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A Relativistic One-Particle Wave-Equation for Arbitrary Spin and Non-Zero Mass

We proceed by a slightly more indirect analogy to the Dirac equation than is customary for the Kemmer-Duffin equation or Bhabha equation, the aim being to avoid a redundant multiplicity in the spin space and hence avoid the necessity for subsidiary conditions.

For the Dirac equation we use a special (Van der Waerden ?) representation,

$$\alpha_\mu = \begin{pmatrix} \tilde{\sigma}_\mu & 0 \\ 0 & \sigma_\mu \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma_\mu = \beta\alpha_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1)$$

$$\tilde{\sigma}_\mu = (\sigma_0, -\vec{\sigma})_\mu \; ;$$

$$g_{\mu\nu} = (+\,-\,-\,-)$$

$$\gamma_\mu k^\mu u(k) = m\, u(k). \qquad (2)$$

$$i\gamma^\mu \partial_\mu \phi(x) = m\phi(x)$$

$$\phi(x) = e^{-ik\cdot x} u(k)$$

If we write

$$u(k) = \begin{pmatrix} \xi(k) \\ \tilde{\eta}(k) \end{pmatrix} \qquad (3)$$

then

$$k\cdot\tilde{\sigma}\, \xi(k) = m\, \tilde{\eta}(k)$$
$$k\cdot\sigma\, \tilde{\eta}(k) = m\, \xi(k). \qquad (4)$$

The solutions can be obtained in terms of their rest frame values, ϕ. Let $p = (m,0,0,0)$ and $B_{k\leftarrow p}$ an arbitrary 2x2 unimodular matrix such that

$$B_{k\leftarrow p} \frac{p\cdot\sigma}{m} B^\dagger_{k\leftarrow p} = \frac{k\cdot\sigma}{m} = B_{k\leftarrow p} B^\dagger_{k\leftarrow p} \qquad (5)$$

The most general form is

$$B_{k\leftarrow p} = R_{k\leftarrow q} A_{q\leftarrow p} \quad (\text{or } A_{k\leftarrow p} R) \qquad (6)$$

where $A_{q\leftarrow p} = \sqrt{\frac{q\cdot\sigma}{m}} = A^\dagger_{q\leftarrow p}$ and $R^{-1}_{k\leftarrow q} = R^\dagger_{k\leftarrow q}$. (7)

That is, $R_{k\leftarrow q}$ represents a rotation of q into k and $A_{q\leftarrow p}$ a pure Lorentz transformation.

We write

$$\xi(k) = B_{k\leftarrow p}\,\phi. \tag{8}$$

Then from (4)

$$\begin{aligned}\tilde{\eta}(k) &= \frac{k\cdot\tilde{\sigma}}{m} B_{k\leftarrow p}\,\phi \\ &= B^{-1\dagger}_{k\leftarrow p} B^{\dagger}_{k\leftarrow p} \frac{k\cdot\tilde{\sigma}}{m} B_{k\leftarrow p}\,\phi \\ \tilde{\eta}(k) &= B^{-1\dagger}_{k\leftarrow p}\,\phi\end{aligned} \tag{9}$$

since

$$\begin{aligned}B^{\dagger}_{k\leftarrow p} \frac{k\cdot\tilde{\sigma}}{m} B_{k\leftarrow p} &= \left(B^{-1}_{k\leftarrow p} \frac{k\cdot\sigma}{m} B^{-1\dagger}_{k\leftarrow p}\right)^{-1} \\ &= \left(\frac{p\cdot\sigma}{m}\right)^{-1} = I\end{aligned}$$

and

$$k\cdot\sigma\; k\cdot\tilde{\sigma} = k\cdot k = m^2. \tag{10}$$

Thus,

$$u_{\pm}(k) = \begin{pmatrix} B_{k\leftarrow p}\,\phi \\ \pm B^{-1\dagger}_{k\leftarrow p}\,\phi \end{pmatrix}, \tag{11}$$

where $\pm$ indicates (pos/neg) energy solutions.*

Now, from (1) and (2) we have

$$\begin{pmatrix} 0 & \frac{\sigma\cdot k}{m} \\ \frac{\tilde{\sigma}\cdot k}{m} & 0 \end{pmatrix} u(k) = \begin{pmatrix} 0 & D^{\frac{1}{2}0}\left(\frac{\sigma\cdot k}{m}\right) \\ D^{0\frac{1}{2}}\left(\frac{\sigma\cdot k}{m}\right) & 0 \end{pmatrix} u(k) = u(k) \tag{12}$$

* For fermions only.

because $\frac{\sigma\cdot k}{m}$ is a unimodular 2x2 matrix and $D^{\frac{1}{2}0}(A) = A$, $D^{0\frac{1}{2}}(A) = D^{\frac{1}{2}0}(\tilde{A}) = \tilde{A}$ for A a 2x2 unimodular matrix.

Our wave equation for arbitrary spin is written analagously to (12)

$$\begin{pmatrix} 0 & D^{j0}\left(\frac{\sigma\cdot P}{m}\right) \\ D^{0j}\left(\frac{\tilde{\sigma}\cdot P}{m}\right) & 0 \end{pmatrix} \phi(x) = \phi(x) \tag{13}$$

where

$$P_\mu = i\partial_\mu - eA_\mu, \tag{14}$$

and the differential operator $D^{j0}\left(\frac{\sigma\cdot P}{m}\right)$ is defined, for example, by

$$D^{10}_{\alpha\beta}\left(\frac{\sigma\cdot P}{m}\right) = \sum_{\substack{\alpha_1\alpha_2 \\ \beta_1\beta_2}} C_{\frac{1}{2}\frac{1}{2}}(1\alpha;\alpha_1\alpha_2)\, C_{\frac{1}{2}\frac{1}{2}}(1\beta;\beta_1\beta_2)\, \times \left(\frac{\sigma\cdot P}{m}\right)_{\alpha_1\alpha_2} \left(\frac{\sigma\cdot P}{m}\right)_{\alpha_2\beta_2} \tag{15}$$

Note that in general $D^{j0}\left(\frac{\sigma\cdot P}{m}\right)$ is a differential operator of order $2j$; hence there are more boundary conditions than usual. By operating on both sides of (13) with the differential operator on the left of (13) we find that

$$\left(\frac{P^\mu P_\mu}{m^2}\right)^{2j} \phi(x) = \phi(x) \qquad (16)$$

is identically satisfied by each of the $2(2j+1)$ components of $\phi(x)$. The extra boundary conditions will be contained in the requirement that in fact the components satisfy the Klein-Gordon equation,

$$\frac{P^\mu P_\mu}{m^2}\phi(x) = \phi(x). \qquad (17)$$

Then the free particle solutions can be written

$$\phi_\pm(x) = e^{-ik\cdot x}\begin{pmatrix} D^{j0}(B_{k\leftarrow p})\,\phi \\ \pm D^{0j}(B_{k\leftarrow p})\,\phi \end{pmatrix}, \qquad (18)$$

where ϕ is a constant $2j+1$ component spinor, hence can be chosen in $2j+1$ independent ways, and $\pm$ indicate $\binom{\text{pos}}{\text{neg}}$ energy solutions (for fermions only).

Alternatively, we could have expressed the definition of the differential operator, analagous to (15) and the requirement (17) in the definition

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int_{(+)} \frac{d^3k}{\omega_k}\, e^{-ik\cdot x} f(k) \qquad (19)$$

$$\begin{pmatrix} 0 & D^{j0}\left(\frac{\sigma\cdot i\partial}{m}\right) \\ D^{0j}\left(\frac{\sigma\cdot i\partial}{m}\right) & 0 \end{pmatrix}\phi(x) = \qquad (20)$$

$$= \frac{1}{(2\pi)^{3/2}}\int_{(+)}\frac{d^3k}{\omega_k}\begin{pmatrix} 0 & D^{j0}\left(\frac{\sigma\cdot k}{m}\right) \\ D^{0j}\left(\frac{\sigma\cdot k}{m}\right) & 0 \end{pmatrix} e^{-ik\cdot x} f(k),$$

where the D^{j0} in the integral is well-defined since $\sigma\cdot k/m$ is a unimodular matrix (2x2).

Thus we have obtained a wave equation with spinor solutions transforming according to the representation $D^{j0}\oplus D^{0j}$ which is the analog of the 4-component spinor formalism for spin-½; and which reduces to it when $j=\frac{1}{2}$.

the spinors are of just the right dimension and hence no subsidiary conditions are required. One may ask why only the representations $\mathcal{D}^{j0}$ and not the more general $\mathcal{D}^{j_1 j_2}$ were considered. The answer is that of course one can do this without strain, but as far as we are able to determine, the more general representations are not necessary for physics. The point is that it is presumably the unitary representations of the inhomogeneous Lorentz group which have physical significance, and our investigations in S-matrix theory have shown that these can be discussed in terms of the $\mathcal{D}^{j0}$ representations of the homogeneous group. It is just a matter of algebra to verify whether this goes through in the case here.

FIELD THEORIES OF ARBITRARY SPIN (FERMIONS)

It is a very straightforward to discuss all of the standard topics in the theory of the Dirac equation such as forming conjugate spinors $\left(\bar{u}(k) = u^{+}(k)\beta,\ \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)$, negative energy

states, charge conjugation, helicity eigenstates, etc. And in certain respects it is quite straightforward to write down field theories for arbitrary spin by directly translating the spin-½ theory. One must expect that, for example, the algebra of calculating traces will be more complicated, but one hopes to minimize this as much as possible by using the group representation property to do calculations in terms of the 2x2 matrix arguments. When it comes to writing down interactions, however, it seems at present that there is some unavoidable complication, which we shall look into a bit later.

To give an idea of how field theory translates we indicate how to write down free fields and propagation functions. We proceed by writing down some normalization and orthogonality relations á lè Schweber ch. 4e.

First we choose an orthonormal set

of spinor solutions to the free particle equation, (18)

$$u_{+m}(k) = \begin{pmatrix} D^{j0}(B_{k\leftarrow p})\,\phi_m \\ D^{j0}(B^{-1\dagger}_{k\leftarrow p})\,\phi_m \end{pmatrix} \qquad (21)$$

$$u_{-m}(k) = \begin{pmatrix} D^{j0}(B_{k\leftarrow p})\,\phi_m \\ -D^{j0}(B^{-1\dagger}_{k\leftarrow p})\,\phi_m \end{pmatrix} \qquad (22)$$

$$\phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \phi_{2j+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad (23)$$

$$\bar{u}_{+m}(k) = \left(\phi_m D^{j0}(B^{-1}_{k\leftarrow p}),\ \phi_m D^{j0}(B^{\dagger}_{k\leftarrow p})\right)$$

$$\bar{u}_{-m}(k) = \left(-\phi_m D^{j0}(B^{-1}_{k\leftarrow p}),\ \phi_m D^{j0}(B^{\dagger}_{k\leftarrow p})\right). \qquad (24)$$

Then we have the orthonormality relations,

$$\bar{u}_{+m}(k)\, u_{+n}(k) = \delta_{mn}$$

$$\bar{u}_{\pm m}(k)\, u_{\mp n}(k) = 0 \qquad (25)$$

$$\bar{u}_{-m}(k)\, u_{-n}(k) = -\delta_{mn}, \quad m,n = 1,\ldots 2j+1;$$

$$\sum_{n=1}^{2j+1}\left[u^{\alpha}_{+n}(k)\,\bar{u}^{\beta}_{+n}(k) - u^{\alpha}_{-n}(k)\,\bar{u}^{\beta}_{-n}(k)\right] = \delta_{\alpha\beta}; \qquad (26)$$

$\alpha, \beta = 1, \cdots, 2(2j+1)$
(spinor components)

To construct positive and negative energy projection operators we note that

$$\begin{pmatrix} 0 & D^{j0}\left(\frac{\sigma\cdot k}{m}\right) \\ D^{j0}\left(\frac{\tilde{\sigma}\cdot k}{m}\right) & 0 \end{pmatrix} u_{+}(k) = u_{+}(k)$$

$$\begin{pmatrix} 0 & D^{j0}\left(\frac{\sigma\cdot k}{m}\right) \\ D^{j0}\left(\frac{\tilde{\sigma}\cdot k}{m}\right) & 0 \end{pmatrix} u_{-}(k) = -u_{-}(k). \qquad (27)$$

Then we define

$$\Lambda_+(k) = \frac{1}{2}\begin{pmatrix} I & D^{j0}\left(\frac{k\cdot\sigma}{m}\right) \\ D^{j0}\left(\frac{k\cdot\tilde{\sigma}}{m}\right) & I \end{pmatrix}, \qquad (28)$$

$$\Lambda_-(k) = \frac{1}{2}\begin{pmatrix} I & -D^{j0}\left(\frac{k\cdot\sigma}{m}\right) \\ -D^{j0}\left(\frac{k\cdot\tilde{\sigma}}{m}\right) & I \end{pmatrix}, \qquad (29)$$

which have the properties

$$\Lambda_\pm(k)\, u_\pm(k) = u_\pm(k);$$

$$\Lambda_\pm(k)\, u_\mp(k) = 0; \qquad (30)$$

$$\Lambda_+(k) + \Lambda_-(k) = I;$$

$$\Lambda_+(k)\Lambda_-(k) = \Lambda_-(k)\Lambda_+(k) = 0;$$

$$[\Lambda_+(k)]^2 = \Lambda_+(k) = \sum_{r=1}^{2j+1} u_{+r}(k) \otimes \bar{u}_{+r}(k);$$

$$[\Lambda_-(k)]^2 = \Lambda_-(k) = -\sum_{r=1}^{2j+1} u_{-r}(k) \otimes \bar{u}_{-r}(k).$$

The calculation of sums and averages over final and initial states goes precisely as usual, and we get (pos. energy case)

$$\sum_{r=1}^{2j+1} \left(\bar{u}^{\,i}_{+r}(k)\, O'\, u^{f}_{+r}(k')\right)\left(\bar{u}^{\,f}_{+r}(k')\, O\, u^{i}_{+r}(k)\right)$$

$$= \bar{u}^{\,i}_+(k)\, O'\, \Lambda_+(k')\, O\, u^{i}_+(k); \qquad (31)$$

$$\frac{1}{2j+1}\sum_{r=1}^{2j+1} \bar{u}^{\,i}_{+r}(k)\, O'\Lambda_+(k')\, O\, u^{i}_{+r}(k) \qquad (32)$$

$$= \frac{1}{2j+1}\,\mathrm{Tr}\left[O'\Lambda_+(k')\, O\, \Lambda_+(k)\right].$$

To write down the free fields and propagation functions is again utterly straightforward.

We define particle and antiparticle creation and destruction operators

$$\text{particle} \quad [b_r(p), b_s^\dagger(p')]_+ = \delta_{rs}\,\delta(p-p') \qquad (33)$$

$$\text{antiparticle} \quad [c_r(p), c_s^\dagger(p')]_+ = \delta_{rs}\,\delta(p-p'),$$

$$r, s = 1, \cdots, 2j+1$$

all other anticommutators vanishing.

The Heisenberg fields are

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int_{(\infty)} d^3p \left(\frac{m}{\omega_{\vec p}}\right)^{\frac12} \sum_{r=1}^{2j+1} \{ b_r(p)\, u_{+r}(p) e^{-ip\cdot x} +$$

$$+ c_r^\dagger(p)\, u_{-r}(p)\, e^{ip\cdot x} \}; \qquad (34)$$

$$\tilde\psi(x) = \frac{1}{(2\pi)^{3/2}} \int_{(\infty)} d^3p \left(\frac{m}{\omega_{\vec p}}\right)^{\frac12} \sum_{r=1}^{2j+1} \{ b_r^\dagger(p)\, \tilde u_{+r}(p) e^{ip\cdot x} +$$

$$+ c_r(p)\, \tilde u_{-r}(p)\, e^{-ip\cdot x} \}. \qquad (35)$$

These fields satisfy our free particle wave equation.

The commutation relations are

$$[\psi_\alpha(x), \tilde\psi_\beta(x')]_+ = -i\, S_{\alpha\beta}(x-x', m) \qquad (36)$$

$$S(x-x';m) = -m \begin{pmatrix} I & D^{j0}\left(\frac{\sigma\cdot i\partial}{m}\right) \\ D^{0j}\left(\frac{\bar\sigma\cdot i\partial}{m}\right) & I \end{pmatrix} \Delta(x-x';m) \qquad (37)$$

$$\Delta(x-x';m) = \frac{-i}{2(2\pi)^3} \int \frac{d^3k}{\omega_{\vec k}} \left(e^{-ik\cdot x} - e^{ik\cdot x}\right). \qquad (38)$$

The (+) and (−) S functions are defined by substituting Δ_+, Δ_- into (37).

We must next look into the question of CPT transformations on the fields, which should be straightforward, and of how to form interactions and bases for the operators O and O' in (31) and (32), which looks nontrivial.