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Euclidean Fermi Fields with a Hermitean Feynman-Kac-Nelson Formula, II*

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Abstract

We present a second heuristic FKN formula for fermions which is more suitable for constructing relativistic fields in the interacting case than our first attempt. The FKN formula remains Hermitean, and the Euclidean Dirac fields remain undoubled. This version is based on a local extension of Osterwalder-Schrader positivity to overlapping, Euclidean time arguments, which is not quite so immediate for us as it was for them. We propose a modified set of axioms for Euclidean Dirac fields, abstracted from the FKN formula. We show from Osterwalder-Schrader positivity that the Schwinger functions relative to the physical Hilbert space are at least well-defined as distributions, and they rigorously correspond to Wightman fields if one admits their existence as analytic functions with the appropriate continuations from Euclidean to Minkowski points. We do not check in this paper the natural conjecture that Nelson's Axiom (A′) implies the continuation.

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Contents

I. Introduction

In an earlier paper [1], hereafter referred to as (I), we presented a Euclidean Dirac (ED) field theory for free fields that required no doubling of particles or spin states, and proposed an interacting theory, based on a Hermitean Feynman-Kac-Nelson (FKN) formula that preserved a certain positivity condition obeyed by the free field, and admitted the construction of a "physical" Hilbert space and a Hermitean, contraction semigroup on it for imaginary time evolution.

Although that theory still seems to us to have an interesting structure, we have not found a way to construct relativistic fields from it, in the interacting case. The present treatment is more promising in that respect.

In (I) we did not fully appreciate that the free ED field in fact obeys *three* apparently distinct analogs of Nelson [2, 3] or Osterwalder-Schrader [4, 5] (OS) positivity (besides Symanzik positivity [6], which is the statement that the smeared fields are operators on a Hilbert space). The positivity condition we used for the interacting theory in (I) was nonlocal, but was a direct analog of Nelson's Markov plus reflection property [2] for bosons. Of the three conditions, it is perhaps the least direct analog of OS positivity (although we called it that), insofar as OS positivity is regarded as an intrinsic property of the *relativistic* field. The connection to the relativistic free field for the positivity condition in (I) is through a nonlocal transformation, which we describe in Appendix E.

There is a second positivity condition, also expressed in terms of the nonlocal, Euclidean field ϕ defined in (I), which we give in Appendix D. From it one reconstructs the dotted and undotted spinor parts of the relativistic free fields.[1]

The positivity condition that we use in this paper, which we call $\overline{\text{OS}}$ positivity, is a direct extension of OS positivity to overlapping time arguments in the Euclidean fields. This was automatic for Osterwalder and Schrader [5], and also in (I), because there the positivity condition was expressed in terms of a unitary, metric operator, which commuted with the Wick expansion. In this paper, we have to argue a bit more to achieve the same end. We discuss $\overline{\text{OS}}$ positivity in Sec. II, and prove it for free ED fields in Appendix A. Continuing in Sec. II, we adopt $\overline{\text{OS}}$-positivity as an axiom for interacting ED fields, and show that the Euclidean time evolution passes to a Hermitean, contraction semigroup on the physical space (proof in Appendix B). We then show that the appropriate objects, to be identified eventually with the Wick rotated, relativistic field operators, are densely defined operators on the natural domain in the physical space, when smeared in time (proof in Appendix C). The analogous result for nonsharp time, boson fields is also true; and as far as we know, the fact that it follows just from OS positivity is new. A similar result is true for the nonlocal, ϕ fields in (I); but they do not have, without some as yet unclear transformation in the interacting case, the necessary properties to guarantee relativistic invariance and locality in case the continuation to Minkowski points should exist. The present treatment remedies that defect. We do not give the details of the proof, because it is a very direct imitation of Nelson [2]; but we indicate at the end of Sec. II what is involved.

[1] We have heard by word of mouth that J. Fröhlich and K. Osterwalder have considered Euclidean, spin one-half fields corresponding to only one type of relativistic spinor index. We do not know if our construction is related to theirs, particularly since we construct *both* types of relativistic spinors from the same Euclidean theory.

The formal parametrizatiion of interacting ED theories, based on $\overline{\text{OS}}$ positivity, is described in Sec. III.[2] It has much the same structure as in (I), with Hermitean, commuting action integrands and a Hermitean FKN formula, while also intuitively admitting the construction of relativistic fields.

In Sec. IV, we list a modified set of axioms for ED fields, including Nelson's Axiom (A′) [2], which we believe to be sufficient for the construction of a Wightman theory. We intend to present the verification of the reconstruction elsewhere. It is rather unrewarding, because presumably no new ideas are needed beyond those of Nelson, and Osterwalder and Schrader, but necessary, because of certain changes of detail. The point is to verify that Axiom (A′) ensures the existence of the continuation from Euclidean to Minkowski points, after which, according to this paper, we are finished. The outright existence of the continuation is presumably the weaker assumption, but there could be a technical advantage for models in making the connection through (A′), or some other axiom.

The notation in this paper is taken from (I), and we refer to (I) for definitions of the free ED fields, conventions for ED matrices, etc. We refer to Eq. (X) in (I) as Eq. (I.X).

II. Osterwalder-Schrader Positivity (Free Fields)

The OS positivity condition is based on the following, general property of Wightman fields, which we write in simplified form by considering only monomials. Let $t_n > \ldots > t_1 > 0$. Let $\psi^\sharp(0,f)$, $f \in \mathcal{S}(\mathbb{R}^3) \otimes \mathbb{C}^4$, be the relativistic, four-component spinor field or its Hermitean adjoint at time zero, and let Ω be the relativistic vacuum. Then

$$e^{-t_1 H}\, \psi^\sharp(0,f_1)\, e^{-(t_2-t_1)H}\, \psi^\sharp(0,f_2)\, \ldots\, e^{-(t_n-t_{n-1})H}\, \psi^\sharp(0,f_n)\, \Omega \tag{1}$$

is a normalizable vector, and

$$\psi^\sharp(it,f) = e^{-tH}\, \psi^\sharp(0,f)\, e^{tH} \tag{2}$$

is an operator defined on such vectors for $t_1 > t > 0$.

Our construction of the two-point function for the free ED field in (I) was designed to arrange, for example,

$$\begin{aligned} & t_n > \ldots > t_1 > 0 > -t_1 > \ldots > -t_n\ : \\ & \langle \Omega_{\mathrm{E}},\ \gamma_5 \psi_{\mathrm{E}}(t_n,\bar f_n) \ldots \gamma_0 \psi^*_{\mathrm{E}}(t_1,\bar f_1)\, \gamma_5 \psi_{\mathrm{E}}(-t_1,f_1) \ldots \gamma_0 \psi^*_{\mathrm{E}}(-t_n,f_n)\ \Omega_{\mathrm{E}} \rangle \\ & \quad = \langle \Omega,\ \psi(-it_n,\bar f_n)\ \ldots\ \psi^*(-it_1,\bar f_1)\, \psi(it_1,f_1) \ldots \psi^*(it_n,f_n)\ \Omega \rangle \\ & \quad > 0\,. \end{aligned} \tag{3}$$

In this expression, all fields are free fields; we have smeared only in three-space; and the formal correspondence between Euclidean and Wightman fields is that of Eq. (I.20).

[2]This theory is actually a return to an unpublished precursor of (I), which we thought unsuccessful because of a problem with Wick ordering. The trick that resolves the problem is based on the local anti-commutation relations.

When we wrote (I), we believed that the positivity of the left-hand side of this expression could be extended to overlapping times $t_1, \dots, t_n$ (after smearing) only at the expense of Wick ordering the positive time, and separately, the negative time fields, so that in the Wick expansion no contractions at equal arguments would occur. The positivity does not extend to overlapping times as it stands.

We have since learned the following. As long as the times are unequal, the $2n$-point function for the ED fields does not change if we put the negative-time fields in transposed order and supply the signature factor of the permutation, because of the local anticommutation relations. The semi-transposed expression, as we show in Appendix A, then remains nonnegative at equal arguments, because of the following identity for the free, two-point function, valid with four-dimensional smearing (e.g., in $L_2(\mathbb{R}^4) \otimes \mathbb{C}^4$) and no support restrictions:

Lemma A.

$$\begin{aligned} &\overline{\langle \Omega_{\mathrm{E}},\ \gamma_5 \psi_{\mathrm{E}}(\bar{f}_1)\, \gamma_0 \psi^*_{\mathrm{E}}(f_2)\ \Omega_{\mathrm{E}} \rangle} \\ &\quad = - \langle \Omega_{\mathrm{E}},\ \gamma_0 \psi^*_{\mathrm{E}}(f_{1\theta})\, \gamma_5 \psi_{\mathrm{E}}(\bar{f}_{2\theta)})\ \Omega_{\mathrm{E}} \rangle \,. \end{aligned} \tag{4}$$

Proof. The notation f_θ indicates reflection of the time argument, Eq. (I.35). After recalling the definition of the two-point function, Eq. (I.17), and that γ_0, γ_5, and γ_{E} are Hermitean, with γ_0 and γ_5 real, according to our convention, and our convention for Fourier transforms, Eq. (I.28), we see that the left-hand side is

$$\begin{aligned} &\int \mathrm{d}\mu_{\mathrm{E}}\ g_1(p)^{\mathrm{Tr}}\, \gamma_5 \left(\lambda I + \gamma_{\mathrm{E}} \cdot p + \mu \gamma_5\right)^{\mathrm{Tr}} \gamma_0 \overline{g_2(p)} \\ &\quad = - \int \mathrm{d}\mu_{\mathrm{E}}\ g_1(p)^{\mathrm{Tr}}\, \gamma_0 \left(\lambda I - \gamma_{\mathrm{E}} \cdot \theta p - \mu \gamma_5\right)^{\mathrm{Tr}} \gamma_5 \overline{g_2(p)} \\ &\quad = - \int \mathrm{d}\mu_{\mathrm{E}}\ g_1(-\theta p)^{\mathrm{Tr}}\, \gamma_0 \left(\lambda I + \gamma_{\mathrm{E}} \cdot p - \mu \gamma_5\right)^{\mathrm{Tr}} \gamma_5 \overline{g_2(-\theta p)} \,. \end{aligned} \tag{5}$$

In the last line, we used the invariance of $\mathrm{d}\mu_{\mathrm{E}}$, λ, and μ under reflections. The last line coincides with the definition of the right-hand side in Lemma A.

We note in passing that Eq. (4) can be used to define an *antiunitary*, Euclidean, "time reflection" operator on free ED fields, which leaves the vacuum invariant. Its action as an automorphism of the restricted Euclidean group is the same as that of the unitary time reflection, Eq. (I.35), except that $U_2 \times U_1$ on the right-hand side is replaced by $\bar{U}_2 \times \bar{U}_1$. Thus, it does not quite have the right to be called "time reflection".

In order to write the extended OS-positivity condition (including overlapping arguments) that results from Lemma A as a bilinear form, with the same combination of fields in both arguments of the form, we introduce some notation. First, in expressions like the left-hand side of Eq. (3), we absorb the factors γ_5 and γ_0 into the test functions and smear in four dimensions; and we re-express the time reflection in terms of the unitary Euclidean time reflection operator Θ defined in Eq. (I.35), which yields for the semi-transposed expression:

$$(-1)^\sigma \left\langle \psi_{\mathrm{E}}(f_1) \dots \psi^*_{\mathrm{E}}(f_n)\, \Omega_{\mathrm{E}},\ \Theta\, \psi_{\mathrm{E}}(iKf_n) \dots \psi^*_{\mathrm{E}}(iKf_1)\ \Omega_{\mathrm{E}} \right\rangle . \tag{6}$$

The matrix K is the ED raising and lowering symbol in Eq. (I.11).

Next, we define an operation on polynomials $\mathcal{P}[\psi_{\mathrm{E}}^{\sharp}(f)]$ in the smeared fields:

$$\mathcal{K}\left\{\mathcal{P}\left[\psi_{\mathrm{E}}^{\sharp}(f)\right]\right\} \equiv \mathcal{P}\left[\psi_{\mathrm{E}}^{\sharp *}(iKf)\right]^{\sigma\mathrm{Tr}}, \tag{7}$$

where the instructions are:

(i) replace each field by its adjoint (no conjugation of test function);

(ii) multiply each test function by iK;

(iii) take the transposed order in each field monomial;

(iv) supply the signature factor of transposition in each field monomial;

(v) do nothing to multiples of the identity.

The operation $\mathcal{K}$ can be regarded as an involution on the algebra of test functions. In Appendix A, we prove the following:

Theorem A ($\overline{\mathrm{OS}}$ positivity)**.** Let $\mathcal{P}_{+}$ be a polynomial in the free fields $\psi_{\mathrm{E}}^{\sharp}$ with test functions having strictly positive time support. Then

$$\left\langle \mathcal{P}_{+}\,\Omega_{\mathrm{E}},\ \Theta\mathcal{K}[\mathcal{P}_{+}]\,\Omega_{\mathrm{E}}\right\rangle \geq 0\,. \tag{8}$$

The reconstruction of the relativistic Hilbert space, and of the continuous, Hermitean, contraction semigroup of imaginary, positive time evolution follows standard lines [4, 5, 7]; one only has to verify a few changes of detail due to the fact that, although we work in a Euclidean Hilbert space, our reflection operation $\Theta\mathcal{K}$ is not an operator because of the factor $\mathcal{K}$.[3]

We state the result in a form that is valid for interacting fields which obey Axioms (i)-(v) in Sec. IV. Let $\mathcal{H}_{\mathrm{E}+}$ be the submanifold of the Euclidean Hilbert space generated from the vacuum by field polynomials smeared with strictly positive time support. When $\overline{\mathrm{OS}}$ positivity is valid, the physical pre-Hilbert space is

$$\mathcal{H}_{\mathrm{phys}} = \mathcal{H}_{\mathrm{E}+}/\,\mathrm{Ker}\,\Theta\mathcal{K}\,, \tag{9}$$

where $\mathrm{Ker}\,\Theta\mathcal{K}$ is the kernel of the sesquilinear form on the field algebra defined by $\Theta\mathcal{K}$. Let $[\mathcal{P}_{+}]\Omega \in \mathcal{H}_{\mathrm{phys}}$ be a notation for the equivalence class defined by $\mathcal{P}_{+}\Omega_{\mathrm{E}}$. Let $U(t)$, $t \geq 0$, be the Euclidean time evolution operator.

Theorem B. Let Axioms (i)-(v) in Sec. IV be valid. Then

$$P^{t}[\mathcal{P}_{+}]\Omega \equiv \left[U(t)]\mathcal{P}_{+}U(t)^{-1}\right]\Omega \tag{10}$$

defines a continuous, Hermitean, contraction semigroup on $\mathcal{H}_{\mathrm{phys}}$, which leaves the physical vacuum Ω invariant.

[3]Hegerfeldt has already remarked, in a note at the end of [7], that it is not necessary to have the reflection operation in his discussion of OS positivity be an operator.

The proof is reviewed in Appendix B.

The relativistic fields are constructed by analytic continuation from the Schwinger functions, which goes through by inspection for the free field. For interacting fields, one must of course ensure that the continuation to Minkowski points exists, and check that it has the right covariance and locality properties. As a preliminary result in that direction, we note that operators $\widehat{\psi^\sharp}(f_+)$, where f_+ has strictly positive, compact time support, later to be identified with the continued, relativistic field operators by the formula

$$\widehat{\psi^\sharp}(f_+) = \sum_a \int \mathrm{d}^4 y\, \psi^\sharp_a(-iy_0, \boldsymbol{y})\, f_{+a}(y)\,, \tag{11}$$

exist on the appropriate domain in $\mathcal{H}_{\text{phys}}$.

To describe that result, we introduce a notation for those Euclidean fields which are to correspond to relativistic fields:

$$\widehat{\psi_{\text{E}}}(f) \equiv \psi_{\text{E}}(\gamma_5 f)\,; \qquad \widehat{\psi^*_{\text{E}}}(f) \equiv \psi^*_{\text{E}}(\gamma_0 f)\,. \tag{12}$$

Theorem C. Let $\psi^\sharp_{\text{E}}$ obey Axioms (i)-(v) in Sec. IV. Let $\mathcal{P}_{>t}$ be any polynomial in $\psi^\sharp_{\text{E}}$ with test functions having time support strictly greater than $t > 0$. Let the time support of $f_{+<t} \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^4$ be strictly between 0 and t. Then the expressions

$$P^{-s}\, \left[\mathcal{P}_{>t}\right]\, \Omega \equiv \left[U(-s)\, \mathcal{P}_{>t}\, U(s)\right]\, \Omega\,; \qquad 0 \le s \le t\,; \tag{13a}$$

$$\widehat{\psi^\sharp}(f_{+<t})\, \left[\mathcal{P}_{>t}\right] \Omega \equiv \left[\widehat{\psi^\sharp_{\text{E}}}(f_{+<t})\, \mathcal{P}_{>t}\right] \Omega\,; \tag{13b}$$

densely define P^{-s} and $\widehat{\psi^\sharp}(f_{+<t})$ as linear operators on $\mathcal{H}_{\text{phys}}$.

The proof is given in Appendix C.

The immediate consequence of Theorem C is that expressions like

$$\begin{aligned} &\langle \Omega_{\text{E}},\ \widehat{\psi^\sharp_{\text{E}}}(y_1) \ldots \widehat{\psi^\sharp_{\text{E}}}(y_n)\ \Omega_{\text{E}} \rangle \\ &\quad = \langle \Omega,\ \psi^\sharp(-iy_{10}, \boldsymbol{y}_1) \ldots \psi^\sharp(-iy_{n0}, \boldsymbol{y}_n)\ \Omega \rangle\,, \\ &y_{10} > \ldots > y_{n0}\,, \end{aligned} \tag{14}$$

are well-defined for interacting $\psi^\sharp_{\text{E}}$ in the sense of distributions (including the time variables), where the right-hand side is defined with respect to the physical space if $\psi^\sharp(-iy_0, \boldsymbol{y})$ is replaced by $\widehat{\psi^\sharp}(y)$.[4]

Of course, a necessary condition for the relativistic construction is that such expressions actually be analytic functions in the strictly ordered y_0 variables, and not just distributions. That is the aim of Nelson's technical axiom (A′) [2], which we have included as Axiom (vi) in Sec. IV. In anticipation of that, we can state the following:

[4]In the physical space, one may consider operators to the right of the dividing line beween positive and negative times to act to the right, and operators to the left of it to act to the left. The dividing line may be put anywhere by a time translation, from the invariance of the vacuum. We emphasize that Theorem C holds even if there is no continuation to Minkowski points.

Theorem D (Wightman reconstruction)**.** Let the ED fields obey Axioms (i)-(v) in Sec. IV, and assume that the sharp time, three-space smeared (in $\mathcal{S}(\mathbb{R}^3)\otimes\mathbb{C}^4$) Schwinger functions

$$S(s_1,\ldots,s_n) = \langle \Omega_{\mathrm{E}},\ \widehat{\psi^{\sharp}_{\mathrm{E}}}(s_1,f_1)\ \ldots\ \widehat{\psi^{\sharp}_{\mathrm{E}}}(s_n,f_n)\ \Omega_{\mathrm{E}}\rangle$$

are analytic in the region $\mathfrak{Re}\, s_1 > \ \ldots\ > \mathfrak{Re}\, s_n$, and exist in the limit on the boundary $\mathfrak{Re}\, s_i = 0$, $\forall i$, as tempered distributions in $\mathfrak{Im}\, s_i$. Then $S(s_1,\ldots,s_n)$ are the Schwinger functions of a Wightman field theory.

The proof is an imitation of Nelson's argument [2], based on the analytic continuation of Euclidean invariance under the action of the infinitesimal generators of the restricted Euclidean group, which yields relativistic invariance, and on the antisymmetry of the Schwinger function, due to the local, Euclidean anticommutation relations, which leads to local anticommutation relations for the Wightman fields, via antisymmetry at Schwinger points, which leads to antisymmetry at Jost points [8, 9]. We do not repeat the argument.

III. Heuristic Parametrization of Interaction

Our procedure for formally parametrizing interacting ED field theories is generally the same as in (I), the essential difference being a modification of the class of formal Euclidean action integrands. With the same notation for Euclidean action as in (I), we now demand the following properties:

(i) $V(y)$ is a local polynomial in free, Euclidean, fermion and boson fields, of only even order in $\psi^{\sharp}_{\mathrm{E}}$.

(ii) $V(y)$ is Hermitean.

(iii) $V(y)$ locally commutes with itself.

(iv) $V(y)$ transforms like a scalar field under the full Euclidean group, with positive signature under reflection.

(v) $\mathcal{K}[V(y)] = V(y)$.

The Yukawa interaction class formally obeys all requirements:

$$V(y) = {:}\psi^{*}_{\mathrm{E}}(y)\gamma_5\psi_{\mathrm{E}}(y){:}\, \mathcal{P}[\phi_{\mathrm{E}}(y)]\,, \tag{15}$$

where ϕ_{E} is a scalar, Euclidean boson field.[5] We interpret the action of $\mathcal{K}$ in this case by formally equating the Wick square to the square minus the vacuum expectation

[5] Because of the γ_5 in the transformation from Euclidean to relativistic fields, this corresponds to scalar coupling in the relativistic interaction. Pseudoscalar coupling to an odd polynomial in a pseudoscalar field is also allowed.

value. Of course, none of the terms in such a local expressiion is well-defined for four-dimensional Euclidean fields, but we note that

$$\begin{aligned} \mathcal{K}[\psi_{\mathrm{E}}^*(y)\Gamma\psi_{\mathrm{E}}(y)] &= \psi_{\mathrm{E}}^*(y)\Gamma\psi_{\mathrm{E}}(y) \\ \mathcal{K}[{:}\psi_{\mathrm{E}}^*(y)\Gamma\psi_{\mathrm{E}}(y){:}] &= {:}\psi_{\mathrm{E}}^*(y)\Gamma\psi_{\mathrm{E}}(y){:} \end{aligned} \tag{16}$$

is true whenever

$$K\,\Gamma^{\mathrm{Tr}}K^{-1} = \Gamma \tag{17}$$

and whenever ψ_{E} and ψ_{E}^* are both regularized by convolution with the same approximation to the delta function. $\mathcal{K}$ does nothing to bosons.

The positive and negative time action exponentials, defined as in (I), obey

$$\mathcal{E} = \mathcal{E}_+\mathcal{E}_- = \mathcal{E}_-\mathcal{E}_+\,, \tag{18a}$$

$$\mathcal{E}_\pm = \exp(-V_\pm) = \mathcal{E}_\pm^* > 0\,, \tag{18b}$$

$$= \Theta\,\mathcal{E}_\mp\,\Theta^{-1}, \tag{18c}$$

$$= \mathcal{K}[\mathcal{E}_\pm]\,. \tag{18d}$$

The action of $\mathcal{K}$ is interpreted here in the sense of formal power series expansion in $V_\pm$.

The $\mathcal{K}$ invariance of $\mathcal{E}_\pm$ follows from the following combinatoric lemma:

Lemma B. Let at least $N-1$ of the field polynomials $\mathcal{P}_1, \ldots, \mathcal{P}_N$ be of only even order in $\psi_{\mathrm{E}}^\sharp$. Then

$$\mathcal{K}[\mathcal{P}_1 \ldots \mathcal{P}_N] = \mathcal{K}[\mathcal{P}_N]\ \ldots\ \mathcal{K}[\mathcal{P}_1]\,. \tag{19}$$

Proof. Let $n_1, \ldots, n_N$ be the orders of the monomials in any term of $\mathcal{P}_1\ \ldots\ \mathcal{P}_N$. Such terms enter differently in the left-hand and right-hand sides of Eq. (19) at most in the product of signature factors

$$(-1)^{\left[\frac{\sum n_i}{2}\right]}\cdot(-1)^{\left[\frac{n_1}{2}\right]}\ \cdots\ (-1)^{\left[\frac{n_N}{2}\right]}\,,$$

where $[X]$ is the integer part of X, and where the first factor is the signature of the total transposition, the remaining factors those of the subtranspositions. If all but one of the n_i's are even, the above quantity is $+1$.

The invariance of $\mathcal{E}_\pm$ now follows from the invariance of $V_\pm$ and the restriction that $V(y)$ be even in $\psi_{\mathrm{E}}^\sharp$.

The interacting, Euclidean Hilbert space $\mathcal{H}_{\mathrm{V}}$ is defined as in (I), by using the $\mathcal{E}$ metric, and interacting fields $\psi_{\mathrm{V}}^\sharp$ are defined as in Eq. (I.50). The adjoint on $\psi_{\mathrm{V}}^{\mathrm{A}}$ is that appropriate to the $\mathcal{E}$ metric. The positive time interacting fields are again formally functions of the positive time free fields:

$$\psi_{\mathrm{V}}^\sharp(f_+) = \mathcal{E}_+^{-\frac{1}{2}}\,\psi_{\mathrm{E}}^\sharp(f_+)\,\mathcal{E}_+^{\frac{1}{2}}\,, \tag{20}$$

where wave function renormalization is omitted for simplicity. As in (I), the symmetry operations of the full Euclidean group remain unitary, and the interacting vacuum $\Omega_{\rm V} = \Omega_{\rm E}$ is invariant.

We define a $\mathcal{K}_{\rm V}$ operation on polynomials in $\psi^{\sharp}_{\rm V}$ by substituting the label V for the label E in Eq. (7). We have to distinguish it from the $\mathcal{K}$ operation on free fields because we are going to use both.

Theorem E ($\overline{\rm OS}$ positivity)**.** Let $V(y)$ obey the conditions (i)-(v). Let $\mathcal{P}_{\rm V+} = \mathcal{P}[\psi^{\sharp}_{\rm V}(f_+)]$ be a polynomial in $\psi^{\sharp}_{\rm V}$ with test functions having strictly positive time support. Then

$$\left\langle \mathcal{P}_{\rm V+}\,\Omega_{\rm V},\ \Theta\mathcal{K}_{\rm V}(\mathcal{P}_{\rm V+})\,\Omega_{\rm V}\right\rangle_{\rm V} \geq 0\,. \tag{21}$$

Proof. Except for a positive normalization factor, the left-hand side, according to the definition Eq. (I.48), is

$$\begin{aligned}
&\left\langle \mathcal{P}_{\rm V+}\,\Omega_{\rm E},\ \mathcal{E}\Theta\mathcal{K}_{\rm V}(\mathcal{P}_{\rm V+})\,\Omega_{\rm E}\right\rangle \\
&\quad= \left\langle \mathcal{P}[\psi^{\sharp}_{\rm V}(f_+)]\,\Omega_{\rm E},\ \mathcal{E}_+\Theta\mathcal{E}_+\,\mathcal{P}[\psi^{\sharp{\rm A}}_{\rm V}(iKf_+)]^{\sigma{\rm Tr}}\ \Omega_{\rm E}\right\rangle \\
&\quad= \left\langle \mathcal{E}_+^{\frac{1}{2}}\mathcal{P}[\psi^{\sharp}_{\rm E}(f_+)]\mathcal{E}_+^{\frac{1}{2}}\,\Omega_{\rm E},\ \Theta\mathcal{E}_+^{\frac{1}{2}}\mathcal{P}[\psi^{\sharp *}_{\rm E}(iKf_+)]^{\sigma{\rm Tr}}\mathcal{E}_+^{\frac{1}{2}}\ \Omega_{\rm E}\right\rangle \\
&\quad= \left\langle \mathcal{E}_+^{\frac{1}{2}}\mathcal{P}[\psi^{\sharp}_{\rm E}(f_+)]\mathcal{E}_+^{\frac{1}{2}}\,\Omega_{\rm E},\ \Theta\mathcal{K}\left\{\mathcal{E}_+^{\frac{1}{2}}\mathcal{P}[\psi^{\sharp}_{\rm E}(f_+)]\mathcal{E}_+^{\frac{1}{2}}\right\}\ \Omega_{\rm E}\right\rangle \\
&\quad\geq 0\,.
\end{aligned} \tag{22}$$

In the next to last line, we used the $\mathcal{K}$ invariance of $\mathcal{E}_+^{\frac{1}{2}}$ and Lemma B, and the last inequality is a formal consequence of $\overline{\rm OS}$ positivity for free fields. It is at this point that we really need the extension of OS positivity to overlapping arguments, because the times in $\mathcal{E}_+^{\frac{1}{2}}$ would overlap those in the field polynomial, even if the times in the polynomial would be kept distinct.

The Hermitean FKN forumula is precisely the same as in Eq. (I.58).

From here, the formal construction of a relativistic theory proceeds along the lines indicated in the discussion of Theorems B and C, up to the point where one needs the existence of the analytic continuation of the Schwinger functions at unequal arguments to Minkowski points.

IV. Axioms for Euclidean Dirac Fields

We now use the notation $\psi_{\rm E}(f)$ and $\psi^*_{\rm E}(f)$ for possibly interacting, Euclidean Dirac fields.

Axioms (i)-(iv) are identical with Axioms (i)-(iv) in Sec. VII of (I). They assert the Euclidean invariance, temperedness, and irreducibility of the fields, the cyclicity, uniqueness, and invariance of the vacuum, and local anticommutation relations.

(v) $\overline{\text{OS}}$ positivity. Define the involution $\mathcal{K}$ on polynomials in the smeared fields $\psi_{\mathrm{E}}^{\sharp}$ by analogy with Eq. (7). Let $\mathcal{P}_+$ be any such polynomial where the test functions have strictly positive time support. Let Ω_{E} be the unique, Euclidean vacuum. Then

$$\left\langle \mathcal{P}_+ \,\Omega_{\mathrm{E}},\ \Theta\mathcal{K}(\mathcal{P}_+)\,\Omega_{\mathrm{E}}\right\rangle \geq 0\,.$$

(vi) Existence of analytic continuation. The fields $\psi_{\mathrm{E}}^{\sharp}$ obey the natural analog of Nelson's Axiom (A′) [2].

In the absence of surprises, Axiom (vi) is a sufficient technical condition for the existence of the analytic continuation to Minkowski points. In that case, as we have already indicated, the relativistic construction goes through, by imitation of arguments of Nelson and/or Osterwalder and Schrader.

V. Concluding Remarks

(i) We would like to see a definition of the FKN formula, Eq. (I.58), from the formal relativistic interaction. The remark in (I) that we expected that to be straightforward was misdirected, to say the least, because it is still not clear whether the parametrization of action in (I) admits interacting Wightman fields at the heuristic level.

(ii) We are not certain that the structure of interaction in (I) is uninteresting for relativistic theories.

(iii) As in (I), cutoffs may be introduced without destroying $\overline{\text{OS}}$ positivity, at the expense of restricted Euclidean invariance. It remains to be checked whether the analog of Osterwalder and Schrader's Feynman-Kac formula with cutoffs [5, Eq. (5.15)], is true. If so, it remains to be seen whether there is a technical advantage in dealing with Hermitean rather than non-Hermitean actions.

(iv) Although we have fewer derivatives in the action than in (I), the Euclidean renormalization problem may still be more divergent than in the corresponding relativistic theory, because of the extra power of four-momentum we put into the Euclidean two-point function to make it positive. Indeed, one could imagine that the Euclidean theory corresponding to a given, renormalizable, relativistic theory might be unrenormalizable; i.e., the Hilbert space Euclidean theory could exist in a cutoff version, but in the limit only the unequal argument Schwinger functions might exist (no Symanzik positivity in the limit).

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Appendix A. Proof of Theorem A

Following Osterwalder and Schrader [5], we introduce a map from test functions over $\mathbb{R}^4$ with positive time support into test functions over $\mathbb{R}^3$. Let $f_+ \in \mathrm{L}_2(\mathbb{R}^4)$ have strictly positive time support. Define

$$\widehat{f}(\boldsymbol{x}) = (2\pi)^{-1} \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, g_+(-i\omega, \boldsymbol{k})\, \mathrm{d}^3 k\,, \tag{A.1}$$

where $g_+(-i\omega, \boldsymbol{k})$ is the analytic continuation of the Fourier transform of f_+. By Osterwalder and Schrader's criterion, $f_+ \in \mathrm{L}_2(\mathbb{R}^4)$ implies $\widehat{f} \in \mathrm{L}_2(\mathbb{R}^3)$.

The notation $\widehat{\psi_{\mathrm{E}}^{\sharp}}$ is defined in Eq. (12).

It is a straightforward calculation from the definition of the two-point functions, Eqs. (I.16, 17), that

$$\langle \Omega_{\mathrm{E}},\ \widehat{\psi_{\mathrm{E}}^{\sharp}}(\bar{f}_+)\, \widehat{\psi_{\mathrm{E}}^{\sharp *}}(f'_{+\theta})\, \Omega_{\mathrm{E}}\rangle = \langle \Omega,\ \psi^{\sharp}(0, \overline{\widehat{f}})\, \psi^{\sharp *}(0, \widehat{f'})\, \Omega\rangle\,, \tag{A.2}$$

where relativistic free fields at time zero appear on the right-hand side.

Let $:\!\mathcal{M}[\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_+)]\!:$ be any Wick monomial in $\widehat{\psi_{\mathrm{E}}^{\sharp}}$ with test functions having strictly positive time supports. If we replace f_+ by $f_{+\theta}$ or $\bar{f}_+$ in the above notation, we indicate the corresponding Wick monomials where reflection of the time argument or complex conjugation is performed on all test functions. Let $:\!\mathcal{M}[\psi^{\sharp}(0, \widehat{f})]\!:$ be the corresponding Wick monomial in relativistic free fields at time zero. Then it follows from the Wick expansion and Eq. (A.2) that

$$\begin{aligned} &\sum_{i,j} \bar{c}_i c_j\, \langle \Omega_{\mathrm{E}},\ :\!\mathcal{M}_i[\widehat{\psi_{\mathrm{E}}^{\sharp *}}(\bar{f}_+)]^{\mathrm{Tr}}\!:\ :\!\mathcal{M}_j[\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_{+\theta})]\!:\ \Omega_{\mathrm{E}}\rangle \\ &\quad= \sum_{i,j} \bar{c}_i c_j\, \langle \Omega,\ :\!\mathcal{M}_i[\psi^{\sharp}(0, \widehat{f})]\!:^*\ :\!\mathcal{M}_j[\psi^{\sharp}(0, \widehat{f})]\!:\ \Omega\rangle \\ &\quad\geq 0\,, \end{aligned} \tag{A.3}$$

where the c_i are arbitrary complex numbers, and there is a transposition of order in the left factor in the first line.

Now consider the following polynomials, at first with no positive time restriction:

$$\mathcal{P}[\widehat{\psi_{\mathrm{E}}^{\sharp *}}(\bar{f})]^{\mathrm{Tr}} \quad \text{and} \quad \mathcal{P}[\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_\theta)]^{\sigma\mathrm{Tr}}.$$

To keep the complex conjugation straight, think of $\mathcal{P}$ as a sum of field monomials, with all numerical coefficients absorbed into test functions. Put the monomials in these expressions into one-to-one correspondence via the correspondence

$$\widehat{\psi_{\mathrm{E}}^{\sharp *}}(\bar{f}) \ \longleftrightarrow\ \widehat{\psi_{\mathrm{E}}^{\sharp}}(f_\theta)\,. \tag{A.4}$$

Corresponding fields appear in the same order in corresponding monomials.

Focus on a corresponding pair of monomials, and make their Wick expansions. Put the Wick monommials in the two expansions into correspondence, and look at their coefficients. A typical pair coming from monomials of order N would be:

$$c_{2i} :\widehat{\psi_{\mathrm{E}}^{\sharp *}}(\bar{f}_j)\ldots\widehat{\psi_{\mathrm{E}}^{\sharp *}}(\bar{f}_1): \;\longleftrightarrow\; (-1)^{\left[\frac{N}{2}\right]} c'_{2i} :\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_{j\theta})\ldots\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_{1\theta}):\,, \tag{A.5}$$

where $2i + j = N$, $[x]$ is the integer part of x, and the coefficients c_{2i} and c'_{2i} are products of i corresponding two-point functions.

Because of the identity in Eq. (4),

$$c'_{2i} = (-1)^i\,\overline{c_{2i}}\,. \tag{A.6}$$

Thus, the right-hand member is equal to

$$(-1)^{\left[\frac{N}{2}\right]}(-1)^{\left[\frac{j}{2}\right]}(-1)^i\,\overline{c_{2i}} :\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_{1\theta})\ldots\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_{j\theta}):\,, \tag{A.7}$$

where we transposed the order of the Wick monomial. But

$$(-1)^{\left[\frac{N}{2}\right]}(-1)^{\left[\frac{j}{2}\right]}(-1)^i = 1\,. \tag{A.8}$$

Thus, putting in the positive time restriction, it follows that

$$\langle \Omega_{\mathrm{E}},\ \mathcal{P}[\widehat{\psi_{\mathrm{E}}^{\sharp *}}(\bar{f}_+)]^{\mathrm{Tr}}\,\mathcal{P}[\widehat{\psi_{\mathrm{E}}^{\sharp}}(f_{+\theta})]^{\sigma\mathrm{Tr}}\ \Omega_{\mathrm{E}}\rangle \geq 0\,. \tag{A.9}$$

because we have reduced the left-hand side to the same form as the left-hand side of Eq. (A.3).

Theorem A now follows by absorbing the real, symmetric matrices γ_0 and γ_5, and applying the definitions of Θ and $\mathcal{K}$.

Appendix B. Proof of Theorem B

The argument is a trivial variation of that due to Osterwalder and Schrader [4], and refined by Hegerfeldt [7].

Let $U(t)$ be the unitary, Euclidean time evolution operator. Then

$$\Theta U(t)\Theta^{-1} = U(-t), \tag{A.10}$$

and for $t \geq 0$, $U(t)$ preserves $\mathcal{H}_{\mathrm{E}+}$. From the definition of $\mathcal{K}$ and the action of $U(t)$,

$$U(t)\,\mathcal{K}[\mathcal{P}]\,U(t)^{-1} = \mathcal{K}[U(t)\mathcal{P}U(t)^{-1}]\,. \tag{A.11}$$

Let $\mathcal{P}_+$ and $\mathcal{P}'_+$ be positive time polynomials. Then

$$\begin{aligned}
&\left\langle U(t)\mathcal{P}'_+\,\Omega_{\mathrm{E}},\ \Theta\mathcal{K}(\mathcal{P}_+)\,\Omega_{\mathrm{E}}\right\rangle \\
&\quad= \left\langle \mathcal{P}'_+\,\Omega_{\mathrm{E}},\ \Theta U(t)\mathcal{K}(\mathcal{P}_+)\,\Omega_{\mathrm{E}}\right\rangle \\
&\quad= \left\langle \mathcal{P}'_+\,\Omega_{\mathrm{E}},\ \Theta\mathcal{K}[U(t)\mathcal{P}_+U(t)^{-1}]\,\Omega_{\mathrm{E}}\right\rangle .
\end{aligned} \tag{A.12}$$

From this formula, we can read off the following:

(i) For $t \geq 0$, $U(t)$ preserves the kernel of $\Theta\mathcal{K}$.

For, $[\mathcal{P}_+]\Omega = 0$ means precisely

$$\left\langle \mathcal{P}'_+ \, \Omega_{\mathrm{E}}, \; \Theta\mathcal{K}(\mathcal{P}_+) \, \Omega_{\mathrm{E}} \right\rangle = 0$$

for all $\mathcal{P}'_+$. Reading backwards, and using the invariance of Ω_{E} in the first line of Eq. (A.12), we see that $[\mathcal{P}_+]\Omega = 0 \;\Rightarrow\; [U(t)\mathcal{P}_+ U(t)^{-1}]\Omega = 0$. Thus, Eq. (10) defines an operator P^t for each $t \geq 0$, which leaves the vacuum Ω invariant.

(ii) P^t is Hermitean.

(iii) Since Eq. (10) is well-defined,

$$P^t P^s = P^{t+s}, \qquad t, s \geq 0 \,. \tag{A.13}$$

Next, to see that P^t is a contraction, we have the estimate of Osterwalder and Schrader, in a form we take from Hegerfeldt [7]:

$$\begin{aligned} &\left\langle P^t[\mathcal{P}_+]\,\Omega, \; P^t[\mathcal{P}_+]\,\Omega \right\rangle \\ &\quad \leq \|[\mathcal{P}_+]\Omega\|^{\sum_0^N 2^{-n}} \cdot \|P^{2^{N+1}t}[\mathcal{P}_+]\Omega\|^{2^{-N}} \\ &\quad \leq \|[\mathcal{P}_+]\Omega\|^{\sum_0^N 2^{-n}} \cdot \left(\|\mathcal{P}_+\Omega_{\mathrm{E}}\| \cdot \|\mathcal{K}(\mathcal{P}_+)\Omega_{\mathrm{E}}\|\right)^{2^{-N}} , \end{aligned} \tag{A.14}$$

which gives the result as the integer $N \to \infty$. The last line contains norms, both in $\mathcal{H}_{\mathrm{phys}}$ and $\mathcal{H}_{\mathrm{E}}$. The essential ingredients in this estimate are the Hermiticity of P^t, the boundedness of Θ and $U(t)$ on $\mathcal{H}_{\mathrm{E}}$, the iterability of the smeared fields on Ω_{E}, the invariance of Ω_{E}, and the fact that $\mathcal{K}$ preserves the test function space.

Finally, we can read from Eq. (A.12) that P^t is weakly continuous on the pre-Hilbert space $\mathcal{H}_{\mathrm{phys}}$. It follows that P^t is strongly continuous on $\mathcal{H}_{\mathrm{phys}}$, because of the contraction semigroup property, and hence strongly continuous on the closure of $\mathcal{H}_{\mathrm{phys}}$ (contraction semigroup property, again).

Appendix C. Proof of Theorem C

To prove that Eq. (13a) defines an operator, we show that $[\mathcal{P}_{>t}]\Omega = 0 \;\Rightarrow\; [U(-s)\,\mathcal{P}_{>t}\,U(s)]\Omega = 0$.

Choose an $\varepsilon > 0$ such that $\mathcal{P}_{>t}$ has time supports $\geq t+\varepsilon$. Note that $U(-s')\,\mathcal{P}_{>t}\,U(s')$ has strictly positive time support if $0 \leq s' < t + \varepsilon$. Then we can compute the physical norm

$$\begin{aligned} &\left\|[U(-s'/2)\,\mathcal{P}_{>t}\,U(s'/2)]\,\Omega\right\|^2 \\ &\quad = \left\langle U(-s'/2)\,\mathcal{P}_{>t}\;\Omega_{\mathrm{E}}, \; \Theta\mathcal{K}[U(-s'/2)\,\mathcal{P}_{>t}\,U(s'/2)]\;\Omega_{\mathrm{E}} \right\rangle \\ &\quad = \left\langle U(-s')\,\mathcal{P}_{>t}\;\Omega_{\mathrm{E}}, \; \Theta\mathcal{K}(\mathcal{P}_{>t})\;\Omega_{\mathrm{E}} \right\rangle \\ &\quad = \left\langle [U(-s')\,\mathcal{P}_{>t}\,U(s')]\;\Omega, \; [\mathcal{P}_{>t}]\;\Omega \right\rangle . \end{aligned} \tag{A.15}$$

This vanishes if $[\mathcal{P}_{>t}]\Omega = 0$. By induction, the same hypothesis gives

$$[U(-s'\textstyle\sum_1^N 2^{-n})\,\mathcal{P}_{>t}\,U(s'\sum_1^N 2^{-n})]\,\Omega = 0\,. \tag{A.16}$$

We can cover every point in the closed interval $[0, t]$ by numbers of the form $s' \sum_1^N 2^{-n}$, so the result follows.

We apply the same criterion to show that Eq. (13b) defines an operator. We may drop the hat because the γ_0 and γ_5 play no role. We compute

$$\begin{aligned}
&\left\| [\psi^\sharp_{\mathrm{E}}(f_{+<t})\,\mathcal{P}_{>t}]\,\Omega \right\|^2 \\
&\quad = \pm \left\langle \psi^\sharp_{\mathrm{E}}(f_{+<t})\,\mathcal{P}_{>t}\,\Omega_{\mathrm{E}},\ \Theta\mathcal{K}(\mathcal{P}_{>t})\,\psi^{\sharp *}_{\mathrm{E}}(iKf_{+<t})\,\Omega_{\mathrm{E}} \right\rangle \\
&\quad = \pm \left\langle \psi^\sharp_{\mathrm{E}}(f_{+<t})\,\mathcal{P}_{>t}\,\Omega_{\mathrm{E}},\ \Theta\psi^{\sharp *}_{\mathrm{E}}(iKf_{+<t})\,\mathcal{K}(\mathcal{P}_{>t})\,\Omega_{\mathrm{E}} \right\rangle \\
&\quad = \pm \left\langle \psi^{\sharp *}_{\mathrm{E}}(i\gamma_{\mathrm{E}0}\bar{f}_{+<t,\theta})\,\psi^\sharp_{\mathrm{E}}(f_{+<t})\,\mathcal{P}_{>t}\,\Omega_{\mathrm{E}},\ \Theta\mathcal{K}(\mathcal{P}_{>t})\,\Omega_{\mathrm{E}} \right\rangle .
\end{aligned} \tag{A.17}$$

We have so far used the definition of $\mathcal{K}$, local anticommutativity, and the action of Θ, without keeping track of signs.

Now we insert the factor $U(-t)U(t)$ on both sides of $\psi^{\sharp *}_{\mathrm{E}}(i\gamma_{\mathrm{E}0}\bar{f}_{+<t,\theta})$ in the last line. The innermost conjugation changes the negative time support function $\bar{f}_{+<t,\theta})$ to positive time support. The rightmost factor $U(t)$ preserves the positive time support of what follows it. The leftmost factor $U(-t)$ we send through to the right-hand side of the scalar product, where it remains $U(-t)$. Then we apply the result for Eq. (13a) to conclude that if $[\mathcal{P}_{>t}]\Omega = 0$, then the last line is zero, thus proving that Eq. (13b) defines an operator.

All that remains is to note that the domain on which the operators in Eqs. (13a) and (13b) are defined is dense. That follows by a simple argument from the fact that all vectors $P^t[\mathcal{P}_+]\Omega$ are in the domain, and $[\mathcal{P}_+]\Omega$ is dense by definition.[6]

Appendix D. Reconstruction of Free, Relativistic, Two-Component Spinor Fields

We show how to compute the relativistic combinations

$$\psi^\sharp_\sigma(f) \equiv \psi^\sharp[(I + \sigma\gamma_5)f/2], \quad \sigma = \pm\,, \tag{A.18}$$

from the free, nonlocal, Euclidean fields $\phi^\sharp$, defined in Eqs. (I.32, 33).

There are several identities which relate the two-point functions of $\phi^\sharp$ to those of $\psi^\sharp$, and some of these even induce identities among the $\phi^\sharp$ two-point functions. For example, at time zero:

$$\langle \Omega_{\mathrm{E}},\ \phi(0, \gamma_0 f)\,\phi^*(0, f')\ \Omega_{\mathrm{E}} \rangle = \langle \Omega_{\mathrm{E}},\ \phi^*(0, \gamma_0 f')\,\phi(0, f)\ \Omega_{\mathrm{E}} \rangle\,. \tag{A.19}$$

[6] If ψ belongs to a dense set, and ϕ is orthogonal to all $P^t\psi$, then $0 = \langle \phi, P^t\psi \rangle = \langle P^t\phi, \psi \rangle \Rightarrow P^t\phi = 0 \Rightarrow \phi = 0$, because $P^t = \exp -tH$ is invertible.

From this identity, one can determine two subalgebras of the time zero $\phi^\sharp$ algebra for which the vacuum is a central state, by restricting to test functions which obey $f = \pm\gamma_0 f$, $f' = \pm\gamma_0 f'$. We are not interested in that particular possibility here, but we do in fact make an algebraic restriction on the class of test functions.

Define $\widehat{\phi}^\sharp_\sigma$ with four-dimensional smearing by

$$\begin{aligned}
&\widehat{\phi}_-(f) = m\,\phi[(I+\gamma_5)f/2]\,,\\
&\widehat{\phi}_+(f) = \phi[i\gamma_{\mathrm{E}}^{\mathrm{Tr}}\cdot\partial(I+\gamma_0)f/2]\,,\\
&\widehat{\phi}^*_-(f) = \phi^*[i\gamma_{\mathrm{E}}\cdot\partial(I-\gamma_0)f/2]\,,\\
&\widehat{\phi}^*_+(f) = m\,\phi^*[(I-\gamma_0)f/2]\,.
\end{aligned} \tag{A.20}$$

Thus, we are effectively restricting $\widehat{\phi}_\sigma$ to test functions which obey $\gamma_0 f = f$, and $\widehat{\phi}^*_\sigma$ to those which obey $\gamma_0 f = -f$. In the van der Waerden representation, that means the upper components of f are equal, respectively, to $\pm$ the lower components of f.

We introduce the notation

$$f_t = f(\boldsymbol{y})\otimes\delta(y_0 - t)$$

for Euclidean test functions at sharp time, with $f\in \mathrm{L}_2(\mathbb{R}^3)\otimes\mathbb{C}^4$. Then, we claim the following identities for the nonvanishing two-point functions:

$$\begin{aligned}
&t>0>-t'\;:\\
&\langle\Omega_{\mathrm{E}},\ \widehat{\phi}^\sharp_\sigma(f_t)\,T\widehat{\phi}^\sharp_{\sigma'}(f'_{t'})\,\Omega_{\mathrm{E}}\rangle\\
&\quad=\langle\Omega,\ \psi^\sharp_\sigma[-it,(I+s(\sharp)\gamma_0)f/2]\,\psi^{\sharp *}_{\sigma'}[it',(I-s(\sharp)\gamma_0)f'/2]\,\Omega\rangle\,,
\end{aligned} \tag{A.21}$$

where T is the unitary time reflection operator defined in Eq. (I.40), and $s(\sharp) = (-)$ if $\sharp = *$ and $(+)$ in the other case. The imaginary time notation is the same as in Eq. (2). In general, this identity is to be understood in the sense of distributions, although those two-point functions containing only $\widehat{\phi}_-$ and $\widehat{\phi}^*_+$ are also directly well-defined as functions, and at $t = t' = 0$, in particular. In the right-hand side of Eq. (A.21), the $(I+\sigma\gamma_5)/2$, for example, picks out the upper or lower components of $(I+s(\sharp)\gamma_0)f/2$.

These fields are nonlocal, just as ϕ is. From Eq. (I.34) it is easy to compute:

$$\begin{aligned}
&\left\{\widehat{\phi}_-(\bar f),\ \widehat{\phi}^*_+(f')\right\} = 0\,,\\
&\left\{\widehat{\phi}_-(\bar f),\ \widehat{\phi}^*_-(f')\right\} = \left\{\widehat{\phi}_+(\bar f),\ \widehat{\phi}^*_+(f')\right\}\\
&\qquad\qquad\qquad\quad = -\int \mathrm{d}\mu_{\mathrm{E}}\, g^*(p)\,\frac{I+\gamma_0}{2}\,\gamma_{\mathrm{E}0}p_0\,\frac{I-\gamma_0}{2}\,g'(p)\,,\\
&\left\{\widehat{\phi}_+(\bar f),\ \widehat{\phi}^*_-(f')\right\} = \frac{2}{m}\int \mathrm{d}\mu_{\mathrm{E}}\, g^*(p)\,\frac{I+\gamma_0}{2}\,\gamma_{\mathrm{E}0}p_0\,\boldsymbol{\gamma}_{\mathrm{E}}\cdot\boldsymbol{p}\,\frac{I-\gamma_0}{2}\,g'(p)\,,
\end{aligned} \tag{A.22}$$

with all other anticommutators vanishing. Except for the first one, the above anticommutators do not generally vanish when f and f' have disjoint supports.

We have not explored whether the positivity condition implied by Eq. (A.21) can be extended to overlapping arguments. Although we are by no means sure it would be fruitless to try, we do no see at the moment a natural way to build these fields into an interacting theory which would formally admit interacting, relativistic fields.

Appendix E. Sharp-Time Reconstruction of Free Dirac Fields

The nonlocal, Euclidean field ϕ, defined in Eqs. (I.32, 33), exists at sharp time. We look for a transformation of the test functions over three-space, $f \to Xf$, which lets us identify the free, Euclidean and Dirac fields at time zero on the time zero subspace of the Euclidean Fock space:

$$\begin{aligned} \phi^*(0, Xf) &= \psi^*(0, f), \\ \phi(0, \overline{Xf}) &= \psi(0, f), \\ \langle \Omega_\mathrm{E},\ \phi(0, \overline{Xf})\ \phi^*(0, Xf')\ \Omega_\mathrm{E} \rangle &= \langle \Omega,\ \psi(0, \bar{f})\ \psi^*(0, f')\ \Omega \rangle\,, \\ \langle \Omega_\mathrm{E},\ \phi^*(0, X\bar{f})\ \phi(0, \overline{Xf'})\ \Omega_\mathrm{E} \rangle &= \langle \Omega,\ \psi^*(0, \bar{f})\ \psi(0, f')\ \Omega \rangle\,. \end{aligned} \tag{A.23}$$

We find solutions which are multiplication by a matrix in p-space: i.e., under the three-dimensional Fourier transform:

$$\begin{aligned} f(\boldsymbol{x}) &\to g(\boldsymbol{k}), \\ Xf(\boldsymbol{x}) &\to X(\boldsymbol{k})g(\boldsymbol{k}), \end{aligned} \tag{A.24}$$

where $X(\boldsymbol{k})$ is a 4×4 matrix. By inspection of the two-point function, Eqs. (I.33), (I.16), such solutions must obey:

$$\begin{aligned} X(\boldsymbol{k})^* \, \frac{I + \gamma_5}{2m} \, X(\boldsymbol{k}) &= (\gamma \cdot k + m)\gamma_0\,, \\ X(\boldsymbol{k})^* \, \frac{I - \gamma_5}{2m} \, X(\boldsymbol{k}) &= \gamma_0(\gamma \cdot k - m)\,. \end{aligned} \tag{A.25}$$

With the van der Waerden convention for Dirac matrices, the solutions are

$$X(\boldsymbol{k}) = \sqrt{m} \begin{pmatrix} U(\boldsymbol{k})\sqrt{k \cdot \sigma} & U(\boldsymbol{k})\sqrt{k \cdot \tilde{\sigma}} \\ U'(\boldsymbol{k})\sqrt{k \cdot \tilde{\sigma}} & -U'(\boldsymbol{k})\sqrt{k \cdot \sigma} \end{pmatrix} \tag{A.26}$$

where U and U' are SU(2)-valued, Lebesgue measureable functions of $\boldsymbol{k}$.

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