SL(2,C) Exponentiation Counterexample

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There are elements of SL(2, C) that cannot be written as the exponential of an element of the Lie algebra. That is not exotic for Lie groups, but as physicists we are not always as systematic about nailing down irrelevant facts as scholarship might demand, and this fact was a mild surprise to me when I first encountered it. This note describes a set of counterexamples to exponentiation in the subgroup $\Gamma \subset \mathrm{SL}(2, \mathrm{C})$ of matrices of the form[1]

$$A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \qquad a, b \in \mathbb{C}\,. \tag{1}$$

First, some notation. Let

$$\boldsymbol{z} = (z_1, z_2, z_3) \tag{2}$$

be a complex three-vector, and let the components of

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \tag{3}$$

be the standard 2×2 Pauli matrices. The three-vector dot products $\boldsymbol{z}\cdot\boldsymbol{z}$ and $\boldsymbol{z}\cdot\boldsymbol{\sigma}$ do not involve conjugation. Any element of the 6-dimensional, real Lie algebra of SL(2, C) can be written as $-i\boldsymbol{z}\cdot\boldsymbol{\sigma}$, where[2]

$$\boldsymbol{z}\cdot\boldsymbol{\sigma} = \begin{pmatrix} z_3 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 \end{pmatrix}, \tag{4}$$

[1] Although I've forgotten his exact formulation, I'm sure I learned about this from one of Eyvind Wichmann's lucid sets of lecture notes.

[2] Here $\boldsymbol{J} = \boldsymbol{\sigma}/2$ generates rotations and $\boldsymbol{K} = -i\boldsymbol{\sigma}/2$ generates boosts.

and it is a Pauli-matrix identity that

$$(\boldsymbol{z}\cdot\boldsymbol{\sigma})^2 = \boldsymbol{z}\cdot\boldsymbol{z}\, I \equiv w^2\, I\,. \tag{5}$$

With the help of this identity, it is easy to compute the exponential of $-i\boldsymbol{z}\cdot\boldsymbol{\sigma}$ from its power series, which converges everywhere:

$$\exp -i\boldsymbol{z}\cdot\boldsymbol{\sigma} = I\,\cos w - i\boldsymbol{z}\cdot\boldsymbol{\sigma}\,\frac{\sin w}{w}\,. \tag{6}$$

The theorem below parametrizes the elements of Γ that are exponentials. The set of elements which are not exponentials is then noted in a corollary.

Theorem. $A = \exp -i\boldsymbol{z}\cdot\boldsymbol{\sigma} \in \Gamma$ if and only if A and $\boldsymbol{z}$ have one of the two forms,

$$\text{I:}\quad A = (-1)^n\, I\,, \qquad \boldsymbol{z}\cdot\boldsymbol{z} = (n\pi)^2, \quad n = \pm1, \pm2, \dots\,, \tag{7a}$$

$$\text{II:}\quad A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \qquad \boldsymbol{z} = \left(i\frac{bz_3}{2\sin z_3}\,, \frac{bz_3}{2\sin z_3}\,, z_3 \right), \tag{7b}$$

$$z_3 \neq n\pi\,, \quad n = \pm1, \pm2, \dots\,.$$

The identity matrix is common to both forms, but the conditions on $\boldsymbol{z}$ are mutually exclusive.

Proof. Let

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \exp -i\boldsymbol{z}\cdot\boldsymbol{\sigma} = I\,\cos w - i\boldsymbol{z}\cdot\boldsymbol{\sigma}\,\frac{\sin w}{w}\,, \tag{8}$$

From the form of $\boldsymbol{z}\cdot\boldsymbol{\sigma}$ in Eq. (4), we must have

$$(z_1 + iz_2)\,\frac{\sin w}{w} = 0\,. \tag{9}$$

There are two, not entirely exclusive cases:

1. $\underline{\sin w/w = 0}$

Note that $w \neq 0$ in this case, because then $\sin w/w = 1$. Thus,

$$\begin{aligned} \sin w &= 0\,, \qquad w = n\pi\,, \qquad n = \pm1, \pm2, \dots\,, \\ \boldsymbol{z}\cdot\boldsymbol{z} &= w^2 = (n\pi)^2, \\ \cos w &= (-1)^n, \\ \exp -i\boldsymbol{z}\cdot\boldsymbol{\sigma} &= (-1)^n\, I\,. \end{aligned} \tag{10}$$

That precisely covers form I.

2. $\underline{z_1 + iz_2 = 0}$

In this case

$$w^2 = \boldsymbol{z}\cdot\boldsymbol{z} = (z_3)^2, \tag{11a}$$

$$\begin{aligned} \exp -i\boldsymbol{z}\cdot\boldsymbol{\sigma} &= I \cos z_3 - i \begin{pmatrix} z_3 & 2z_1 \\ 0 & -z_3 \end{pmatrix} \frac{\sin z_3}{z_3} \\ &= \begin{pmatrix} \exp -iz_3 & -2iz_1 \dfrac{\sin z_3}{z_3} \\ 0 & \exp iz_3 \end{pmatrix}. \end{aligned} \tag{11b}$$

When $\sin z_3/z_3 = 0$, we have $z_3 = n\pi$, $n = \pm 1, \pm 2, \dots$, and $\boldsymbol{z}\cdot\boldsymbol{z} = (n\pi)^2$, a special case of form I.

When $\sin z_3/z_3 \neq 0$, we precisely cover form II, which can be split into two subforms, each with unrestricted b:

$$z_3 = 0\,, \qquad (a, b) = (1, -2iz_1), \tag{12a}$$

$$z_3 \neq 0, \pm\pi, \pm 2\pi, \dots, \qquad (a, b) = \left(\exp -iz_3, -2iz_1 \frac{z_3}{\sin z_3}\right). \tag{12b}$$

□

Corollary. $A \in \Gamma$ is not an exponential if and only if it has the form

$$A = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, \qquad b \neq 0\,. \tag{13}$$

Proof. According to the theorem, the pairs $(a, b) \in \mathbb{C}^2$ for which A corresponds to an exponential of form I belong to the set

$$E_{\mathrm{I}} = \{(a, b)\colon\ a = \pm 1 \quad \text{and} \quad b = 0\}, \tag{14a}$$

and for form II, according to Eqs. (12a) and (12b), to the set[3]

$$\begin{aligned} E_{\mathrm{II}} &= \{(a, b)\colon\ a = 1 \quad \text{or} \quad \sin z_3 \neq 0\} \\ &= \{(a, b)\colon\ a = 1 \quad \text{or} \quad a \neq \pm 1\}. \end{aligned} \tag{14b}$$

[3] The two sets have the point $(a, b) = (1, 0)$ in common.

Since

$$\neg E_{\mathrm{I}} = \{(a,b):\ a \neq \pm 1 \quad \text{or} \quad b \neq 0\}, \\ = \left\{(a,b):\ a \neq \pm 1\right\} \bigcup \left\{(a,b):\ b \neq 0\right\}, \tag{15a}$$

and

$$\neg E_{\mathrm{II}} = \left\{(a,b):\ a \neq 1\right\} \bigcap \left\{(a,b):\ a = \pm 1\right\} \\ = \left\{(a,b):\ a = -1\right\}, \tag{15b}$$

the negation of $E_{\mathrm{I}} \cup E_{\mathrm{II}}$ is

$$\neg E_{\mathrm{I}} \cap \neg E_{\mathrm{II}} = \left\{(a,b):\ a = -1 \quad \text{and} \quad b \neq 0\right\}. \tag{16}$$

□