The Classical Relativistic Elastic String*

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August 10, 2011

Abstract

The theory of an elastic string in classical special relativity is considered. Standard elasticity concepts are applied to parametrize the elastic tensor, which is put into a Lagrangian in order to define canonical variables. A few specific models for the elastic potential energy as a function of elastic strain are examined, with brief comments about their possible suitability for canonical quantization.

*This LaTeX version has only cosmetic changes from the original, handwritten manuscript, dated September 15, 1974, except for the abstract and introduction, added in July, 2007, and new string models, added in January, 2009, and reparametrized in July, 2011.

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1 Introduction

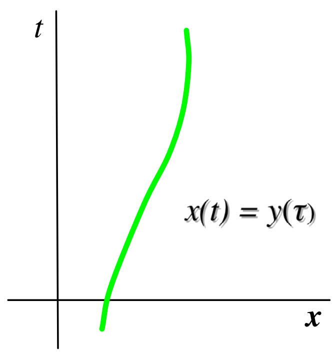
This article lays out the theory of a classical, elastic string in special relativity. It is self-contained, and aims to develop a possible foundation for the canonical quantization of strings in conventional space-time. It does not include an actual attempt at quantization.

Although we reinvented it for ourselves, the formulation of elasticity for a string was not new in 1974, when we finished the body of this work. Thus we only claim that Sections 2–5 establish our notation, not that they break new ground.

New models were added to Section 7 in January, 2009. Model parametrization was revised in July, 2011, to make it easier to discuss the properties sought for models: stability, nontachyonicity, simplicity of or freedom from constraints, and simplicity of the four-momentum density expressed in canonical variables.

Two of the models from the original manuscript remain the most promising, those for the dual string and the dual string with mass.

2 Kinematics of point motion



We may parametrize the orbit of a point, massive particle in Minkowski space by

$$x(t) = (t, \boldsymbol{x}(t))\,. \tag{2.1}$$

Along the orbit, $\mathrm{d}x$ is assumed to be positive timelike. In units where the speed of light c is unity, the velocity

$$v(t) = (1, \boldsymbol{v}(t))\,, \quad \boldsymbol{v}(t) \equiv \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t), \tag{2.2}$$

is proportional to a four vector.

The proper time elapse between times t_0 and t is

$$\tau(t) - \tau(t_0) = \int_{t_0}^{t} \frac{\mathrm{d}t'}{\gamma(|\boldsymbol{v}(t')|)}\,, \qquad \gamma(v) \equiv \frac{1}{\sqrt{1-v^2}}\,, \qquad |v| < 1\,. \tag{2.3}$$

This quantity is Poincaré invariant. We take the active view. First of all, translations: $\boldsymbol{v}$ evaluated at the image of a point on the orbit is unchanged by a four-dimensional translation. Secondly, we claim that if $x' = (t', \boldsymbol{x}'(t')) = \Lambda x$, $\Lambda \in \mathrm{L}^{\uparrow}$, and if t' and t'_0 are the images of t and t_0 on the orbit under Λ, then

$$\tau'(t') - \tau'(t'_0) = \int_{t'_0}^{t'} \frac{\mathrm{d}t''}{\gamma(|\boldsymbol{v}'(t'')|)} = \tau(t) - \tau(t_0)\,. \tag{2.4}$$

Proof. Note that $\mathrm{d}t/\mathrm{d}t' = (\mathrm{d}t'/\mathrm{d}t)^{-1} > 0$ if $\Lambda \in \mathrm{L}^{\uparrow}$. Now

$$\frac{\mathrm{d}}{\mathrm{d}t'}\left(t', \boldsymbol{x}'(t')\right) = \Lambda \frac{\mathrm{d}}{\mathrm{d}t'}(t, \boldsymbol{x}(t)) = \Lambda \frac{\mathrm{d}t}{\mathrm{d}t'}(1, \boldsymbol{v}(t))\,. \tag{2.5}$$

Hence

$$1 - v'^2 = \left(\frac{\mathrm{d}t}{\mathrm{d}t'}\right)^2 (1 - v^2)\,, \qquad \frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\gamma(v)}{\gamma(v')}\,. \tag{2.6}$$

The result follows by a change of variable in the integral. □

We choose to parametrize

$$x(t) = y(\tau)\,, \qquad t(\tau) - t(\tau_0) \equiv \int_{\tau_0}^{\tau} \gamma\, \mathrm{d}\tau'\,, \tag{2.7}$$

where γ is now understood as a function of τ: $\gamma = \gamma(|\boldsymbol{v}(t(\tau)|)$. We have chosen arbitrary values τ_0 and $t_0 = t(\tau_0)$ at which we set $x(t_0) = y(\tau_0)$. We compute from the parametrization:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\,. \tag{2.8}$$

We can now see that $y(\tau)$ is a four-vector:

$$y'(\tau) = \Lambda y(\tau) = \big(t'(\tau), \boldsymbol{x}'[t'(\tau)]\big)\,. \tag{2.9}$$

Proof. That $\Lambda y(\tau) = \big(t', \boldsymbol{x}'(t')\big)$ is true by definition. The claim is that $t' = t'(\tau)$, where

$$t'(\tau) - t'(\tau_0) = \int_{\tau_0}^{\tau} \gamma(|\boldsymbol{v}'|)\, \mathrm{d}\tau'\,, \tag{2.10}$$

and $t'(\tau_0) \equiv t'_0$, the image of t_0 under Λ. That is so because:

(a) If we define $t'(\tau)$ from $\Lambda y = x'$, we get

$$\frac{\mathrm{d}t'}{\mathrm{d}\tau} = \frac{\mathrm{d}t'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\gamma'}{\gamma}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma'\,, \tag{2.11}$$

where we computed $\mathrm{d}t/\mathrm{d}\tau$ from the unprimed definition $y(\tau) = x(t)$ above. This coincides with $\mathrm{d}t'/\mathrm{d}\tau$ computed from the primed integral definition in Eq. (2.10).

(b) Furthermore $t'(\tau_0) = t'_0$ by definition. Hence $t'(\tau) = t'$, the image of $t(\tau)$, for all τ, where $t'(\tau)$ is defined by the primed integral. □

We define the four-velocity

$$u(\tau) = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \gamma\, \frac{\mathrm{d}x}{\mathrm{d}t} = (\gamma, \gamma\, \boldsymbol{v}(t))\,. \tag{2.12}$$

It is a Minkowski unit vector,

$$u{\cdot}u = \gamma^2(1 - v^2) = 1\,, \tag{2.13}$$

and a four-vector,

$$u'(\tau') = \frac{\mathrm{d}x'}{\mathrm{d}\tau'}(\tau') = \Lambda \frac{\mathrm{d}x}{\mathrm{d}\tau}(\tau) = \Lambda u(\tau)\,. \tag{2.14}$$

The four-acceleration is also a four-vector:

$$a(\tau) \equiv \frac{\mathrm{d}u}{\mathrm{d}\tau}(\tau)\,, \qquad a(\tau)\cdot u(\tau) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\, u\cdot u = 0\,. \tag{2.15}$$

The latter equation shows that $a(\tau)$ is spacelike. Finally, we compute:

$$\begin{aligned} a(\tau) = \frac{\mathrm{d}t}{\mathrm{d}\tau}\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\gamma, \gamma\boldsymbol{v}) = \gamma\left(\frac{v}{(1-v^2)^{\frac{3}{2}}},\ \frac{\boldsymbol{v}v}{(1-v^2)^{\frac{3}{2}}} + \gamma\boldsymbol{a}\right) \\ = \left(\frac{v}{(1-v^2)^2},\ \frac{\boldsymbol{v}v}{(1-v^2)^2} + \frac{\boldsymbol{a}}{1-v^2}\right), \\ \boldsymbol{a} \equiv \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,. \end{aligned} \tag{2.16}$$

3 Deformation

We introduce a reference, or undistorted string, which lies at rest on a straight line in three-dimensional space. It has length l, and points on it are parametrized isometrically by a body coordinate λ, $-l/2 \le \lambda \le l/2$.

We give a family of motions in Minkowski space, one for each of the body points λ:

$$x(t, \lambda) = \big(t, \boldsymbol{x}(t, \lambda)\big)\,. \tag{3.1}$$

We assume that $x(t, \lambda)$ is sufficiently smooth in t and λ. At time t, each body point moves with velocity

$$\boldsymbol{v}(t, \lambda) = \frac{\partial \boldsymbol{x}}{\partial t}(t, \lambda)\,, \tag{3.2}$$

and acceleration

$$\boldsymbol{a}(t, \lambda) = \frac{\partial \boldsymbol{v}}{\partial t}(t, \lambda)\,. \tag{3.3}$$

The state of distortion at time t may be described by the family of vectors $\partial\boldsymbol{x}(t, \lambda)/\partial\lambda$. In particular, local dilatation is described by

$$\left(\frac{\delta|\boldsymbol{x}|}{\delta\lambda}\right)^2 \equiv \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\,. \tag{3.4}$$

We need a covariant description of this information. And in the relativistic regime, we may also wonder about a distortion in relative proper time elapse (aging) of neighboring points on the string. As an initial step, we parametrize the orbits with proper time instead of t:

$$y(\tau, \lambda) \equiv x(t, \lambda)\,. \tag{3.5}$$

For now, we defer the question of the convention for the initial τ's as a function of λ. Where no confusion seems likely, we follow the conventions that $\partial y/\partial\lambda$ means τ is fixed and $\partial x/\partial\lambda$ means t is fixed, while partial derivatives with resptect to t or τ are always with λ fixed.

The family of Minkowski orbits $y(\tau, \lambda)$ defines a two-dimensional surface in $\mathbb{R}^4$ whose shape can be described by giving at each point two tangent vectors, with appropriate normalization. We choose the tangent vectors to be four-vectors.

The natural choice for the first tangent vector is the four-velocity:

$$u(\tau, \lambda) = \frac{\partial y}{\partial \tau}(\tau, \lambda)\,. \tag{3.6}$$

The second, linearly independent tangent vector could be chosen as $\partial y/\partial\lambda$. However, we choose instead the component of this vector which is orthogonal to u:

$$\eta(\tau, \lambda) = \frac{\partial y}{\partial \lambda} - u\, u\cdot \frac{\partial y}{\partial \lambda}\,. \tag{3.7}$$

The question then arises what information we throw away by that choice. That question is related to the choice of initial proper times as a function of λ. Suppose we define

$$\tau(t, \lambda) = \tau(t_0, \lambda) + \int_{t_0}^{t} \frac{\mathrm{d}t'}{\gamma(|\boldsymbol{v}(t', \lambda)|)}\,. \tag{3.8}$$

The function $\tau_0(\lambda) \equiv \tau(t_0, \lambda)$ is at our disposal. If in a given frame we should start the string at time t_0 from rest in the undistorted configuration, it might be natural to choose $\tau_0(\lambda) = \tau_0$ to be independent of λ in that frame. But let's leave $\tau_0(\lambda)$ arbitrary.

Then

$$\frac{\partial y}{\partial \lambda}(\tau, \lambda) = \frac{\partial x}{\partial t}(t, \lambda)\ \left.\frac{\partial t}{\partial \lambda}\right|_{\tau} + \frac{\partial x}{\partial \lambda}(t, \lambda)\,. \tag{3.9}$$

Also,

$$\begin{aligned} \left.\frac{\partial \tau}{\partial \lambda}\right|_{\tau} = 0 &= \frac{\partial \tau}{\partial t}(t, \lambda)\ \left.\frac{\partial t}{\partial \lambda}\right|_{\tau} + \left.\frac{\partial \tau}{\partial \lambda}\right|_{t} \\ &= \frac{1}{\gamma}\ \left.\frac{\partial t}{\partial \lambda}\right|_{\tau} + \left.\frac{\partial \tau}{\partial \lambda}\right|_{t}\,. \end{aligned} \tag{3.10}$$

Thus,

$$\frac{\partial y}{\partial \lambda} = \frac{\partial x}{\partial \lambda} - u\ \left.\frac{\partial \tau}{\partial \lambda}\right|_{t}, \tag{3.11}$$

and from Eq. (3.7),

$$\eta(\tau, \lambda) = \frac{\partial x}{\partial \lambda} - u\, u\cdot \frac{\partial x}{\partial \lambda}\,. \tag{3.12}$$

We see that η is totally independent of any particular $\tau_0(\lambda)$, which according to Eq. (3.11) enters $\partial y/\partial\lambda$ only through $\partial\tau/\partial\lambda|_t$. The quantity $\partial\tau/\partial\lambda|_t$ also measures the distortion in relative age of neighboring body points λ, and that is the information which is absent in η.

Note that in η, the first term,

$$\frac{\partial x}{\partial\lambda} = \left(0, \frac{\partial \boldsymbol{x}}{\partial\lambda}\right), \tag{3.13}$$

is all that survives in the nonrelativistic limit, since $\boldsymbol{u}$ is of order $\boldsymbol{v}/c$ in units where c is not necessarily one.

Our invariant measure of stretch relative to the undistorted body is thus

$$\begin{aligned} -\eta\cdot\eta &= -\frac{\partial y}{\partial\lambda}\cdot\frac{\partial y}{\partial\lambda} + \left(u\cdot\frac{\partial y}{\partial\lambda}\right)^2 \\ &= \left.\frac{\partial \boldsymbol{x}}{\partial\lambda}\right|_t\cdot\left.\frac{\partial \boldsymbol{x}}{\partial\lambda}\right|_t + \frac{\left(\boldsymbol{v}\cdot\left.\frac{\partial \boldsymbol{x}}{\partial\lambda}\right|_t\right)^2}{1-v^2}\,. \end{aligned} \tag{3.14}$$

4 Material conservation

Let the undistorted rest mass per unit length of the string be $\sigma_0(\lambda)$. The rest mass density field is then

$$\rho(x) = \iint \delta[x - y(t,\lambda)]\,\gamma(\tau,\lambda)\,\sigma_0(\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda\,, \tag{4.1}$$

where the τ integration is from $-\infty$ to $+\infty$, and the λ integration from $-l/2$ to $+l/2$. This can also be written

$$\begin{aligned} \rho(x) &= \iint \delta[x - x(t,\lambda)]\,\gamma\,\sigma_0(\lambda)\,\frac{\mathrm{d}t}{\gamma}\,\mathrm{d}\lambda \\ &= \int \delta[\boldsymbol{x} - \boldsymbol{x}(x_0,\lambda)]\,\sigma_0(\lambda)\,\mathrm{d}\lambda\,. \end{aligned} \tag{4.2}$$

To see that the definition is correct, let $\delta V(x_0)$ be a volume in $\mathbb{R}^3$ which contains the image of the body segment $\delta\lambda$ at time x_0. Then

$$\int_{\delta V(x_0)} \rho(x_0,\boldsymbol{x})\,\mathrm{d}^3x = \int_{\delta\lambda} \sigma_0(\lambda)\,\mathrm{d}\lambda\,, \tag{4.3}$$

which is the rest mass of the segment $\delta\lambda$.

The definition of $\rho(x)$ implies the continuity equation as an identity for distributions; i.e., let $\boldsymbol{v}(x)$ be any continuous function which agrees with $\boldsymbol{v}(t,\lambda)$ whenever $x = x(t,\lambda)$:

$$\boldsymbol{v}[x(t,\lambda)] = \boldsymbol{v}(t,\lambda)\,; \tag{4.4}$$

then

$$\partial_t \rho(x) + \boldsymbol{\nabla}\cdot[\boldsymbol{v}(x)\rho(x)] = 0\,. \tag{4.5}$$

Proof. Let $f(x) \in \mathcal{S}(\mathbb{R}^4)$ (Schwartz test functions). Then

$$\begin{aligned}\int f[\partial_t\rho + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\rho)]\,\mathrm{d}^4x &= -\int \big[(\partial_t f)\rho + (\boldsymbol{\nabla} f)\cdot\boldsymbol{v}\rho\big]\,\mathrm{d}^4x \\ &= -\iint u\cdot\frac{\partial f}{\partial y}[y(\tau,\lambda)]\,\sigma_0(\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda \\ &= -\iint \frac{\partial}{\partial\tau} f[y(\tau,\lambda)]\,\sigma_0(\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda \\ &= \iint f[y(\tau,\lambda)]\,\frac{\partial}{\partial\tau}\,\sigma_0(\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda = 0\,. \quad \square\end{aligned} \tag{4.6}$$

The rest mass density $\rho(x)$ is not a scalar field because of Lorentz contraction of the volume element. If we define $\gamma(x)$ in terms of $\boldsymbol{v}(x)$, we get a scalar field by

$$\frac{\rho(x)}{\gamma(x)} = \iint \delta[x - y(\tau,\lambda)]\,\sigma_0(\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda\,. \tag{4.7}$$

The covariant kinetic stress-energy-momentum tensor field is

$$\begin{aligned}K^{\mu\nu}(x) &= \iint \delta[x - y(\tau,\lambda)]\,\sigma_0(\lambda)\,u^\mu(\tau,\lambda)\,u^\nu(\tau,\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda \\ &= \frac{\rho(x)}{\gamma(x)}\,u^\mu(x)\,u^\nu(x)\,.\end{aligned} \tag{4.8}$$

In the last line, $u(x)$ is defined in terms of $\boldsymbol{v}(x)$, and is thus arbitrary away from the world surface of the string.

We get two material identities for $K^{\mu\nu}$, true in the sense of distributions, which one again proves by smoothing with a test function:

$$\begin{aligned}\partial_\mu K^{\mu\nu} &= \iint \delta[x - y(\tau,\lambda)\,\sigma_0(\lambda)]\,\frac{\partial u^\nu}{\partial\tau}(\tau,\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda \\ &= \frac{\rho(x)}{\gamma(x)}\,a^\mu(x)\,, \qquad a^\mu[y(\tau,\lambda)] = \frac{\partial u^\nu}{\partial\tau}(\tau,\lambda)\,,\end{aligned} \tag{4.9a}$$

$$\begin{aligned}u_\nu\,\partial_\mu K^{\mu\nu} &= \iint \delta[x - y(\tau,\lambda)]\,\sigma_0(\lambda)\,u\cdot a\,\mathrm{d}\tau\,\mathrm{d}\lambda \\ &= 0\,.\end{aligned} \tag{4.9b}$$

5 Elastic Tensor

We parametrize the class of elastic strings by giving stress-energy-momentum tensors of the form

$$T^{\mu\nu}(x) = K^{\mu\nu}(x) + E^{\mu\nu}(x)\,. \tag{5.1}$$

The elastic tensor $E^{\mu\nu}$ describes the effect of elastic forces in the string.

The equations of motion will then be

$$\partial_\mu T^{\mu\nu}(x) = f^\nu(x)\,, \tag{5.2}$$

where f^ν is the external force per unit volume. It is constrained to obey

$$u(x)\cdot f(x) = 0 \tag{5.3}$$

for x on the world surface, at which points $u(x) = u(\tau, \lambda)$, in order to have the proper balance between power supplied and work done by f. Thus we demand

$$u_\nu\, \partial_\mu T^{\mu\nu} = 0\,, \tag{5.4}$$

which implies

$$u_\nu\, \partial_\mu E^{\mu\nu} = 0\,, \tag{5.5}$$

because $K^{\mu\nu}$ identically drops out.

To deserve the name "elastic tensor", we demand that $E^{\mu\nu}$ depend only on the distortion variable η and the kinematic variable u, plus an explicit dependence on λ, to allow for variable "spring constants". We make the *Ansatz*:

$$\begin{aligned} E^{\mu\nu}(x) = \iint \mathrm{d}\tau\, \mathrm{d}\lambda\, \delta[x - y(\tau, \lambda)] \\ \times \left[e_{00}\, u^\mu\, u^\nu + e_{01}\, u^\mu\, \eta^\nu + e_{10}\, \eta^\mu\, u^\nu + e_{11}\, \eta^\mu\, \eta^\nu\right]\,, \end{aligned} \tag{5.6}$$

where $e_{\sigma\sigma'}$ can depend only on $\eta\cdot\eta$ and λ. We do not know offhand how to rule out the remaining covariant possibility $\epsilon^{\mu\nu\lambda\rho}u_\lambda\eta_\rho$, and so there may be less than full generality at this point.

After applying the above conservation law in the sense of distributions, we find

$$\begin{aligned} 0 = u_\nu \Bigg\{ \frac{\partial}{\partial\tau}\left(e_{00}\, u^\nu + \, e_{01}\, \eta^\nu\right) + \frac{\partial}{\partial\lambda}\left(e_{10}\, u^\nu + \, e_{11}\, \eta^\nu\right) \\ - \frac{\partial}{\partial\tau}\left[u\cdot\frac{\partial y}{\partial\lambda}\left(e_{10}\, u^\nu + \, e_{11}\, \eta^\nu\right)\right]\Bigg\}\,. \end{aligned} \tag{5.7}$$

To derive this, we used the fact that

$$\eta\cdot\frac{\partial f}{\partial y}[y(\tau, \lambda)] = \left(\frac{\partial}{\partial\lambda} - u\cdot\frac{\partial y}{\partial\lambda}\frac{\partial}{\partial\tau}\right) f[y(\tau, \lambda)]\,. \tag{5.8}$$

Simplifying:

$$
\begin{aligned}
0 &= \frac{\partial e_{00}}{\partial \tau} + e_{01}\, u\cdot\frac{\partial \eta}{\partial \tau} + \frac{\partial e_{10}}{\partial \lambda} - \frac{\partial e_{10}}{\partial \tau}\, u\cdot\frac{\partial y}{\partial \lambda} - e_{10}\, a\cdot\frac{\partial y}{\partial \lambda} \\
&\qquad\quad + e_{11}\left(u\cdot\frac{\partial \eta}{\partial \lambda} - u\cdot\frac{\partial y}{\partial \lambda}\, u\cdot\frac{\partial \eta}{\partial \tau}\right) \\
&= \frac{\partial e_{00}}{\partial \tau} - e_{01}\, a\cdot\eta + \frac{\partial e_{10}}{\partial \lambda} - \frac{\partial e_{10}}{\partial \tau}\, u\cdot\frac{\partial y}{\partial \lambda} - e_{10}\, a\cdot\eta \\
&\qquad\quad + e_{11}\left(u\cdot\frac{\partial \eta}{\partial \lambda} + u\cdot\frac{\partial y}{\partial \lambda}\, a\cdot\eta\right).
\end{aligned}
\tag{5.9}
$$

Further note that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial\, \eta\cdot\eta}{\partial \tau} &= \eta\cdot\frac{\partial u}{\partial \lambda} - \eta\cdot a\, u\cdot\frac{\partial y}{\partial \lambda} \\
&= -u\cdot\frac{\partial \eta}{\partial \lambda} - \eta\cdot a\, u\cdot\frac{\partial y}{\partial \lambda}
\end{aligned}
\tag{5.10}
$$

which leads to

$$
0 = \frac{\partial e_{00}}{\partial \tau} - \frac{e_{11}}{2}\,\frac{\partial\, \eta\cdot\eta}{\partial \tau} - e_{01}\, a\cdot\eta - e_{10}\, a\cdot\eta + \frac{\partial e_{10}}{\partial \lambda} - \frac{\partial e_{10}}{\partial \tau}\, u\cdot\frac{\partial y}{\partial \lambda}\,. \tag{5.11}
$$

As the canonical class of elastic strings, we choose[1]

$$
e_{01} = e_{10} = 0\,. \tag{5.12}
$$

This gives

$$
\frac{\partial e_{00}}{\partial \tau} = \frac{\partial e_{00}}{\partial\, \eta\cdot\eta}\,\frac{\partial \eta\cdot\eta}{\partial \tau} = \frac{e_{11}}{2}\,\frac{\partial\, \eta\cdot\eta}{\partial \tau}\,, \tag{5.13}
$$

$$
e_{11}(\lambda, \eta\cdot\eta) = 2\,\frac{\partial e_{00}}{\partial\, \eta\cdot\eta}(\lambda, \eta\cdot\eta)\,, \tag{5.14}
$$

$$
\begin{aligned}
E^{\mu\nu} &= \iint \mathrm{d}\tau\, \mathrm{d}\lambda\, \delta[x - y(\tau, \lambda)] \\
&\qquad \times \left[e_{00}(\lambda, \eta\cdot\eta)\, u^\mu\, u^\nu + e_{11}(\lambda, \eta\cdot\eta)\, \eta^\mu\, \eta^\nu\right],
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
\partial_\mu E^{\mu\nu} &= \iint \mathrm{d}\tau\, \mathrm{d}\lambda\, \delta[x - y(\tau, \lambda)] \\
&\qquad \times \left[\frac{\partial}{\partial \tau}\left(e_{00}\, u^\mu - u\cdot\frac{\partial y}{\partial \lambda}\, e_{11}\, \eta^\mu\right) + \frac{\partial}{\partial \lambda}\left(e_{11}\, \eta^\mu\right)\right].
\end{aligned}
\tag{5.16}
$$

[1]Note that $e_{01} = -e_{10} =$ constant in λ and η^2 would also cancel e_{01} and e_{10} from the equation.

The e_{00} term in $E^{\mu\nu}$ has the same structure as $K^{\mu\nu}$. It represents the flow of elastic potential energy, whch gives a correction to the rest mass density. The e_{11} term is the elastic stress term, and embodies the familiar statement characterizing elastic media that stress is the gradient of elastic potential energy with respect to strain.

We have already seen that there are other solutions, with $\sigma_{01} = -\sigma_{10} \neq 0$. Just as in the analogous situation with three-dimensional elastic media [1], we think they correspond to velocity-dependent elastic forces, which may persist in the nonrelativistic limit. We put such solutions outside the canonical class.

6 Canonical variables

In the variables τ, λ the equations of motion can be written (no external force):

$$\begin{aligned} 0 &= \frac{\partial}{\partial\tau}\left[\sigma_0(\lambda)\, u^\mu + e_{00}\, u^\mu - u\cdot\frac{\partial y}{\partial\lambda}\, e_{11}\, \eta^\mu\right] + \frac{\partial}{\partial\lambda}\left(e_{11}\,\eta^\mu\right), \\ e_{11} &= -\frac{\partial e_{00}}{\partial D}, \qquad D \equiv -\frac{\eta\cdot\eta}{2}. \end{aligned} \tag{6.1}$$

These equations of motion can be written as Euler-Lagrange equations of the form

$$\frac{\partial}{\partial\tau}\frac{\delta L}{\delta\dfrac{\partial y^\mu}{\partial\tau}} + \frac{\partial}{\partial\lambda}\frac{\delta L}{\delta\dfrac{\partial y^\mu}{\partial\lambda}} - \frac{\delta L}{\delta y^\mu} = 0, \tag{6.2}$$

corresponding to the stationary action integral[2]

$$I = \iint L\left(y, \frac{\partial y}{\partial\tau}, \frac{\partial y}{\partial\lambda}, \tau, \lambda\right) \mathrm{d}\tau\, \mathrm{d}\lambda, \tag{6.3}$$

if we choose

$$L = \frac{1}{2}\left(\sigma_0 + e_{00}\right) u\cdot u + \frac{1}{2}\, e_{00} \tag{6.4a}$$

or

$$L = \left(\sigma_0 + e_{00}\right)\sqrt{u\cdot u}, \tag{6.4b}$$

subject to the constraint $u\cdot u = 1$.

In either case, we find[3]

$$\frac{\delta L}{\delta u^\mu} = \left(\sigma_0 + e_{00}\right) u_\mu + \frac{\partial e_{00}}{\partial D}\frac{\delta D}{\delta u^\mu}, \qquad \frac{\delta D}{\delta u^\mu} = u\cdot\frac{\partial y}{\partial\lambda}\,\eta_\mu, \tag{6.5a}$$

$$\frac{\delta L}{\delta\dfrac{\partial y^\mu}{\partial\lambda}} = \frac{\partial e_{00}}{\partial D}\frac{\delta D}{\delta\dfrac{\partial y^\mu}{\partial\lambda}}, \qquad \frac{\partial D}{\partial\dfrac{\partial y^\mu}{\partial\lambda}} = -\eta_\mu. \tag{6.5b}$$

[2]With finite limits in the τ integration.

[3]Derivatives of D in these expressions are taken before the four-velocity constraint is applied.

Note that

$$u\cdot\frac{\partial y}{\partial\lambda} = u\cdot\frac{\partial x}{\partial\lambda} - \left.\frac{\partial\tau}{\partial\lambda}\right|_t$$
$$= -\boldsymbol{u}\cdot\frac{\partial\boldsymbol{x}}{\partial\lambda} - \left.\frac{\partial\tau}{\partial\lambda}\right|_t . \tag{6.6}$$

From Eqs. (3.12) and (3.13),

$$\boldsymbol{u}\cdot\frac{\partial\boldsymbol{x}}{\partial\lambda} = \boldsymbol{u}\cdot\boldsymbol{\eta} - \boldsymbol{u}\cdot\boldsymbol{u}\,\boldsymbol{u}\cdot\frac{\partial\boldsymbol{x}}{\partial\lambda}\,, \tag{6.7}$$

which implies

$$\boldsymbol{u}\cdot\frac{\partial\boldsymbol{x}}{\partial\lambda} = \frac{\boldsymbol{u}\cdot\boldsymbol{\eta}}{\gamma^2} = \frac{\eta_0}{\gamma}\,; \tag{6.8}$$

so we get the expression for the canonical four-momentum:

$$\Pi^\mu(\tau,\lambda) = \frac{\delta L}{\delta u_\mu} = \left(\sigma_0 + e_{00}\right)u^\mu - e_{11}\,u\cdot\frac{\partial y}{\partial\lambda}\,\eta^\mu \tag{6.9a}$$

$$= \left(\sigma_0 + e_{00}\right)u^\mu + e_{11}\left(\frac{\eta^0}{\gamma} + \left.\frac{\partial\tau}{\partial\lambda}\right|_t\right)\eta^\mu . \tag{6.9b}$$

By inspection of Eqs. (4.8) and (5.15), we find

$$T^{\mu0}(0,\boldsymbol{x}) = \int \mathrm{d}\lambda\,\delta[\boldsymbol{x} - \boldsymbol{x}(0,\lambda)]\left[\Pi^\mu(\tau,\lambda) - e_{11}\left.\frac{\partial\tau}{\partial\lambda}\right|_t\eta^\mu\right]_{t=0}$$
$$= \int \mathrm{d}\lambda\,\delta[\boldsymbol{x} - \boldsymbol{x}(0,\lambda)]\;(\Pi^\mu)_{t=0}\,, \tag{6.10}$$

if in the last line we adopt the convention

$$\left.\frac{\partial\tau}{\partial\lambda}\right|_t = 0\,. \tag{6.11}$$

The infinitesimal generators of the Poincaré group are:

$$P^\mu = \int \mathrm{d}^3x\;T^{\mu0}(0,\boldsymbol{x})\,, \tag{6.12a}$$

$$M^{\mu\nu} = \int \mathrm{d}^3x\;\left(x^\mu T^{\nu0} - x^\nu T^{\mu0}\right)_{t=0}\,. \tag{6.12b}$$

To save writing, put

$$\boldsymbol{x}(\lambda) \equiv \boldsymbol{x}(0,\lambda)\,, \tag{6.13a}$$

$$\Pi^\mu(\lambda) \equiv \Pi^\mu(\tau,\lambda)_{t=0}\,. \tag{6.13b}$$

Then we get:

$$P^{\mu} = \int \mathrm{d}\lambda\, \Pi^{\mu}(\lambda)\,, \tag{6.14a}$$

$$M^{0j} = -\int \mathrm{d}\lambda\; x^{j}(\lambda)\,\Pi^{0}(\lambda)\,, \tag{6.14b}$$

$$M^{ij} = \int \mathrm{d}\lambda\; \left[x^{i}(\lambda)\,\Pi^{j}(\lambda) - x^{j}(\lambda)\,\Pi^{i}(\lambda)\right]\,. \tag{6.14c}$$

The canonical variables are $\boldsymbol{x}(\lambda)$ and $\boldsymbol{\Pi}(\lambda)$. One would expect eventually to quantize the theory by imposing canonical commutation relations:

$$\left[x^{i}(\lambda), \Pi^{j}(\lambda')\right] = i\,\delta_{ij}\,\delta(\lambda-\lambda')\,. \tag{6.15}$$

It is thus important to express Π^0 as a function of the canonical variables in the classical theory.

As an initial step, we compute from Eq. (6.9b):

$$\Pi\!\cdot\!\Pi = \left(\sigma_0 + e_{00}\right)^2 + \left(e_{11}\,\frac{\eta_0}{\gamma}\right)^2 \eta\!\cdot\!\eta\,. \tag{6.16}$$

This suggests that we need canonical expressions for the variables $(\eta_0/\gamma)^2$ and $\eta\!\cdot\!\eta$, on the latter of which e_{00} and e_{11} depend. This can be done by two identities:

$$\frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda} = -\eta\!\cdot\!\eta - \left(\frac{\eta_0}{\gamma}\right)^2, \tag{6.17a}$$

$$\boldsymbol{\Pi}\!\cdot\!\frac{\partial \boldsymbol{x}}{\partial\lambda} = \frac{\eta_0}{\gamma}\left(\sigma_0 + e_{00} - e_{11}\,\eta\!\cdot\!\eta\right). \tag{6.17b}$$

To compute these identities we used the following:

$$\boldsymbol{u}\!\cdot\!\boldsymbol{\eta} = \gamma\,\eta_0\,, \qquad \boldsymbol{u}\!\cdot\!\frac{\partial \boldsymbol{x}}{\partial\lambda} = \frac{\eta_0}{\gamma}\,, \qquad \boldsymbol{\eta}\!\cdot\!\frac{\partial \boldsymbol{x}}{\partial\lambda} = -\eta\!\cdot\!\eta\,. \tag{6.18}$$

The last of these is valid at $t=0$, with our convention for $\partial\tau/\partial\lambda$.

Putting these identities together, we get $\eta\!\cdot\!\eta$ in terms of $\left(\boldsymbol{\Pi}\!\cdot\!\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2$ and $\frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}$ by solving

$$\left(\boldsymbol{\Pi}\!\cdot\!\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2 = -\left(\eta\!\cdot\!\eta + \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)\left(\sigma_0 + e_{00} - e_{11}\,\eta\!\cdot\!\eta\right)^2. \tag{6.19}$$

Then we have in canonical form

$$\left[\Pi^0(\lambda)\right]^2 = \left(\sigma_0 + e_{00}\right)^2 - e_{11}^2\,\eta\!\cdot\!\eta\left(\eta\!\cdot\!\eta + \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right) + \boldsymbol{\Pi}\!\cdot\!\boldsymbol{\Pi}\,, \tag{6.20}$$

by substituting the solution for $\eta\!\cdot\!\eta$ into e_{00} and e_{11}.

7 Models

To solve for $\eta\cdot\eta$ in terms of canonical variables, we consider amenable models for the potential energy function e_{00}. It is convenient to write e_{00} not as a function of $-\eta\cdot\eta$, but of its square root, and to scale out a σ_0 factor. In the following, we suppress any explicit λ dependence:

$$\xi \equiv \sqrt{-\eta\cdot\eta}\,, \qquad 0 \le \xi \le \infty\,, \tag{7.1a}$$

$$e_{00} \equiv \sigma_0\, w(\xi)\,, \tag{7.1b}$$

$$e_{11} = -\frac{\sigma_0}{\xi}\,\frac{\partial w}{\partial \xi} \equiv -\frac{\sigma_0}{\xi}\, w'(\xi)\,, \tag{7.1c}$$

$$\sigma_0\, h(\xi) \equiv \sigma_0 + e_{00} - e_{11}\,\eta\cdot\eta = \sigma_0\left(1 + w - \xi w'\right)\,, \tag{7.1d}$$

$$\left(\frac{\boldsymbol{\Pi}}{\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2 = \left(\xi^2 - \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)[h(\xi)]^2\,. \tag{7.1e}$$

If we like some function $h(\xi)$ because it admits a pleasant solution of the last equation for ξ in terms of canonical variables, we can find a corresponding potential energy function w by solving the inhomogeneous differential equation

$$w - \xi w' = h - 1\,. \tag{7.2}$$

The solution of the homogeneous equation is $w = \kappa\xi$, $w' = \kappa$, with integration constant κ; so the inhomogeneous solution is unique if either w or w' is specified at the undistorted value $\xi = 1$. The general inhomogeneous solution and its first two derivatives are

$$w(\xi) = \kappa\xi - 1 - \xi\int_1^\xi \frac{h(x)}{x^2}\,\mathrm{d}x\,, \qquad \kappa = w(1) + 1\,, \tag{7.3a}$$

$$w'(\xi) = \kappa - \frac{h(\xi)}{\xi} - \int_1^\xi \frac{h(x)}{x^2}\,\mathrm{d}x\,, \tag{7.3b}$$

$$w''(\xi) = -\frac{h'(\xi)}{\xi}\,. \tag{7.3c}$$

With this parametrization, Eq. (6.20) gives the following for the scaled square of the canonical four-momentum:

$$\frac{\Pi\cdot\Pi}{\sigma_0^2} = (1+w)^2 - w'^2\left(\xi^2 - \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right), \tag{7.4a}$$

$$= (1+w)^2 - \frac{w'^2}{h^2}\left(\frac{\boldsymbol{\Pi}}{\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2, \tag{7.4b}$$

$$= 2(w+1)\,h - h^2 + w'^2\,\frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\,, \tag{7.4c}$$

where the second version results from Eq. (7.1e). The Hamiltonian density Π_0 can be extracted from these.

Here are two basic properties of the potential energy that we look for, expressed alternatively as properties of w or of h:

- *nontachyonicity*: The requirement that the kinetic plus elastic rest mass density be nonnegative is expressed as

$$w + 1 \geq 0 \qquad \Longleftrightarrow \qquad \kappa \geq \int_1^{\xi} \frac{h(x)}{x^2}\,\mathrm{d}x\,. \tag{7.5a}$$

  At least tachyonic regions should not be reachable via the equations of motion from initial conditions of interest.

- *stability*:

$$w' = 0 \qquad \Longleftrightarrow \qquad \kappa = \frac{h}{\xi} + \int_1^{\xi} \frac{h(x)}{x^2}\,\mathrm{d}x\,. \tag{7.5b}$$

$$w'' > 0 \qquad \Longleftrightarrow \qquad h' < 0\,. \tag{7.5c}$$

Now we try some examples.

(a) The dual string. Let $h = 0$. Then

$$\begin{aligned} w + 1 &= \kappa\,\xi\,, \qquad \kappa > 0\,, \\ w' &= \kappa\,, \qquad w'' = 0\,. \end{aligned} \tag{7.6}$$

Note that the $-\sigma_0$ term in the elastic part of the rest mass density, $e_{00} = \sigma_0\,(\kappa\xi - 1)$, exactly cancels the kinetic term. To requirement $\kappa > 0$ avoids tachyons. The model is not conventionally stable, because $w' = \kappa \neq 0$ holds for all ξ. The potential energy function rises linearly from its totally compressed value at $\xi = 0$ through the undistorted reference value at $\xi = 1$ to infinity at the totally stretched $\xi = \infty$. If started at rest, the string will shrink to a point.[4]

By Eq. (7.1e) the canonical momentum has only transverse components,

$$\mathbf{\Pi}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda} = 0\,; \tag{7.7}$$

so canonical quantization has to deal with a constraint. Because $h = 0$, Eq. (7.1e) doesn't give an equation to solve for the canonical form of ξ. That's not needed because from Eq. (7.4c), the squared canonical four-momentum is given by

$$\frac{\Pi\cdot\Pi}{\sigma_0^2} = \kappa^2 \frac{\partial \boldsymbol{x}}{\partial \lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda}\,, \tag{7.8}$$

and has no explicit ξ dependence.

[4]In finite time?

I have shown that this model is identical to the Nambu dual string [2, 3], where any nonzero rest mass density is due to elastic stretch energy. It is known that there are classical motions where the string rotates about a perpendicular axis, with the ends moving at the speed of light.[5]

(b) The dual string with mass. Because of widespread interest in the dual model it seems sensible to try a minimal modification that avoids the transversality constraint. We propose $h(\xi) = \alpha =$ constant:

$$\begin{aligned} w + 1 &= (\kappa - \alpha)\,\xi + \alpha\,, \qquad \kappa > \alpha > 0\,, \\ w' &= \kappa - \alpha\,, \qquad w'' = 0\,. \end{aligned} \tag{7.9}$$

The behavior of the potential energy function is then qualitatively the same as before, with the same instability property and the same avoidance of tachyons; but the kinetic term does not cancel; and rest mass density at totally compressed $\xi = 0$ is nonzero. Of course $\alpha = 0$ recovers the usual dual model.

The canonical momentum is not constrained, and we have

$$\xi^2 = \frac{\partial \boldsymbol{x}}{\partial \lambda} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda} + \left(\frac{\boldsymbol{\Pi}}{\alpha \sigma_0} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda} \right)^2 . \tag{7.10}$$

The scaled, squared canonical four-momentum is given by:

$$\begin{aligned} \frac{\Pi \cdot \Pi}{\sigma_0^2} &= \alpha^2 + (\kappa - \alpha)^2 \frac{\partial \boldsymbol{x}}{\partial \lambda} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda} \\ &\qquad + 2(\kappa - \alpha)\alpha \sqrt{\frac{\partial \boldsymbol{x}}{\partial \lambda} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda} + \left(\frac{\boldsymbol{\Pi}}{\alpha \sigma_0} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda} \right)^2}\,. \end{aligned} \tag{7.11}$$

(c) A stable string. We look for a potential energy function that describes a string in stable equilibrium in the reference configuration, with restoring forces towards equilibrium. We choose $h(\xi)$ to get a quadratic equation for ξ^2. Thus, let

$$\begin{aligned} h &= \frac{\alpha}{\xi^2}\,, \qquad w + 1 = \left(\kappa - \frac{\alpha}{3} \right) \xi + \frac{\alpha}{3\xi^2}\,, \\ w' &= \kappa - \frac{\alpha}{3} - \frac{2\alpha}{3\xi^3}\,, \qquad w'' = \frac{2\alpha}{\xi^4}\,. \end{aligned} \tag{7.12}$$

To have no stress at $\xi = 1$, we put $\alpha = \kappa$. The second derivative is always positive when $\kappa > 0$, which makes the minimum of w at $\xi = 1$ unique. Thus

$$w + 1 = \frac{\kappa}{3} \left(2\xi + \frac{1}{\xi^2} \right), \qquad w' = \frac{2\kappa}{3} \left(1 - \frac{1}{\xi^3} \right), \qquad w'' = \frac{2\kappa}{\xi^4}\,. \tag{7.13}$$

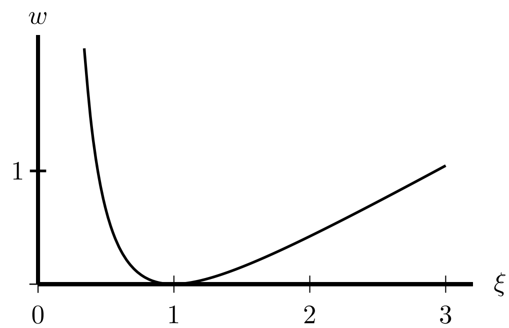
[5] I have not located the notes that, if I remember correctly, verified these claims from my original manuscript.

For the minimum value,

$$w(1) + 1 = \kappa > 0 \qquad \Rightarrow \qquad \sigma_0 + e_{00} > 0\,; \tag{7.14}$$

i.e., there is never a tachyonic region.

The plot below shows w for the special case $\kappa = 1$.



model (c) with $\kappa = 1$

In this model ξ is a solution of the equation

$$\left(\frac{\mathbf{\Pi}}{\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2 = \left(\xi^2 - \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)\frac{\kappa^2}{\xi^4}\,. \tag{7.15}$$

Hence[6]

$$\xi^2 = \frac{1 \pm \left[1 - 4\left(\frac{\mathbf{\Pi}}{\kappa\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2 \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right]^{\frac{1}{2}}}{2\left(\frac{\mathbf{\Pi}}{\kappa\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2}\,. \tag{7.16}$$

There is a constraint, which could pose a problem for canonical quantization:[7]

$$\left(\frac{\mathbf{\Pi}}{\kappa\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda}\right)^2 \frac{\partial \boldsymbol{x}}{\partial\lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial\lambda} \le \frac{1}{4}\,. \tag{7.17}$$

(d) A tachyonic, stable string. To avoid the constraint on the size of the longitudinal canonical momentum, we try

$$\begin{aligned} h &= \kappa\xi\,, \\ w + 1 &= \kappa(\xi - \xi\ln\xi)\,, \qquad w' = -\kappa\ln\xi\,, \qquad w'' = -\frac{\kappa}{\xi}\,. \end{aligned} \tag{7.18}$$

[6] Use the minus sign to make ξ^2 regular at zero denominator.

[7] We don't know whether the equations of motion preserve the constraint.

There is a unique minimum at $\xi = 1$ if $\kappa < 0$, but unfortunately that point then has a tachyonic neighborhood. We therefore reject this model, even though the solution for ξ is unconstrained:

$$\xi^2 = \frac{1}{2}\left[\frac{\partial \boldsymbol{x}}{\partial \lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda} + \sqrt{\left(\frac{\partial \boldsymbol{x}}{\partial \lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda}\right)^2 + 4\left(\frac{\boldsymbol{\Pi}}{\kappa\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda}\right)^2}\,\right]. \tag{7.19}$$

(e) Tachyon removed from string (d). To remove the tachyonic region in the preceding model, we shift w upward by a positive constant α:

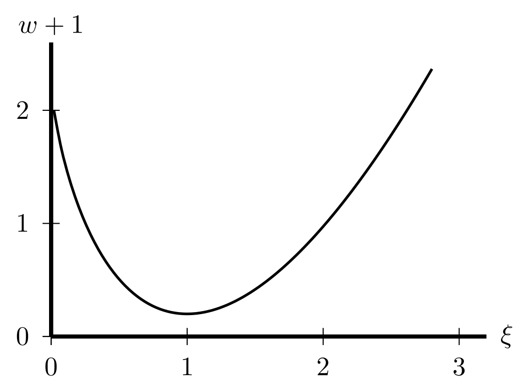
$$\begin{aligned} h &= \kappa\xi + \alpha\,, \\ w + 1 &= \kappa(\xi - \xi\ln\xi) + \alpha\,, \qquad w' = -\kappa\ln\xi\,, \qquad w'' = -\frac{\kappa}{\xi}\,. \end{aligned} \tag{7.20}$$

The unique minimum at $\xi = 1$ remains for $\kappa < 0$, but there is no tachyon as long as $\alpha \geq -\kappa$. From Eq. (7.1e) the canonical equation for ξ is quartic:

$$\left(\frac{\boldsymbol{\Pi}}{\kappa\sigma_0}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda}\right)^2 = \left(\xi^2 - \frac{\partial \boldsymbol{x}}{\partial \lambda}\cdot\frac{\partial \boldsymbol{x}}{\partial \lambda}\right)\left(\xi + \frac{\alpha}{\kappa}\right)^2. \tag{7.21}$$

We have not analyzed whether it leads to a constraint.

The plot below shows $w + 1$ for $\kappa = -2$ and $\alpha = 2.2$.



model (e) with $\kappa = -2.0$, $\alpha = 2.2$

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