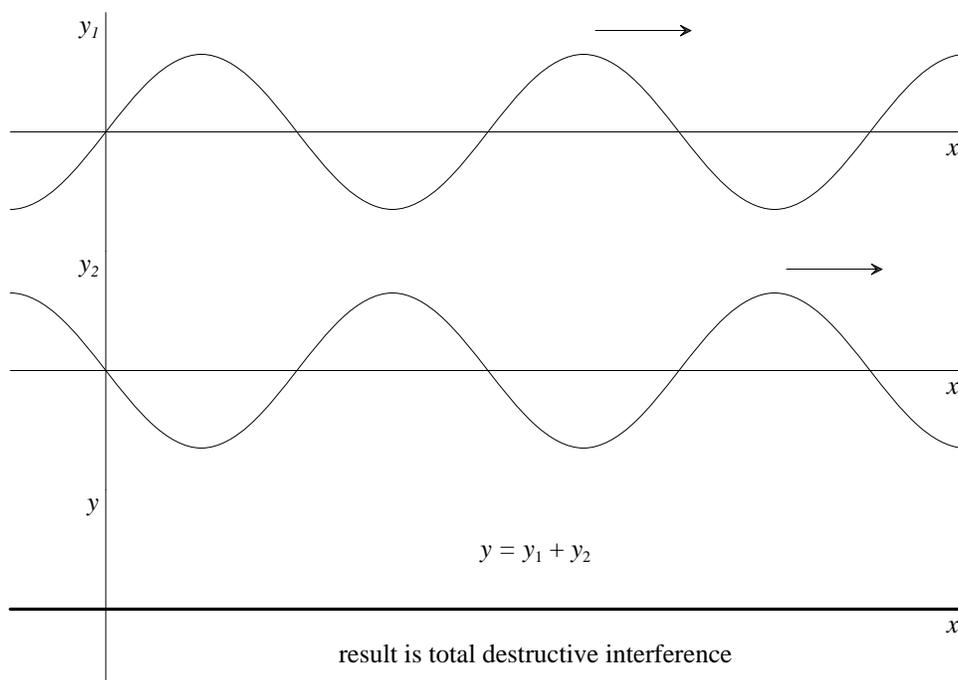
Even as the shapes move to the right, they're always a half wavelength out of step, and still exactly cancel everywhere.[5]

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[5]Since it's tricky (not impossible) to physically do this particular kind of superposition on a string, we might better look to other kinds of waves for examples, such as sound waves.

Exercise: Show from the mathematical formula that this destructive interference result holds when $\phi = 2\pi(n + \frac{1}{2})$, where n is any positive or negative integer.

When the phase difference $\phi = 2\pi n$, the two waves are exactly in step, and we get *constructive* interference at all positions x. We leave it as an exercise for the student to draw snapshots analogous to those above, and to verify that the mathematics also works:

$$y = 2A \cos \frac{n\pi}{2} \sin\left(kx - \omega t - \frac{n\pi}{2}\right)$$
$$= 2A \sin(kx - \omega t)$$

Sound Wave Examples from Serway: The examples of destructive and constructive interface described in the previous section are atypical in that we are often interested in interference that occurs at a single point, where waves of the same wavelength and frequency arrive by different paths. In the two examples from Serway reviewed in the lecture,[6] we interfered two sinusoidal waves with phases

$$\theta_1 = \frac{2\pi}{\lambda} r_1 - \omega t, \qquad \theta_2 = \frac{2\pi}{\lambda} r_2 - \omega t,$$

where r_1 and r_2 were the lengths of the paths traveled by the two waves before they arrived at the same point, assuming they started out in phase.

What matters then is the phase difference:

$$\Delta\phi = \theta_1 - \theta_2 = \left(\frac{2\pi}{\lambda} r_1 - \omega t\right) - \left(\frac{2\pi}{\lambda} r_2 - \omega t\right)$$
$$= \frac{2\pi}{\lambda}\left(r_1 - r_2\right)$$
$$\Delta\phi = 2\pi n \quad \text{(constructive interference)}$$
$$= 2\pi(n + \frac{1}{2}) \quad \text{(destructive interference)}$$

In terms of path difference, this translates into

$$r_1 - r_2 = n\lambda \quad \text{(constructive interference)}$$
$$r_1 - r_2 = (n + \frac{1}{2})\lambda \quad \text{(destructive interference)}$$

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[6]One was a speaker emitting waves into a tube that split into two bent tubes, one with a variable length, then recombined into a single tube at a reciever; and the other was two speakers driven in phase by the same source, emitting sound waves that interfered at the same point after traveling along different straight paths.

The mathematics is essentially the same as before; we superpose

$$y = A \sin \theta_1 + A \sin \theta_2 = A \sin \theta_1 + A \sin(\theta_1 - \Delta\phi)$$

$$= 2A \cos \frac{\Delta\phi}{2} \sin\left(kr_1 - \omega t - \frac{\Delta\phi}{2}\right)$$

10.2   Standing Waves

Coming back to strings, let's superpose two waves with the same frequency, wavelength, and amplitude, but traveling in *opposite* directions, which is even fairly easy to do on real strings:

$$y_1 = A \sin(kx - \omega t), \qquad y_2 = A \sin(kx + \omega t)$$

$$y = y_1 + y_2$$

$$= A\left(\sin kx \cos \omega t - \cos kx \sin \omega t + \sin kx \cos \omega t + \cos kx \sin \omega t\right)$$

$$= 2A \sin kx \cos \omega t$$

We know of course that this obeys the wave equation for speed $v = \omega/k$.

Notice that the space and time dependences have factorized. This wave is no longer going anywhere. Below we show two snapshots, one at time $t = 0$, and the other a little later, where $0 < \cos \omega t < 1$. Prominent features of the shape such as the peaks and *nodes* (places where the wave is zero) do not move to the right or left. The *antinodes* (places where the amplitude has maximum magnitude) move up and down with angular frequency ω. Such waves are called *standing waves*.
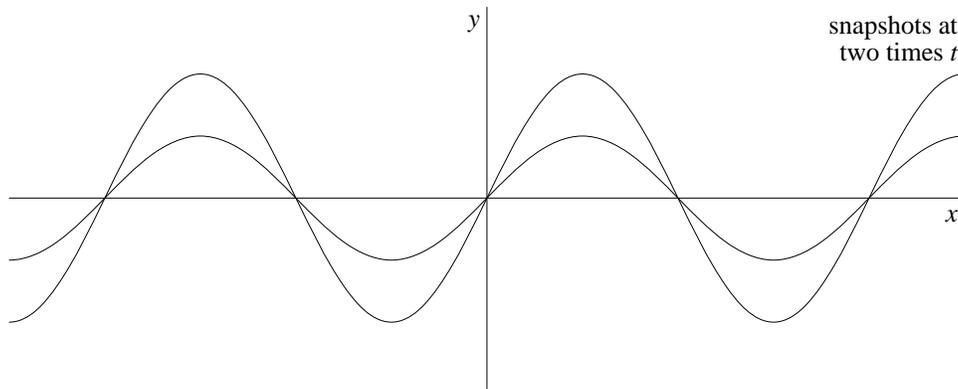
The nodes are determined by $\sin kx = 0$:

$$kx = \frac{2\pi}{\lambda}x = 0, \pm\pi, \pm2\pi, \cdots = n\pi$$

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$$\boxed{x = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots \quad \text{(nodes)}}$$

They are separated by a half wavelength.

The antinodes are determined by $\sin kx = \pm 1$:

$$kx = \frac{2\pi}{\lambda}x = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \cdots = (2n+1)\frac{\pi}{2}$$

$$\boxed{x = \left(n + \tfrac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots \quad \text{(antinodes)}}$$

They are also separated by a half wavelength.

There is stored energy in the up and down motion, but no transmission of energy along the string.

10.3 Boundary Conditions and Normal Modes

Standing wave nodes and antinodes play a special role in problems with *boundary conditions*. When the ends of a string, say at $x = 0$ and $x = L$, are tied down so they can't move, they are forced to be nodes. The solutions $y(x, t)$ of the wave equation for such a problem are said to obey the boundary conditions $y(0, t) = y(L, t) = 0$, and the mathematical problem is to find all solutions for $0 \leq x \leq L$ which obey those boundary conditions. Analogous problems occur in quantum mechanics.

We already know a lot of solutions from our discussion of nodes in harmonic standing waves. Let's classify them by drawing pictures. The first two solutions are shown below, the trivial solution, with $y \equiv 0$, and the *fundamental* solution or first harmonic, where the first node to the right of $x = 0$ is at the end, $x = L$.

$L = \lambda/2$
fundamental or 1st harmonic

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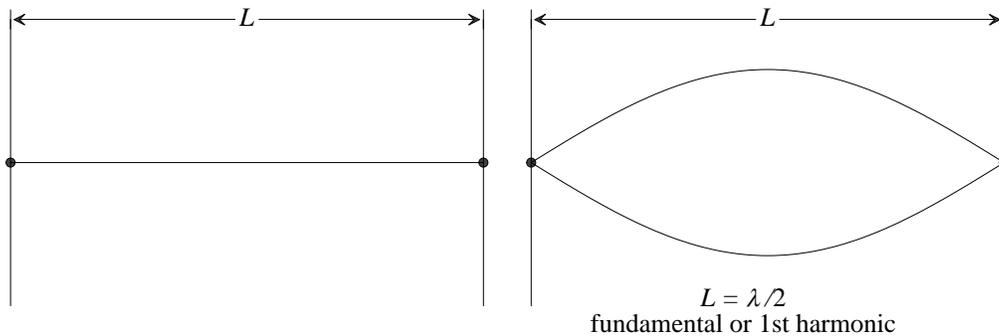
The next two are called the second and third harmonics, with one and two nodes, respectively, between the ends.

$L = \lambda$
2nd harmonic

$L = 3\lambda/2$
3rd harmonic
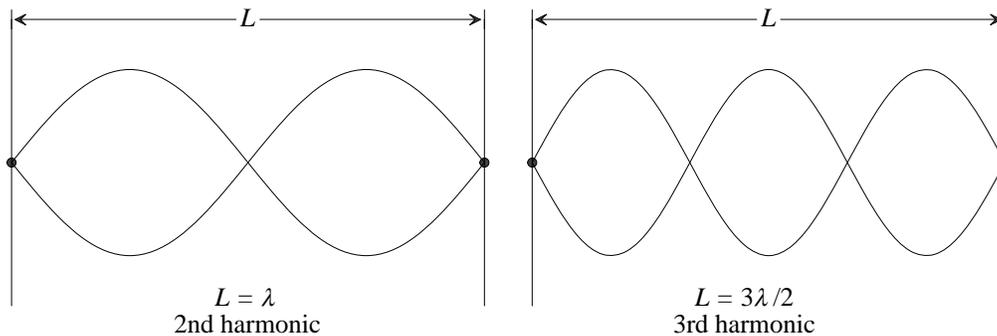
The pattern is clear. At each step, we fit one more node corresponding to one more half wavelength between the ends. The rule for the wavelengths and frequencies of the standing waves is

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, \dots$$

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

Note that the harmonic frequencies go up in equal steps from the fundamental frequency: $f_n = nf_1$. The wavenumbers and angular frequencies are

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}, \qquad \omega_n = 2\pi f_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

The frequencies f_n are called *normal mode* frequencies, and the corresponding solutions of the wave equation are called normal mode solutions, or normal modes:

$$y_n = A \sin k_n x \cos \omega_n t = A \sin \frac{n\pi}{L}x \cos \frac{n\pi v}{L}t$$

The mathematicians tell us that these are in fact *all* of the solutions, in the sense that any solution can be written as a superposition of these.

A guitar string is a familiar example of this situation. The dominant tone that we hear[7] when a guitar string is plucked is the fundamental frequency, higher for

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[7]Of course what we hear is a sound wave driven by the string wave and amplified by the guitar cavity.

lighter strings under higher tension, lower for heavier strings under less tension. For a given string, higher fundamental frequencies result when the guitarist effectively shortens the string, keeping the tension fixed, by pressing it against a fret.

The string actually vibrates as a superposition of the fundamental and higher harmonic overtones; that is, it is a complex standing wave. A basic part of the reason that the fundamental sounds loudest is that the initial shape of the string just after it is plucked is close enough to that of the fundamental mode that higher harmonics have relatively small amplitudes in the complex wave. In the "cultural example" below,[8] we indicate how this happens for the triangle shape, which occurs just after the string is plucked at its middle. (Another effect is that sufficiently high harmonic frequencies are inaudible to humans.)

Cultural example (the triangle wave): At time $t = 0$ suppose the shape of the string is that of the triangle shown below.
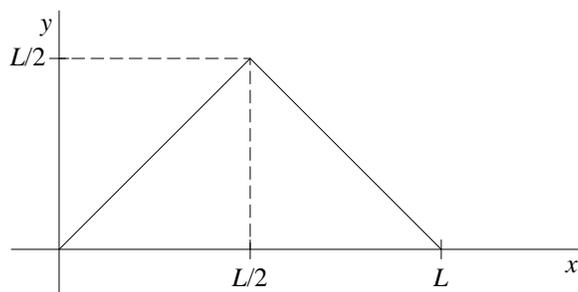
The function in this snapshot is

$$y(x, 0) = x, \qquad 0 \le x \le \frac{L}{2}$$

$$= L - x, \qquad \frac{L}{2} \le x \le L$$

The Fourier series expansion of this function can be shown to be the infinite sum:

$$y(x, 0) = \frac{4L}{\pi^2} \left(\frac{1}{1^2} \sin \frac{\pi x}{L} - \frac{1}{3^2} \sin \frac{3\pi x}{L} + \frac{1}{5^2} \sin \frac{5\pi x}{L} - \dots \right)$$

The solution of the wave equation is then:

$$y(x, t) = \frac{4L}{\pi^2} \left(\frac{1}{1} \sin \frac{\pi x}{L} \cos \frac{\pi v t}{L} - \frac{1}{9} \sin \frac{3\pi x}{L} \cos \frac{3\pi v t}{L} + \frac{1}{25} \sin \frac{5\pi x}{L} \cos \frac{5\pi v t}{L} - \dots \right)$$

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[8]We call this a cultural example because the mathematics is beyond the scope of this course, but it's easy to understand what the mathematical result means.

In this case, the second, fourth, sixth, etc., harmonics are missing, and the amplitude of the first contributing overtone (the third harmonic) is down by a factor of 9 from the fundamental, with even more dramatic suppression for higher overtones.