Summary of Local Baker-Wightman Invariance for Euclidean Functional Field Equations*

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1 Introduction

Baker and Wightman[1] introduced the idea that the class of solutions of the functional field equation may be enlarged, and the triviality problem for four dimensions possibly avoided, by exploiting its invariance under the choice of contours

*February 18, 2021: The body of this document is an almost literal transcription of the original manuscript, with the above date, entitled "Notes on the Euclidean ϕ^4 functional field equation." The table of contents was added in April, 1994, and the introduction in March, 2008.

[1] G. A. Baker and A. S. Wightman, "Trying to Violate Coupling Constant Bounds in ϕ^4_ν Quantum Field Theory," *in* Progress in Quantum Field Theory, eds. H. Ezawa and S. Kamefuchi, (Elsevier Science Publishers, 1986), pp. 15–29.

in the lattice version of the Euclidean path space integration. We call this *Baker-Wightman invariance*. They discussed a ferromagnetic and antiferromagnetic mixture, and showed that unfortunately it violated the cluster property.

These notes summarize our local extension of their idea, called *local Baker-Wightman invariance*, which aims to restore the cluster property. It was the starting point for an honors thesis by Michael K. Weiss,[2] and was used by the author to explain why, in an ultralocal model, the continuum limit from the lattice was trivial, but the Euclidean functional field equation nevertheless had a nontrivial solution obeying the cluster property.[3]

2 Continuum field equation

Bare generating functional:

$$\mathcal{E}_0(J_0) = \frac{\int \mathrm{d}\phi_0 \, \exp\left[\left(-\frac{1}{2}\langle\, \phi_0(-\Delta + \mu_0^2)\phi_0 \,\rangle - \frac{1}{4}\lambda_0 \langle\, \phi_0^4 \,\rangle + \langle\, \phi_0 J_0 \,\rangle\right)/\hbar c\right]}{\int \mathrm{d}\phi_0 \, \exp\left[(\cdots)/\hbar c\right]\big|_{J_0=0}}$$

$$\langle\, f \,\rangle \equiv \int f(x)\, \mathrm{d}^4 x$$

Renormalization:

$$\mu_0^2 = \mu^2 + \delta\mu^2\,, \qquad \lambda_0 = \lambda + \delta\lambda\,, \qquad \phi_0 = \sqrt{Z}\phi\,, \qquad J_0 = J/\sqrt{Z}$$

$$\mathcal{E}_0(J_0) = \mathcal{E}(J)$$

$$= \frac{\int \mathrm{d}\phi \, \exp\left[\left(-\frac{1}{2}Z\langle\, \phi(-\Delta + \mu^2 + \delta\mu^2)\phi \,\rangle - \frac{1}{4}(\lambda + \delta\lambda)Z^2\langle\, \phi^4 \,\rangle + \langle\, \phi J \,\rangle\right)/\hbar c\right]}{\int \mathrm{d}\phi \, \exp[(\cdots)/\hbar c]\big|_{J=0}}$$

[2] Michael K. Weiss, "Standard and Nonstandard ϕ^4 Theories *with* An Introduction to Quantum Field Theory," honors thesis for the Bachelor of Science degree, University of Michigan, April 15, 1994.

[3] David N. Williams, "Triviality and Nontriviality of Ultralocal, Euclidean Φ^4," unreleased manuscript, 1985.

Field equation: Define

$$\mathcal{E}(J) = \exp[L(J)/\hbar c] = \exp[L_0(J_0)/\hbar c] = \mathcal{E}_0(J_0)$$

$$L(J) = \text{connected generating functional}$$

$$\phi(x) = \frac{\delta L}{\delta J(x)} = \frac{1}{\sqrt{Z}} \frac{\delta L_0}{\delta J_0(x)} = \frac{1}{\sqrt{Z}}\,\phi_0(x) = \text{“effective field”}$$

The field equation results from the formal integration by parts identity: let

$$Z(J) = \int \mathrm{d}\phi\ \exp\left[\frac{-S(\phi) + \langle\, \phi J\,\rangle}{\hbar c}\right]$$

$$\mathcal{E}(J) \equiv Z(J)/Z(0)$$

$$\begin{aligned} 0 &= \int \mathrm{d}\phi\ \hbar c \frac{\delta}{\delta\phi} \exp\left[\frac{-S(\phi) + \langle\, \phi J\,\rangle}{\hbar c}\right] \\ &= \int \mathrm{d}\phi\ \left(-\frac{\delta S}{\delta \phi} + J\right) \exp\left[\frac{-S(\phi) + \langle\, \phi J\,\rangle}{\hbar c}\right] \\ &= \left[-\frac{\delta S}{\delta\phi}\left(\hbar c \frac{\delta}{\delta J}\right) + J\right] Z(J) \end{aligned}$$

$$\boxed{\left[\frac{\delta S}{\delta\phi}\left(\hbar c \frac{\delta}{\delta J}\right) - J\right] \mathcal{E}(J) = 0}$$

When expressed in terms of $L(J)$ and ϕ (effective), this becomes:

$$\boxed{\begin{aligned} J &= -Z\Delta\phi + \left(\mu^2 + \delta\mu^2\right) Z\phi \\ &\qquad + (\lambda + \delta\lambda)\, Z^2 \left[\phi^3 + 3\hbar c\, \phi\delta\phi + (\hbar c)^2 \delta^2\phi\right] \\ \delta\phi &\equiv \frac{\delta\phi(x)}{\delta J(x)}, \qquad \delta^2\phi \equiv \frac{\delta^2\phi(x)}{\delta J(x)\delta J(x)} \end{aligned}}$$

In the above, J, ϕ, μ^2, and λ are renormalized. This is a perfectly good starting point for perturbative renormalization, and in many ways is quite efficient.

The bare version is

$$\boxed{\begin{aligned} J_0 &= -\Delta\phi_0 + \mu_0^2\,\phi_0 + \lambda_0\left[\phi_0^3 + 3\hbar c\,\phi_0\delta\phi_0 + (\hbar c)^2\delta^2\phi_0\right] \\ \delta\phi_0 &\equiv \frac{\delta\phi_0(x)}{\delta J_0(x)}\,, \qquad \delta^2\phi_0 \equiv \frac{\delta^2\phi_0(x)}{\delta J_0(x)\delta J_0(x)} \end{aligned}}$$

3 Discretized field equation

For simplicity of notation, we look at the bare theory for $(1+0)$ time-space dimensions, and we put $\hbar c = 1$.

$$Z(J) = \int\prod_i \mathrm{d}\phi_i\;\exp\sum_j\left[-\frac{(\phi_{j+1}-\phi_j)^2}{2a^2} - \frac{\mu_0^2}{2}\,\phi_j^2 - \frac{\lambda_0}{4}\,\phi_j^4 + \phi_j J_j\right] a^d$$

where a = lattice spacing, $a^d = a$ "=" $\mathrm{d}^d x$. Integration by parts gives the field equation:

$$0 = \int\prod_i \mathrm{d}\phi_i\;\left[\frac{(\Delta\phi)_\ell}{a^2} - \mu_0^2\,\phi_\ell - \lambda_0\,\phi_\ell^3 + J_\ell\right]\exp[\cdots]$$

$$(\Delta\phi)_\ell \equiv \phi_{\ell+1} - 2\phi_\ell + \phi_{\ell-1}$$

$$0 = \left[-\left(\frac{\Delta}{a^2}\frac{\partial}{\partial J}\right)_\ell + \mu_0^2\,\frac{\partial}{\partial J_\ell} + \lambda_0\,\frac{\partial^3}{\partial J_\ell^3}\right] Z(J)$$

The corresponding equation in terms of the effective field can be easily written down. We leave things this way for now to discuss our local version of Baker and Wightman's invariance idea.

Note that if ϕ_j at one or more sites j is complex instead of real, the integration by parts identity and its expression in terms of $\partial/\partial J$ remains unchanged. The only technical point is the convergence of the integration. For pure imaginary ϕ_j, this too is satisfied (at least for a finite lattice), because the $\lambda_0\phi_j^4$ terms dominate the exponential, and they remain positive.

Thus, any linear combination of $Z(J)$'s with arbitrary selections of real and imaginary ϕ_j's satisfies the field equation. We next discuss the choice of these combinations necessary for reality and cluster properties.

4 Reality

Reality conditions may be discussed quite generally. Let $S(\phi) = S(\phi_1, \phi_2, ...)$ be any function of fields on the lattice (any dimension) which obeys the following:

(i) $S(\phi)$ is an entire function of its arguments that is real when all ϕ_j's are real, so that under complex conjugation

$$S(\phi_1, \phi_2, ...)^* = S(\phi_1^*, \phi_2^*, ...)\,;$$

(ii) $\int d\phi_1 d\phi_2 \cdots \exp[-S(\phi) + \langle\, \phi J\, \rangle]$ is well defined for any assignment of real and pure imaginary ϕ_j's at the various sites.

Then

$$\frac{\partial}{\partial J_{\ell_1}} \cdots \frac{\partial}{\partial J_{\ell_n}}\, i^N \int d\phi_1 d\phi_2 \cdots \exp[-S(\phi) + \langle\, \phi J\, \rangle]$$

is real for real J's and $N =$ the number of pure imaginary ϕ's.

Proof:

$$\left[i^N \int d\phi_1 d\phi_2 \cdots \exp[-S(\phi) + \langle\, \phi J\, \rangle] \right]^*$$

$$= (-i)^N \int d\phi_1^* d\phi_2^* \cdots \exp[-S(\phi^*) + \langle\, \phi^* J\, \rangle]$$

$$= (-i)^N (-1)^N \int d\phi_1 d\phi_2 \cdots \exp[-S(\phi^*) + \langle\, \phi^* J\, \rangle]$$

Now for pure imaginary $\phi_j^* = -\phi_j$ make the change of variables in the integration $\phi_j' = -\phi_j$,

$$\int_{-i\infty}^{i\infty} d\phi_j = \int_{-i\infty}^{i\infty} d\phi_j'\,,$$

and we find that $[\quad]^* = [\quad]$. Derivatives with respect to real J_ℓ's are therefore also real.

Note that this covers the case of Euclidean ϕ^4 theories with cubic terms in the action. That is, the only role of evenness in the action $S(\phi)$ is in the evenness and positivity of the dominant term, to make the integrals converge.

So far, we have learned that real linear combinations of

$$i^N \int \mathrm{d}\phi \, \exp[-S(\phi) + \langle \, \phi J \, \rangle]$$

over selections of real and imaginary fields at each site are real and obey the field equation.

5 Cluster

The reasoning here is handwaving, but surely determines a *necessary* condition for the cluster property. We assume an infinite lattice, with no convergence problems for an arbitrary assignment of real and imaginary fields.

Then the normalized functional $\mathcal{E}(J) = Z(J)/Z(0)$ gives rise to the cluster property when the n-point functions obey

$$\langle \, \phi_{i_1} \cdots \phi_{i_m} \phi^a_{j_1} \cdots \phi^a_{j_n} \, \rangle \xrightarrow[a \to \infty]{} \langle \, \phi_{i_1} \cdots \phi_{i_m} \, \rangle \langle \, \phi_{j_1} \cdots \phi_{j_n} \, \rangle \, ,$$

where a is a space-time translation (in Euclidean theory), and where

$$\langle \, \phi_{i_1} \cdots \phi_{i_m} \, \rangle = \left[\int \mathrm{d}\phi \, e^{-S(\phi)} \, \phi_{i_1} \cdots \phi_{i_m} \right] / Z(0) \, , \quad \text{etc.}$$

The nonindependence or nonvanishing correlation at finite translations a is due only to the gradient terms in the action, and we expect it to remain a small probability business at large a, independent of the sign or reality of the quadratic, nearest or next-nearest neighbor terms in the action. Thus, we expect that every term in a sum over reality assignments in ϕ^4 will factorize at large separations.

A linear combination will not, however, factorize into a product of combinations corresponding to the same $\mathcal{E}(J)$ unless chosen judiciously. In particular, the relative weight of the real and imaginary selections of fields must be the same at each site.

To describe that, we parametrize the combinations as follows, in terms of real fields with explicit factors of i. A *reality configuration* $\{q\}$ is a sequence

$$\{q\} = \{q_1, q_2, \dots \}$$

indexed by sites, where $q_i = 1$ or i. To each site in the configuration we assign a weight α_N when $q_i = 1$ (real, normal), and α_A when $q_i = i$ (imaginary, abnormal). Here α_N and α_A are real, and α_N is the same at each real site, while α_A is the same at each imaginary site.

Define

$$\begin{aligned} Z(J) &= \sum_{\{q\}} \int \prod_i \mathrm{d}\phi_i \; \alpha_\mathrm{N}^{\#\mathrm{reals}} \alpha_\mathrm{A}^{\#\mathrm{imags}} \\ &\qquad \times \exp\left[-S(\phi_1 q_1, \phi_2 q_2, \cdots) + \sum_i \phi_i q_i J_i \right], \\ \mathcal{E}(J) &= Z(J)/Z(0)\,. \end{aligned}$$

Instead of a single path integral, we have here a distribution of path integrals over reality configurations.

Finally, we write this out in the (1+0)-dimensional case (bare quantities):[4]

$$\boxed{\begin{aligned} Z(J) = \sum_{\{q\}} \int \prod_i \mathrm{d}\phi_i \, \alpha_\mathrm{N}^{\#\mathrm{reals}} \alpha_\mathrm{A}^{\#\mathrm{imags}} \\ \times \exp -a \sum_j \Bigg[\frac{(\phi_{j+1} q_{j+1} - \phi_j q_j)^2}{2a^2} \\ + \frac{\mu_0^2}{2}\, \phi_j^2 q_j^2 + \frac{\lambda_0}{4}\, \phi_j^4 q_j^4 - \phi_j q_j J_j \Bigg] \end{aligned}}$$

[4]It would have been cleaner to write this in terms of normalized functionals with weights obeying $\alpha_\mathrm{N} + \alpha_\mathrm{A} = 1$.