Effective Criticality and Stability of the Double Well Coincident Pair Configuration*

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Abstract

We show that the coincident instanton–anti-instanton pair configuration in the $(1+0)$-dimensional, double well model, defined according to a theory presented elsewhere [1], is an effective critical point of the Euclidean action if it exists and has regularity properties; and is stable if it is also unique. On the way to showing stability we prove a positivity preserving property of the $(1+0)$-dimensional double well classical field equation with external sources.

1. Introduction

In ref. [1] we propose a definition of nondilute, multi-instanton configurations for the double well model,[1] which is induced by the ambiguities in the usual collective coordinate parameterization of the Euclidean path integral, and which is aimed at systematizing the asymptotics of small $\hbar$. By the double well model, we mean a Euclidean ϕ^4 field theory in $(1+0)$-dimensional spacetime, or its equivalent one-dimensional quantum anharmonic oscillator.

Multi-instantons are finite action solutions $\phi^s(t)$ of the classical field equation,

$$-\ddot{\phi}^s - \mu^2 \phi^s + \lambda (\phi^s)^3 = \sum_{i=1}^{n} \alpha_i \xi^{s_i}, \qquad \lambda, \mu^2 > 0, \tag{1.1}$$

with sources that are linear superpositions of zero mode eigenfunctions, centered at times $s = (s_1, \ldots, s_n)$,

$$\xi^{s_i}(t) = \operatorname{sech}\left[\mu \left(t - s_i \right) / \sqrt{2} \right], \tag{1.2}$$

subject to exactly n orthogonality constraints

$$\int_{-\infty}^{\infty} \phi^s(t)\, \xi^{s_i}(t)\, \mathrm{d}t \equiv \langle \phi^s, \xi^{s_i} \rangle = 0, \qquad i = 1, \ldots, n. \tag{1.3}$$

The Lagrange multipliers α_i are to be determined by the constraints in (1.3).

We call ϕ^s a *multi-instanton*, the number n the *instanton number* (the abstract number of kinks plus anti-kinks associated with any finite action configuration[2] via orthogonality with zero modes), and the components of s the instanton and anti-instanton *positions*.

From now on, the word *instanton* is used generically either for instantons or anti-instantons.

In saddle point calculations based on multi-instantons, there comes a stage where an integration over $\mathrm{d}^n s$ occurs, with a factor $\exp[-S(\phi^s)/\hbar]$ depending on $\hbar$ and s, where S is the action

$$S(\phi) = \int_{-\infty}^{\infty} \left[\frac{\dot{\phi}^2}{2} + \frac{\lambda}{4} \left(\phi^2 - \frac{\mu^2}{\lambda} \right)^2 \right] \mathrm{d}t. \tag{1.4}$$

We use the language *effective critical points* for the extrema of $S(\phi^s)$ in the variable s. In that language, certain departures from the dilute gas approximation result from stable, effective critical points at values of s where not all of the instanton positions are far separated.

In this paper we investigate the simplest of these nondilute configurations, the coincident instanton–anti-instanton pair, ϕ^s with $s = (s_1, s_2)$ and $s_1 = s_2$. With reasonable

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[1]Because this paper is a sequel to [1], we have not repeated the list of references given there on the double well model, the collective coordinate formalism, and other approaches to multiple counting and approximate classical solutions.

[2]Or indeed with any configuration in the path space.

technical assumptions, we show that such configurations are indeed effective critical points. The argument may admit generalization to configurations for any n, not just $n = 2$; but we have not carried that through.

We show further that, if the solutions for $n = 2$ are unique, the coincident pair is stable, in which case it would give rise to a correction to the dilute gas. We do not have a generalization of this stability result to larger n; and we also do not have a uniqueness theory for multi-instantons.

The stability result is based partly on a positivity property of the equation

$$-\ddot{\phi} - \mu^2 \phi + \lambda \phi^3 = f \,, \tag{1.5}$$

where f is a real, nonpositive or nonnegative external source, a sufficiently smooth and rapidly decreasing function of t. In ref. [3] it was shown under broad technical assumptions that nonlinear equations for $\phi(x)$ with $x \in \mathbb{R}^n$ of the form

$$\left(-\Delta + \mu^2\right)\phi + F(\phi) = f \,, \qquad \mu^2 \geq 0 \,, \tag{1.6}$$

are positivity-preserving when $aF(a) \geq 0$ for all values of a. That is, if f is always nonnegative or nonpositive then so is ϕ.

Equation (1.5) does not have that form, and yet there is a positivity property if we restrict to solutions ϕ that obey the same boundary conditions at $t = \pm\infty$. That is, $\dot{\phi} \to 0$ as $|t| \to 0$ and $\phi \to +\mu/\sqrt{\lambda}$ at both ends $t = \pm\infty$, or $\phi \to -\mu/\sqrt{\lambda}$ at both ends; ϕ has even instanton number.

Since the style of this paper is to assume any reasonable technical properties, we do not give a rigorous proof; but we state the lemma and give a short proof containing the key ideas in sect. 2.

Much of the argument in this paper depends on coincidence properties of the projector onto the span of zero modes, which can be worked out explicitly. The results are summarized in sect. 3.

Section 4 contains the proof that the coincident pair, if existing and regular, is an effective extremum of the action; and in sect. 5 we supply the proof that the further assumption of uniqueness implies stability.

Finally, in sect. 6 we comment briefly on the significance of the pair configuration, in view of a numerical study which is a sequel to this paper [2].

2. Even Instanton Number Preserves Positivity

The two possibilities for finite action and even instanton number are $\phi \to \pm\mu/\sqrt{\lambda}$ as $|t| \to \infty$. Let us define χ, respectively, by

$$\phi = \chi \pm \mu/\sqrt{\lambda} \,. \tag{2.1}$$

Then χ, $\dot{\chi} \to 0$ as $|t| \to \infty$. The field equation (1.5) may be rewritten:

$$-\ddot{\chi} + 2\mu^2 \chi + \lambda\chi^3 = f \mp 3\mu\sqrt{\lambda}\,\chi^2 \,. \tag{2.2}$$

We prove the following:

Positivity Lemma. Let ϕ be a sufficiently regular, finite action solution of (1.5), with boundary values having the same sign at $t = \pm\infty$. Let χ be defined by (2.1), according to the boundary conditions. Then if f is positive or negative semidefinite, sufficiently smooth, and sufficiently decreasing at $|t| = \infty$, χ is also semidefinite with the same sign as f.

The proof goes in two steps:

First, if f is negative respectively, positive, then the right hand side of (2.2) is negative, respectively, positive semidefinite, and the left hand side comes under the theory of eq. (1.6) quoted earlier. It follows from Theorem 5 of ref. [3] that χ has the same sign as f, with regularity and fall off restrictions on f much milder than we need in this paper (f may be a measure in the Sobolev space H_1).

Second, if f is positive, respectively, negative, we consider eq. (1.5) directly instead of (2.2); and we do the standard trick of mapping the problem onto a classical mechanics problem in one dimension. Thus, let $\mu = \lambda = 1$, and let $x(t) = \phi(t)$ represent the position of a classical particle of unit mass with the double hill potential energy

$$U(x) = -(x^2 - 1)^2/4, \tag{2.3}$$

and with a time-dependent, external force, $-f(t)$. Then (1.5) describes such a particle with mechanical energy

$$E(t) = \dot{x}^2/2 + U(x). \tag{2.4}$$

The boundary conditions are such that $E(\pm\infty) = 0$.

We discuss the case $f > 0$ and $x(\pm\infty) = +1$, leaving the other case to the reader. Then the external force is leftward. The particle cannot start out from $x = +1$ at $t = -\infty$ towards the left, because the always leftward external force would give it nonzero mechanical energy at the left peak; and it would escape to $x = -\infty$ at $t = +\infty$, violating finite action. If it starts to the right, it must stay forever to the right, because if it ever crosses the rightmost peak moving towards the left, it will again escape.

Therefore, $x \geq 1$ and $\chi \geq 0$.

The technical requirements to make the second part of the argument rigorous are enough smoothness of ϕ to be able to discuss the classical turning points, and enough fall off and regularity of f to be able to discuss the energy integral from $t = -\infty$ for the external force. Those properties of f are amply obeyed by the C^∞, exponentially decreasing source we use in this paper.

In fact, the classical mechanics analogy could have been used to give the result of the first part of the proof, aside from the technical conditions accessible through [3].

3. The Source Projection at Coincidence

Major tools in the analysis are the $L^2(-\infty, \infty)$ orthogonal projection operator P^s onto the span of ξ^{s_1} and ξ^{s_2} (or of ξ^a and $d\xi^a/da$, in case $s_1 = s_2 = a$), and its derivatives with respect to s.

Let $s_1 \neq s_2$, and define

$$\eta_\pm = (\xi^{s_2} \pm \xi^{s_1})/N_\pm, \tag{3.1}$$

where $N_\pm$ is a positive normalization chosen so that $\langle \eta_+, \eta_+ \rangle = \langle \eta_-, \eta_- \rangle = 1$. Of course $\langle \eta_+, \eta_- \rangle = 0$. The only property of $\xi^s(t) = \xi(t - s)$ that we need in this section is that ξ is an even, L^2 function with sufficiently many derivatives also in L^2.

It saves writing to assume, with no loss of generality, that

$$\|\xi\| \equiv (\langle \xi, \xi \rangle)^{\frac{1}{2}} = 1. \tag{3.2}$$

Let us also define

$$D = \frac{\partial}{\partial s_2} - \frac{\partial}{\partial s_1}, \tag{3.3}$$

which is twice the derivative with respect to $s_2 - s_1$ in the coordinates $(s_1 + s_2)/2$, $s_2 - s_1$.

It is sufficient for us to evaluate everything at $s_1 = 0$, and we shall then simply write "lim" for the limit $s_2 \to 0$. The following comes from straightforward Taylor expansions, becoming an order of magnitude more tedious with each derivative:

$$\lim \eta_+ = \xi, \qquad \lim \eta_- = -\dot{\xi}/\|\dot{\xi}\|, \tag{3.4}$$

$$\lim D\eta_+ = \lim D\eta_- = 0, \tag{3.5}$$

$$\lim D^2\eta_+ = \ddot{\xi} + \xi\|\dot{\xi}\|^2, \tag{3.6}$$

$$\lim D^2\eta_- = -\frac{\dot{\xi}\|\ddot{\xi}\|^2 + \dddot{\xi}\|\dot{\xi}\|^2}{3\|\dot{\xi}\|^3}. \tag{3.7}$$

All limits may be taken as strong limits, if ξ has enough strong derivatives. When ξ is proportional to the zero mode eigenfunction (1.2), it is C^∞ in the strong sense.

Now let P, DP, and D^2P be the limits of the projection operator and its derivatives P^s, DP^s, and D^2P^s, as $s_2 \to s_1 = 0$. We find

$$P = |\xi\rangle\langle\xi| + \frac{|\dot{\xi}\rangle\langle\dot{\xi}|}{\|\dot{\xi}\|^2}, \tag{3.8}$$

$$DP = 0, \tag{3.9}$$

$$D^2P = |\ddot{\xi} + \xi\|\dot{\xi}\|^2\rangle\langle\xi| + \frac{|\dot{\xi}\|\ddot{\xi}\|^2 + \dddot{\xi}\|\dot{\xi}\|^2\rangle\langle\dot{\xi}|}{3\|\dot{\xi}\|^4} + \text{h.c.} \tag{3.10}$$

The analogous results hold when $s_2 \to s_1 \neq 0$.

4. Coincidence is Critical

We assume a translation covariant family of finite action solutions ϕ^s to (1.1) and (1.3) exists for $n = 2$, labeled continuously and differentiably by $s = (s_1, s_2)$. Translation covariance means

$$\phi^{(s_1, s_2)}(t - a) = \phi^{(s_1 + a, s_2 + a)}(t). \tag{4.1}$$

The system of equations (1.1) and (1.3) is translation covariant, and if ϕ^s is a solution, so is $\phi^{(s_1 + a, s_2 + a)}$. The action is translation invariant, so $S(\phi^s)$ depends only on $s_2 - s_1$,

and criticality and stability may be discussed entirely in terms of the derivatives D and D^2.

Consider

$$DS(\phi^s) = \langle J(\phi^s),\, D\phi^s \rangle, \tag{4.2}$$

where we define

$$J(\phi) \equiv -\ddot{\phi} + \lambda\phi^3 - \mu^2\phi. \tag{4.3}$$

Let $E^s = I - P^s$ be the projection operator complementary to P^s. Then the orthogonality condition (1.3) can be restated:

$$\phi^s = E^s\phi^s, \tag{4.4}$$

which is to be interpreted in a dual sense because ϕ^s is not in L^2, due to its approach to $\pm\mu/\sqrt{\lambda}$ at $|t| = \infty$. The field equation (1.1) implies

$$E^s J(\phi^s) = 0; \tag{4.5}$$

i.e., E^s kills the source of ϕ^s.

Note that

$$D\phi^s = D\left(E^s\phi^s\right) = \left(DE^s\right)\phi^s + E^s D\phi^s, \tag{4.6}$$

$$\langle J,\, D\phi^s \rangle = \langle J,\, (DE^s)\phi^s \rangle + \langle E^s J,\, D\phi^s \rangle = \langle J,\, (DE^s)\phi^s \rangle. \tag{4.7}$$

Then (3.9) implies that

$$\lim_{s_2 \to s_1} DS\left(\phi^s\right) = 0. \tag{4.8}$$

To summarize, the essential features of the argument were the existence of a translation invariant, regular family of finite action solutions ϕ^s, and property (3.9) of the source projector. The analog of (3.9) for projectors P^s with $s = \left(s_1, \ldots, s_n\right)$ should be straightforward to discuss. A sufficient condition for a critical point, under the same regularity and translation covariance assumptions, would then be the vanishing of the gradient of P^s with respect to the difference variables formed from s.

5. Uniqueness Gives Stability

The argument for stability is more involved. First, we get a convenient expression for the second derivative:

$$D^2 S(\phi^s) = \langle DJ(\phi^s),\, D\phi^s \rangle + \langle J(\phi^s),\, D^2\phi^s \rangle. \tag{5.1}$$

Let us abbreviate $\phi \equiv \phi^s$, $J \equiv J(\phi^s)$. Upon twice differentiating the identity for all s, $\langle J,\, \phi \rangle = 0$, we find

$$2\langle DJ,\, D\phi \rangle + \langle D^2 J,\, \phi \rangle + \langle J,\, D^2\phi \rangle = 0; \tag{5.2}$$

so

$$\begin{aligned}
D^2 S &= -\langle DJ,\, D\phi \rangle - \langle D^2 J,\, \phi \rangle \\
&= \frac{1}{2}\langle J,\, D^2\phi \rangle - \frac{1}{2}\langle D^2 J,\, \phi \rangle,
\end{aligned} \tag{5.3}$$

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where the last line comes from adding the preceding line to (5.1).

Let us now insert $\phi = E\phi$ and $J = PJ$ into the two terms above, and use $PE = EP = 0$ to get

$$\langle J,\, D^2\phi \rangle = \langle J,\, \left(D^2 E\right)\phi + 2(DE)D\phi \rangle , \tag{5.4}$$

$$\langle D^2 J,\, \phi \rangle = \langle \left(D^2 P\right) J + 2(DP)DJ,\, \phi \rangle . \tag{5.5}$$

At $s_1 = s_2$, since $DE = DP = 0$, we get

$$\langle J,\, D^2\phi \rangle = -\langle J,\, \left(D^2 P\right)\phi \rangle , \tag{5.6}$$

$$\langle D^2 J,\, \phi \rangle = \langle \left(D^2 P\right) J,\, \phi \rangle = \langle J,\, \left(D^2 P\right)\phi \rangle ; \tag{5.7}$$

so at $s_1 = s_2$

$$D^2 S = -\langle J,\, \left(D^2 P\right)\phi \rangle . \tag{5.8}$$

At $s_1 = s_2 = 0$, assuming ϕ^s is smooth enough in s, the source is a finite linear combination of ξ and $\dot{\xi}$:

$$J = \alpha\xi + \beta\dot{\xi} . \tag{5.9}$$

Thus, from eq. (3.10) for $D^2 P$, we find

$$D^2 S = -\alpha\langle \ddot{\xi},\, \phi \rangle - \frac{\beta}{3}\langle \dddot{\xi},\, \phi \rangle . \tag{5.10}$$

Up to now, we have not assumed uniqueness, for a particular choice of boundary conditions at $|t| = \infty$. Since we are assuming two and only two values s_1 and s_2 for which the orthogonality condition (1.3) is obeyed (which means one zero of order two when $s_1 = s_2$), the boundary conditions on a finite action solution of (1.1) are of the same sign type. We take $\phi = \mu/\sqrt{\lambda}$ at $|t| = \infty$, leaving the other case to the reader; and we assume there is but one family ϕ^s of solutions to (1.1) and (1.3) with n exactly 2.

It follows at $s_1 = s_2 = 0$ that $\beta = 0$, for if the solution $\phi(t)$ is not even then $\phi_-(t) \equiv \phi(-t)$ obeys

$$J(\phi_-) = \alpha\xi - \beta\dot{\xi} , \tag{5.11}$$

as well as the two constraints $\langle \phi_-,\, \xi \rangle = \langle \phi_-,\, \dot{\xi} \rangle = 0$ and the boundary conditions. That violates uniqueness.

Notice that

$$\langle \ddot{\xi},\, \phi \rangle = \left. \frac{d^2}{da^2}\langle \xi^a,\, \phi \rangle \right|_{a=0} \tag{5.12}$$

is nonvanishing, because by hypothesis the function $\langle \xi^a,\, \phi \rangle$ has a single zero of order two at $a = 0$. As $|a| \to \infty$, that function is positive, because it is the convolution of the positive, zero mode pulse ξ with a function ϕ that is positive and bounded away from zero at $|t| = \infty$. Therefore, $\langle \xi^a,\, \phi \rangle$ is non-negative; and

$$\langle \ddot{\xi},\, \phi \rangle > 0 . \tag{5.13}$$

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The conclusion so far is that, if the multi-instanton pair configurations for positive boundary conditions are unique, then $\alpha < 0$ at $s_1 = s_2$ means stability (from (5.10), $D^2 S > 0$), while $\alpha > 0$ means instability.

Here we invoke the positivity lemma from sect. 2. If $\alpha > 0$, then by inspection of the field equation (5.9), with $\beta = 0$, it follows that the source $\alpha \xi$ is positive; so $\phi \geq \mu/\sqrt{\lambda} > 0$ for all t, and

$$\langle \phi, \xi \rangle > 0 . \tag{5.14}$$

Hence, the orthogonality constraint cannot be satisfied, $\alpha > 0$ is ruled out, and the solution must be stable.

To summarize the argument for stability, we found that the existence and regularity of a translation invariant family of finite action pair solutions of (1.1), (1.3) implied criticality at coincidence and led to a simple form in (5.10) for the second derivative of the action, that the additional assumption of uniqueness led to the absence of $\dot{\xi}$ in the source, that $\langle \ddot{\xi}, \phi \rangle$ had to be positive for positive boundary conditions at $|t| = \infty$, and that, thanks to the positivity lemma, the orthogonality constraint could then be satisfied only in the stable case.

6. Discussion

In a sequel to this paper [2], we present a numerical study of the nondilute pair configuration which demonstrates its existence and its stability at coincidence. There is no uniqueness proof, but also no indication of nonuniqueness. The action of that family of solutions at coincidence turns out to be

$$S_2(\text{coinc}) = 0.82047 \, S_2(\text{dilute}) , \tag{6.1}$$

where $S_2(\text{dilute}) = 2S_1$ is the action of a far separated pair, double the one-kink action. Our definition of the multi-instanton pair therefore has a solution that agrees with the general recognition that nondilute instantons in the double well model attract [4].

Thus we expect the leading coincident pair correction to the dilute gas, proportional to $\exp[-S_2(\text{coinc})/\hbar)]$, to be somewhat larger than second order in the leading, exponentially small one-instanton behavior. This kind of systematics is problematic in that there will be higher "loop" corrections that are powers of $\hbar$ (or the coupling) times the one-instanton exponential,[3] much larger than the weak coupling exponential squared.

Nevertheless, the square of the one-instanton exponential has been kept in the past [5,6] as the next step in the exponentiation to the dilute gas; and that is the role that we conjecture for the coincident pair. It should exponentiate into another constituent of the instanton gas, including far separated single instantons and coincident pairs. Indeed, it is tempting to conjecture a similar role for the n-instanton, coincident configuration, in a dilute gas of single and coincident multi-instantons of all orders.

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[3] See remarks by Zinn-Justin [4, p. 126], on the necessity for partial summation, and the example of the ground state energy [4, p. 131].

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